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EXISTENCE OF THREE WEAK SOLUTIONS FOR KIRCHHOFF-TYPE PROBLEMS WITH VARIABLE EXPONENT AND NONHOMOGENEOUS NEUMANN CONDITIONS

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Abstract. We study the existence of at least three weak solutions for a class of differential equations with $p(x)$-Kirchhoff-type and subject to perturbations of nonhomogeneous Neumann conditions. Our technical approach is based on variational methods. Some applications and examples illustrate the obtained results.
Key Words and Phrases: Variable exponent Sobolev spaces, $p(x)$-Kirchhoff-type problems, three weak solutions, variational methods.
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1. Introduction

The aim of this paper is to establish the existence of at least three weak solutions for the following perturbed $p(x)$-Kirchhoff-type problem

$$\begin{cases} T(u) = \lambda f(x, u(x)), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = \mu g(\gamma(u(x))), & \text{on } \partial\Omega \end{cases} \qquad (P^{f,g}_{\lambda,\mu})$$

where

$$T(u) = M\left(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)dx\right)(-\Delta_{p(x)}u + \alpha(x)|u|^{p(x)-2}u),$$

$\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, $M : [0, +\infty[\to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$, $p \in C(\bar{\Omega})$, $\lambda > 0$, $\mu \geq 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function,

$g : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\alpha \in L^\infty(\Omega)$, with $ess \inf_\Omega \alpha > 0$, v is the outer unit normal to $\partial\Omega$ and $\gamma : W^{1,p(x)}(\Omega) \to L^{p(x)}(\partial\Omega)$ is the trace operator.

The Kirchhoff equation refers back to Kirchhoff [37] in 1883 in the study on the oscillations of stretched strings and plates, suggested as an extended version of the classical D'Alembert's wave equation by taking into account the effects of the changes in the length of the string during the vibrations. Kirchhoff's model like the problem $(P^{f,g}_{\lambda,\mu})$ model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. Lions in [39] has proposed an abstract framework for the Kirchhoff-type equations. After the work by Lions, various problems of Kirchhoff-type have been widely investigated. We refer the reader to the papers [30, 42, 43, 51, 54] and the references therein.

The $p(x)$-Laplacian is a meaningful generalization of the p-Laplacian. The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. The necessary framework for the study of these problems is represented by the function spaces with variable exponent $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problems of this type is the model of motion of electrorheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [52, 58]. Materials which require such advanced theories have been under experimental studies from the 1950 onwards. The first important discovery on electrorheological fluids was contributed by Willis Winslow in 1949. The viscosity of these fluids depends upon the electric field of the fluids. He discovered that the viscosity of such fluids as instance lithium polymetachrylate in an electrical field is an inverse relation to the strength of the field. The field causes string-like formations in the fluid, parallel to the field. They can increase the viscosity five orders of magnitude. This event is called the Winslow effect. For a general account of the underlying physics see [31] and for some technical applications [46]. Electrorheological fluids also have functions in robotics and space technology. Many experimental researches have been done chiefly in the USA, as in NASA laboratories. Problems with variable exponent growth conditions also appear in the mathematical modelling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the filtration processes of an ideal barotropic gas through a porous medium [2, 3]. Another application of these equations is in image processing [17], in which the variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. For background and recent results on problems with variable exponent, we refer the reader to [1, 10, 11, 14, 21, 28, 29, 32, 33, 34, 41, 45, 47, 48, 49, 50, 53] and the references therein for details. For example, Fan and Ji in [28] by applying a variational principle due to Ricceri and the theory of the variable exponent Sobolev spaces, proved the existence of infinitely many solutions of the following Neumann problem involving the $p(x)$-Laplacian of the following form

$$\begin{cases}-div(|\nabla u|^{p(x)-2}\nabla u)+\lambda(x)|u|^{p(x)-2}u=f(x,u)+g(x,u), & \text{in } \Omega,\\ \frac{\partial u}{\partial \gamma}=0, & \text{on } \partial\Omega.\end{cases}$$

Mihăilescu in [41] by using as the main tool a variational principle due to Ricceri, together with the Palais-Smale property, studied the existence of at least three solutions for the following Neumann problem:

$$\begin{cases}-div(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=\lambda f(x,u), & \text{for } x\in\Omega,\\ \frac{\partial u}{\partial v}=0, & \text{for } x\in\partial\Omega\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded domain with a smooth boundary, $\lambda>0$ is a real number, p is a continuous function on $\overline{\Omega}$ with $\inf_{y\in\overline{\Omega}} p(y)>N$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Cammaroto et al. in [14] based on a three critical points theorem established by Ricceri, obtained weak solutions for a Neumann problem involving the $p(x)$-Laplacian. Bonanno and Chinnì in [11] by applying variational methods under appropriate growth conditions on the nonlinearity, obtained the existence of multiple solutions for nonlinear elliptic Dirichlet problems with variable exponent. D'Aguì and Sciammetta in [21] based on variational methods, established the existence of an unbounded sequence of weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$, in the case $M\equiv 1$.

Recently, the study of $p(x)$-Kirchhoff type problems has been an interesting topic. We refer the reader to [15, 16, 18, 22, 23, 24, 25, 27, 35, 56] for an overview of and references on this subject. For example, Dai and Hao in [22] by means of a direct variational approach and the theory of the variable exponent Sobolev spaces, we establish conditions ensuring the existence and multiplicity of solutions for the $p(x)$-Kirchhoff-type problem with Dirichlet boundary data. Chung in [18] by using the mountain pass theorem combined with the Ekeland variational principle, obtained at least two distinct, non-trivial weak solutions for the $p(x)$-Kirchhoff type equation

$$\begin{cases}-M(\int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}dx)div(|\nabla u|^{p(x)-2}\nabla u)=\lambda f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega\end{cases}$$

where $M:\mathbb{R}^+\to\mathbb{R}$ is a continuous function, $p\in C_+(\overline{\Omega})$ and $f:\Omega\times\mathbb{R}^+\to\mathbb{R}$ is a Carathéodory function, satisfying some certain conditions, λ is a parameter. In [25] multiplicity results for the problem $(P^{f,g}_{\lambda,\mu})$, in the case $\lambda=\mu$ were established. In fact, using variational methods and critical point theory the existence results for the problem under algebraic conditions with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear term were ensured. Furthermore, by combining two algebraic conditions on the nonlinear term which guarantees the existence of two solutions, applying the mountain pass theorem given by Pucci and Serrin the existence of third solution for the problem was proved while in [24] based on variational methods the existence of at least one weak solution for the same problem was discussed.

We also refer the reader to the paper [36, 40] in which using variational methods the existence of multiple solutions for nonlocal systems of $(p(x),q(x))$-Kirchhoff type was discussed.

The existence and multiplicity of solutions for stationary higher order problems of Kirchhoff type (in n-dimensional domains, $n \geq 1$) were also investigated in some recent papers, using variational methods like the symmetric mountain pass theorem in [19] and a three critical point theorem in [6]. Moreover, in [4, 5], some evolutionary higher order Kirchhoff problems were studied, mainly focusing on the qualitative properties of the solutions.

For a thorough discussion on the subject, we also refer the reader to [7, 12].

Motivated by the above facts, in the present paper, by using two kinds of three critical points theorems obtained in [13, 8] which we recall in the next section (Theorems 2.1 and 2.2), we ensure the existence of at least three weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$ under mutually independent conditions; see Theorems 3.1 and 3.2. In fact, in Theorem 3.1, we require that the primitive F of the function f is p-sublinear at infinity and satisfies appropriate local growth condition. In Theorem 3.2 we obtain at least three non-negative weak solutions uniformly bounded with respect to λ, and three non-negative weak solutions, respectively, under a suitable sign hypothesis on the function f, an appropriate growth conditions on the potential F in a bounded interval, and without assuming asymptotic condition at infinity on the function f for every non-negative continuous function g. We present Examples 3.5 and 3.6 to illustrate Theorems 3.1 and 3.2, respectively. Theorem 3.8 is a special case of Theorem 3.1. In Theorem 3.9 we present an application of Theorem 3.2. As special cases of Theorems 3.1 and 3.2, we obtain Theorems 3.10 and 3.11 considering the case $p(x) = p > N$.

Compared to the previous results, we give some new assumptions to obtain the existence of at least three weak solutions of $(P^{f,g}_{\lambda,\mu})$. Recent related works are generalized.

We refer to the recent monograph by Molica Bisci, Rădulescu and Servadei [44] for related problems concerning the variational analysis of solutions of some classes of nonlocal problems.

The paper is organized as follows. In Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to our abstract results.

2. Preliminaries

Our main tools are the following three critical points theorems. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed.

Theorem 2.1 ([13, Theorem 3.6]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$.*
Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that

$$(a_1) \qquad \frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} < \frac{\Psi(\overline{v})}{\Phi(\overline{v})};$$

(a_2) *for each* $\lambda \in \Lambda_r := \left] \dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \dfrac{r}{\sup\limits_{\Phi(u)\le r} \Psi(u)} \right[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X*.*

Theorem 2.2 ([8, Corollary 3.1])**.** *Let* X *be a reflexive real Banach space,* $\Phi : X \to \mathbb{R}$ *be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on* X^**,* $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf\limits_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. *for each* $\lambda > 0$ *and for every* $u_1, u_2 \in X$ *which are local minima for the functional* $\Phi - \lambda\Psi$ *and such that* $\Psi(u_1) \ge 0$ *and* $\Psi(u_2) \ge 0$*, one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \ge 0.$$

Assume that there are two positive constants r_1, r_2 *and* $\overline{v} \in X$*, with* $2r_1 < \Phi(\overline{v}) < \frac{r_2}{2}$*, such that*

$$(b_1) \qquad \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})};$$

$$(b_2) \qquad \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{v})}{\Phi(\overline{v})}.$$

Then, for each

$$\lambda \in \left] \frac{3}{2}\frac{\Phi(\overline{v})}{\Psi(\overline{v})},\ \min\left\{ \frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)},\ \frac{\frac{r_2}{2}}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)} \right\} \right[,$$

the functional $\Phi - \lambda\Psi$ *has at least three distinct critical points which lie in* $\Phi^{-1}(]-\infty, r_2[)$*.*

Theorems 2.1 and 2.2 have been successfully used to ensure the existence of at least three solutions for perturbed boundary value problems in the papers [12, 20].

Here and in the sequel, $\text{meas}(\Omega)$ denotes the Lebesgue measure of the set Ω, and we also assume that $p \in C(\bar{\Omega})$ verifies the following condition:

$$N < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty. \tag{2.1}$$

Let Ω be a bounded domain of $\mathbb{R}^N$, denote:

$$L^{p(x)}(\Omega) := \Big\{ u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < +\infty \Big\},$$

$$L^{p(x)}(\partial\Omega) := \Big\{ u : \partial\Omega \to \mathbb{R} \text{ measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma < +\infty \Big\}.$$

We can introduce the norms on $L^{p(x)}(\Omega)$ and $L^{p(x)}(\partial\Omega)$ by:

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{\beta > 0 : \int_\Omega |\frac{u(x)}{\beta}|^{p(x)}dx \le 1\Big\},$$

$$\|u\|_{L^{p(x)}(\partial\Omega)} = \inf\Big\{\beta > 0 : \int_{\partial\Omega} |\frac{u(x)}{\beta}|^{p(x)}d\sigma \le 1\Big\}$$

where $d\sigma$ is the surface measure on $\partial\Omega$.

Let X be the generalized Lebesgue-Sobolev space $W^{1,p(x)}(\Omega)$ defined by putting $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega)\Big\}$$

and it can be equipped with the norm:

$$\|u\|_{W^{1,p(x)}(\Omega)} := \|u\|_{L^{p(x)}(\Omega)} + \||\nabla u|\|_{L^{p(x)}(\Omega)}. \tag{2.2}$$

It is well known (see [29]) that, in view of (2.1), both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces. Moreover, since $\alpha \in L^\infty(\Omega)$, with $\alpha^- := ess \inf_{x\in\Omega} \alpha(x) > 0$ is assumed, then the following norm

$$\|u\|_\alpha = \inf\Big\{\sigma > 0 : \int_\Omega \big(\alpha(x)|\frac{u(x)}{\sigma}|^{p(x)} + |\frac{\nabla u(x)}{\sigma}|^{p(x)}\big)dx \le 1\Big\},$$

on $W^{1,p(x)}(\Omega)$ is equivalent to that introduce in (2.2). Since $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ (see [29] or [38]) and $p^- > N$, $W^{1,p(x)}(\Omega)$ is continuously embedded in $C^0(\bar\Omega)$ and one has

$$\|u\|_{C^0(\bar\Omega)} \le k_{p^-}\|u\|_{W^{1,p^-}(\Omega)}.$$

When Ω is convex, an explicit upper bound for the constant k_{p^-} is

$$k_{p^-} \le 2^{\frac{p^- -1}{p^-}} \max\left\{ \left(\frac{1}{\|\alpha\|_1}\right)^{\frac{1}{p^-}}, \frac{d}{N^{\frac{1}{p^-}}}\left(\frac{p^- - 1}{p^- - N}\text{meas}(\Omega)\right)^{\frac{p^- -1}{p^-}} \frac{\|\alpha\|_\infty}{\|\alpha\|_1}\right\}$$

where $\|\alpha\|_1 = \int_\Omega \alpha(x)dx$ and $\|\alpha\|_\infty = \sup_{x\in\Omega} \alpha(x)$ and $d = diam(\Omega)$ (see [9, Remark 1]). On the other hand, taking into account that $p^- \le p(x)$, [38, Theorem 2.8] ensures that $L^{p(x)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ and the constant of such embedding does not exceed $1+\text{meas}(\Omega)$. So, one has

$$\|u\|_{W^{1,p^-}(\Omega)} \le (1 + \text{meas}(\Omega))\|u\|_{W^{1,p(x)}(\Omega)} \le (1 + \text{meas}(\Omega))\|u\|_\alpha.$$

In conclusion, put

$$c = k_{p^-}(1 + \text{meas}(\Omega)),$$

it results

$$\|u\|_{C^0(\bar\Omega)} \le c\|u\|_\alpha \tag{2.3}$$

for each $u \in W^{1,p(x)}(\Omega)$.

Set

$$F(x,t)=\int_0^t f(x,\xi)d\xi \quad \text{for all } (x,t)\in\Omega\times\mathbb{R},$$

$$G(t)=\int_0^t g(\xi)d\xi \quad \text{for all } t\in\mathbb{R}$$

and

$$\widetilde{M}(t)=\int_0^t M(\xi)d\xi \quad \text{for all } t\geq 0.$$

We say that $u\in W^{1,p(x)}(\Omega)$ is a weak solution of the problem $(P^{f,g}_{\lambda,\mu})$ if

$$M\left(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)}+\alpha(x)|u(x)|^{p(x)}\big)dx\right)$$
$$\int_\Omega \big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)+\alpha(x)|u(x)|^{p(x)-2}u(x)v(x)\big)dx$$
$$-\lambda\int_\Omega f(x,u(x))v(x)dx-\mu\int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))d\sigma=0$$

for every $v\in W^{1,p(x)}(\Omega)$.

Proposition 2.3 ([26, Proposition 2.4]). *Let*

$$\rho_\alpha(u)=\int_\Omega[|\nabla u|^{p(x)}+\alpha(x)|u|^{p(x)}]dx$$

for $u\in W^{1,p(x)}(\Omega)$, we have

(1) $\|u\|_\alpha\geq 1\Longrightarrow \|u\|_\alpha^{p^-}\leq\rho_\alpha(u)\leq\|u\|_\alpha^{p^+}$;
(2) $\|u\|_\alpha\leq 1\Longrightarrow \|u\|_\alpha^{p^+}\leq\rho_\alpha(u)\leq\|u\|_\alpha^{p^-}$.

For our convenience, set

$$G^\theta := a(\partial\Omega)\max_{|\xi|\leq\theta}G(\xi) \quad \text{for all } \theta>0$$

where $a(\partial\Omega)=\int_{\partial\Omega}d\sigma$ and

$$G_\eta := a(\partial\Omega)\inf_{t\in[0,\eta]}G(t) \quad \text{for all } \eta\geq 1.$$

If g is sign-changing, then clearly $G^\theta\geq 0$ and $G_\eta\leq 0$.

3. Main results

Fixing two positive constants $\theta\geq c$ and $\eta\geq 1$ such that

$$\frac{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1}{\displaystyle\int_\Omega F(x,\eta)dx}<\frac{m_0\theta^{p^-}}{c^{p^-}p^+\displaystyle\int_\Omega\sup_{|t|\leq\theta}F(x,t)dx}$$

where c is given by (2.3) and choosing

$$\lambda \in \Lambda := \left] \frac{\frac{m_1 \eta^{p^+}}{p^-}\|\alpha\|_1}{\int_\Omega F(x,\eta)dx}, \frac{m_0 \theta^{p^-}}{c^{p^-} p^+ \int_\Omega \sup_{|t|\le\theta} F(x,t)dx} \right[,$$

set $\delta_{\lambda,g}$ given by

$$\min\left\{ \frac{m_0\theta^{p^-} - \lambda c^{p^-} p^+ \int_\Omega \sup_{|t|\le\theta} F(x,t)dx}{c^{p^-} p^+ G^\theta}, \left| \frac{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1 - \lambda\int_\Omega F(x,\eta)dx}{\min\{0, G_\eta\}} \right| \right\}$$

and

$$\overline{\delta}_{\lambda,g} := \min\left\{ \delta_{\lambda,g},\ \frac{1}{\max\left\{0, \frac{c^{p^-} p^+ a(\partial\Omega)}{m_0} \limsup_{t\to+\infty} \frac{G(t)}{|t|^{p^-}}\right\}} \right\} \tag{3.1}$$

where we read $\gamma/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,g} = +\infty$ when

$$\limsup_{t\to+\infty} \frac{G(t)}{|t|^{p^-}} \le 0,$$

and $G_\eta = G^\theta = 0$.

We present our first existence result as follows.

Theorem 3.1. *Assume that there exist two positive constants $\theta \ge c$ and $\eta \ge 1$ with*

$$\theta < \sqrt[p^-]{\|\alpha\|_1} c\eta$$

such that

(A$_1$) $$\frac{\int_\Omega \sup_{|t|\le\theta} F(x,t)dx}{\theta^{p^-}} < \frac{p^- m_0}{m_1 c^{p^-} p^+ \|\alpha\|_1} \frac{\int_\Omega F(x,\eta)dx}{\eta^{p^+}};$$

(A$_2$) $$\mathrm{meas}(\Omega) c^{p^-} \limsup_{t\to+\infty} \frac{\sup_{x\in\Omega} F(x,t)}{|t|^{p^-}} \le \Theta$$
where
$$\Theta = \frac{c^{p^-} p^+ \int_\Omega \sup_{|t|\le\theta} F(x,t)dx}{m_0 \theta^{p^-}}.$$

Then, for each $\lambda \in \Lambda$ and for every continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$\limsup_{t\to+\infty} \frac{G(t)}{|t|^{p^-}} < +\infty,$$

there exists $\overline{\delta}_{\lambda,g} > 0$ given by (3.1) such that, for each $\mu \in [0, \overline{\delta}_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ admits at least three distinct weak solutions in $W^{1,p(x)}(\Omega)$.

Proof. Fix λ, g and μ as in the conclusion. Let X be the Sobolev space $W^{1,p(x)}(\Omega)$. In order to apply Theorem 2.1 to our problem. Consider the functionals Φ, Ψ for every $u \in X$, defined by

$$\Phi(u) = \widetilde{M}\left(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)dx\right) \tag{3.2}$$

and

$$\Psi(u) = \int_\Omega F(x, u(x))dx + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(u(x)))d\sigma, \tag{3.3}$$

and put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in X$. Let us prove that the functionals Φ and Ψ satisfy the required the conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)dx + \frac{\mu}{\lambda}\int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))d\sigma$$

for every $v \in X$, and $\Psi' : X \to X^*$ is compact. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)(v) =& M\left(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)dx\right)\\ &\int_\Omega \big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) + \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)\big)dx\end{aligned}$$

for every $v \in X$. We prove that Φ' admits a continuous inverse on X^*. Assuming $\|u\|_\alpha > 1$, we have

$$\Phi'(u)(u) \geq m_0\|u\|_\alpha^{p^-},$$

and since $p^- > 1$, it follows that Φ' is coercive. Since Φ' is the Fréchet derivative of Φ, it follows that Φ' is continuous and bounded. Using the elementary inequality [55]

$$|x-y|^\gamma \leq 2^\gamma(|x|^{\gamma-2}x - |y|^{\gamma-2}y)(x-y) \quad \text{if } \gamma \geq 2,$$

for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $N \geq 1$, we obtain for all $u, v \in X$ such that $u \neq v$,

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle > 0,$$

which means that Φ' is strictly monotone. Thus Φ' is injective. Consequently, thanks to Minty-Browder theorem [57], the operator Φ' is an surjection and has an inverse $\Phi'^{-1} : X^* \to X$, and one has Φ'^{-1} is continues. Furthermore, Φ is sequentially weakly lower semicontinuous. Choose $r := \frac{m_0}{p^+}(\frac{\theta}{c})^{p^-}$ and put $w(x) = \eta$ for all $x \in \Omega$. Clearly $w \in X$. Hence, we have definitively,

$$\frac{m_0\eta^{p^-}}{p^+}\|\alpha\|_1 \leq \Phi(w) \leq \frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1. \tag{3.4}$$

Then, from the condition $\theta < \sqrt[p^-]{\|\alpha\|_1}c\eta$, we get $0 < r < \Phi(w)$. Moreover, for all $u \in X$ with $\Phi(u) < r$, then, owing to [14, Proposition 2.2], one has

$$\|u\|_\alpha \leq \max\big\{(p^+r)^{\frac{1}{p^+}}, (p^+r)^{\frac{1}{p^-}}\big\}.$$

By (2.3) one has $\|u\|_\infty \le c\|u\|_\alpha$, from the definition of r, it follows that

$$\Phi^{-1}(-\infty, r] = \{u \in X; \Phi(u) \le r\} \subseteq \{u \in X; |u| \le \theta\},$$

and it follows that

$$\begin{aligned}\Psi(u) &\le \sup_{u\in\Phi^{-1}(-\infty,r]} \int_\Omega F(x,u(x))dx + \frac{\mu}{\lambda} \sup_{u\in\Phi^{-1}(-\infty,r]} \int_{\partial\Omega} G(\gamma(u(x)))d\sigma \\ &\le \int_\Omega \sup_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}\int_{\partial\Omega} \max_{|t|\le\theta} G(t)d\sigma \\ &= \int_\Omega \sup_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda} G^\theta\end{aligned}$$

for every $u \in X$ such that $\Phi(u) \le r$. Then

$$\sup_{\Phi(u)\le r} \Psi(u) \le \int_\Omega \sup_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda} G^\theta.$$

On the other hand, we have

$$\begin{aligned}\Psi(w) &= \int_\Omega F(x,w(x))dx + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(w(x)))d\sigma \\ &\ge \int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda} G_\eta.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \Psi(u)}{r} \qquad (3.5)\\ &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r]} \int_\Omega F(x,u(x))dx + \sup_{u\in\Phi^{-1}(-\infty,r]} \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(u(x)))d\sigma}{r} \\ &\le \frac{\int_\Omega \sup_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda} G^\theta}{\frac{m_0}{p^+}(\frac{\theta}{c})^{p^-}} \qquad (3.6)\end{aligned}$$

and

$$\begin{aligned}\frac{\Psi(w)}{\Phi(w)} &\ge \frac{\int_\Omega F(x,w(x))dx + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(w(x)))d\sigma}{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1} \\ &\ge \frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda} G_\eta}{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1}. \qquad (3.7)\end{aligned}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{m_0\theta^{p^-} - \lambda c^{p^-} p^+ \int_\Omega \sup_{|t|\le\theta} F(x,t)dx}{c^{p^-} p^+ G^\theta},$$

this means

$$\frac{\int_\Omega \sup\limits_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\frac{m_0}{p^+}(\frac{\theta}{c})^{p^-}} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1 - \lambda\int_\Omega F(x,\eta)dx}{G_\eta},$$

this means

$$\frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta}{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1} > \frac{1}{\lambda}.$$

Then,

$$\frac{\int_\Omega \sup\limits_{|t|\le\theta} F(x,t)dx + \frac{\mu}{\lambda}G^\theta}{\frac{m_0}{p^+}(\frac{\theta}{c})^{p^-}} < \frac{1}{\lambda} < \frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta}{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1}. \tag{3.8}$$

Hence, from (3.5)-(3.8), the condition (a_1) of Theorem 2.1 is fulfilled. Finally, since $\mu < \overline{\delta}_{\lambda,g}$, we can fix $l > 0$ such that

$$\limsup_{t\to+\infty}\frac{G(t)}{|t|^{p^-}} < l,$$

and $\mu l < \dfrac{m_0}{a(\partial\Omega)c^{p^-}p^+}$. Therefore, there exists a function $\varrho \in \mathbb{R}$ such that

$$G(t) \le l|t|^{p^-} + \varrho, \tag{3.9}$$

for every $t \in \mathbb{R}$. Now, fix $0 < \epsilon < \frac{1}{\lambda\Theta}(\frac{m_0}{p^+} - \mu a(\partial\Omega)c^{p^-}l)$. From (A_2) there is a function $\tau \in \mathbb{R}$ such that

$$\frac{\text{meas}(\Omega)c^{p^-}}{\Theta}F(x,t) \le \epsilon|t|^{p^-} + \tau, \tag{3.10}$$

for every $x \in \Omega$ and $t \in \mathbb{R}$. Recalling (2.3), from (3.9) and (3.10), for each $u \in X$, we have

$$\begin{aligned}
\Phi(u) - \lambda\Psi(u) &= \widetilde{M}\left(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)dx\right) \\
&- \lambda\int_\Omega F(x,u(x))dx - \mu\int_{\partial\Omega} G(\gamma(u(x)))d\sigma \\
&\ge \frac{m_0}{p^+}\|u\|_\alpha^{p^-} - \lambda\epsilon\frac{\Theta}{\text{meas}(\Omega)c^{p^-}}\int_\Omega |u(x)|^{p^-}dx \\
&- \lambda\frac{\Theta}{\text{meas}(\Omega)c^{p^-}}\text{meas}(\Omega)\tau - \mu l\int_{\partial\Omega}|u(x)|^{p^-}d\sigma - \mu l a(\partial\Omega)\varrho \\
&\ge \left(\frac{m_0}{p^+} - \lambda\Theta\epsilon - \mu a(\partial\Omega)c^{p^-}l\right)\|u\|_\alpha^{p^-} - \lambda\frac{\Theta}{c^{p^-}}\tau - \mu l a(\partial\Omega)\varrho.
\end{aligned}$$

This leads to the coercivity of the functional I_λ, and the condition (a_2) of Theorem 2.1 is verified. From (3.5)-(3.8) one also has

$$\lambda \in \left] \frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup\limits_{\Phi(u)\le r} \Psi(u)} \right[.$$

Theorem 2.1 assures the existence of three critical points for the functional I_λ (with $\overline{v} = w$), which are solutions of the problem $(P^{f,g}_{\lambda,\mu})$ and we have the conclusion. $\square$

Now, we state a variant of Theorem 3.1 in which no asymptotic condition on the nonlinear term is requested. In such a case f and g are supposed to be non-negative.

For our first goal, let us fix positive constants $\theta_1 \ge c$, θ_2 and $\eta \ge 1$ such that

$$\frac{3}{2}\frac{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1}{\displaystyle\int_\Omega F(x,\eta)dx} < \frac{m_0}{p^+c^{p^-}} \min\left\{\frac{\theta_1^{p^-}}{\displaystyle\int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx}, \frac{\theta_2^{p^-}}{2\displaystyle\int_\Omega \sup_{|t|\le\theta_2} F(x,t)dx}\right\},$$

and take

$$\lambda \in \Lambda' := \left] \frac{3}{2}\frac{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1}{\displaystyle\int_\Omega F(x,\eta)dx}, \frac{m_0}{p^+c^{p^-}} \min\left\{\frac{\theta_1^{p^-}}{\displaystyle\int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx}, \frac{\theta_2^{p^-}}{2\displaystyle\int_\Omega \sup_{|t|\le\theta_2} F(x,t)dx}\right\} \right[.$$

Theorem 3.2. *Assume that there exist three positive constants $\theta_1 \ge c$, θ_2 and $\eta \ge 1$, with $\theta_1 < \sqrt[p^-]{\frac{\|\alpha\|_1}{2}}c\eta$ and $\sqrt[p^-]{\frac{2m_1p^+\|\alpha\|_1}{p^-m_0}}c\eta^{\frac{p^+}{p^-}} < \theta_2$ such that*

(B$_1$) *$f(x,t) \ge 0$ for each $(x,t) \in \Omega \times [-\theta_2, \theta_2]$;*

(B$_2$)

$$\max\left\{\frac{\displaystyle\int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx}{\theta_1^{p^-}}, \frac{2\displaystyle\int_\Omega \sup_{|t|\le\theta_2} F(x,t)dx}{\theta_2^{p^-}}\right\} < \frac{2}{3}\frac{p^-m_0}{m_1c^{p^-}p^+\|\alpha\|_1}\frac{\displaystyle\int_\Omega F(x,\eta)dx}{\eta^{p^+}}.$$

Then, for each $\lambda \in \Lambda'$ and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta^*_{\lambda,g} > 0$ given by

$$\min\left\{\frac{m_0\theta_1^{p^-} - c^{p^-}p^+\lambda\displaystyle\int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx}{c^{p^-}p^+G^{\theta_1}}, \frac{m_0\theta_2^{p^-} - 2c^{p^-}p^+\lambda\displaystyle\int_\Omega \sup_{|t|\le\theta_2} F(x,t)dx}{2c^{p^-}p^+G^{\theta_2}}\right\}$$

such that, for each $\mu \in [0, \delta^*_{\lambda,g})$, the problem $(P^{f,g}_{\lambda,\mu})$ admits at least three distinct non-negative weak solutions u_i for $i = 1, 2, 3$, such that

$$0 \le u_i(x) < \theta_2, \quad \forall\, x \in \Omega,\ (i = 1, 2, 3).$$

Proof. Our aim is to apply Theorem 2.2 to our problem. We consider the auxiliary problem

$$\begin{cases} T(u) = \lambda \hat{f}(x,u), & \text{on } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = \mu g(\gamma(u(x))), & \text{in } \partial\Omega \end{cases} \tag{$P^{\hat{f},g}_{\lambda,\mu}$}$$

where $\hat{f}:\Omega\times\mathbb{R}\to\mathbb{R}$ is an L^1-Carathéodory function, defined as follows

$$\hat{f}(x,\xi) = \begin{cases} f(x,0), & \text{if } \xi < -\theta_2, \\ f(x,\xi), & \text{if } -\theta_2 \le \xi \le \theta_2, \\ f(x,\theta_2), & \text{if } \xi > \theta_2. \end{cases}$$

If any solution of the problem $(P^{f,g}_{\lambda,\mu})$ satisfies the condition $-\theta_2 \le u(x) \le \theta_2$ for every $x\in\Omega$, then, any weak solution of the problem $(P^{\hat{f},g}_{\lambda,\mu})$ clearly turns to be also a weak solution of $(P^{f,g}_{\lambda,\mu})$. Therefore, for our goal, it is enough to show that our conclusion holds for $(P^{f,g}_{\lambda,\mu})$. Fix λ, g and μ as in the conclusion and take Φ, Ψ and X as in the proof of Theorem 3.1. We observe that the regularity the assumptions of Theorem 2.2 on Φ and Ψ are fulfilled. Then, our aim is to verify (b_1) and (b_2). To this end, choose $w(x)=\eta$ for all $x\in\Omega$, as well as

$$r_1 := \frac{m_0}{p^+}\left(\frac{\theta_1}{c}\right)^{p^-} \quad \text{and } r_2 := \frac{m_0}{p^+}\left(\frac{\theta_2}{c}\right)^{p^-}.$$

By considering the conditions $\theta_1 < \sqrt[p^-]{\frac{\|\alpha\|_1}{2}}c\eta$ and $\sqrt[p^-]{\frac{2m_1p^+\|\alpha\|_1}{p^-m_0}}c\eta^{\frac{p^+}{p^-}} < \theta_2$, we have $2r_1 < \Phi(w) < \dfrac{r_2}{2}$. Since $\mu < \delta^*_{\lambda,g}$ and $G_\eta \ge 0$, taking (3.4) into account, one has

$$\begin{aligned}
&\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u)}{r_1} \\
= \quad & \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_1)} \int_\Omega F(x,u(x))dx + \sup\limits_{u\in\Phi^{-1}(-\infty,r_1)} \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(u(x)))d\sigma}{r_1} \\
\le \quad & \frac{\int_\Omega \sup\limits_{|t|\le\theta_1} F(x,t)dx + \frac{\mu}{\lambda}G^{\theta_1}}{\frac{m_0}{p^+}(\frac{\theta_1}{c})^{p^-}} \\
< \quad & \frac{1}{\lambda} < \frac{2}{3}\frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta}{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1} \\
\le \quad & \frac{2}{3}\frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(\eta))d\sigma}{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1} \\
\le \quad & \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}
\end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
&\frac{2 \sup\limits_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u)}{r_2} \\
&= 2\frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_2)} \int_\Omega F(x,u(x))dx + \sup\limits_{u\in\Phi^{-1}(-\infty,r_2)} \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(u(x)))d\sigma}{r_2} \\
&\le 2\frac{\int_\Omega \sup\limits_{|t|\le\theta_2} F(x,t)dx + \frac{\mu}{\lambda}G^{\theta_2}}{\frac{m_0}{p^+}(\frac{\theta_2}{c})^{p^-}} \\
&< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda}G_\eta}{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1} \\
&\le \frac{2}{3}\frac{\int_\Omega F(x,\eta)dx + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(\eta))d\sigma}{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1} \\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}
$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are satisfied. Finally, we prove that $\Phi - \lambda\Psi$ satisfies the assumption *2.* of Theorem 2.2. For this, let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$. We want to prove that they are non-negative. Let u_0 be a weak solution of the problem $(P^{f,g}_{\lambda,\mu})$. Arguing by a contradiction, assume that the set $\mathcal{A} = \{x \in \Omega : u_0(x) < 0\}$ is non-empty and of positive measure. Put $\bar{v}(x) = \min\{0, u_0(x)\}$ for all $x \in \Omega$. Clearly, $\bar{v} \in X$ and one has

$$
\begin{aligned}
&M\left(\int_\Omega \frac{1}{p(x)}\big(|\nabla u_0(x)|^{p(x)} + \alpha(x)|u_0(x)|^{p(x)}\big)dx\right)\left(\int_\Omega |\nabla u_0(x)|^{p(x)-2}\nabla u_0(x)\nabla\bar{v}(x)dx\right. \\
&\left.+\int_\Omega \alpha(x)|u_0(x)|^{p(x)-2}u_0(x)\bar{v}(x)dx\right) \\
&-\lambda\int_\Omega f(x,u_0(x))\bar{v}(x)dx - \mu\int_{\partial\Omega} g(\gamma(u_0(x)))\gamma(\bar{v}(x))d\sigma = 0.
\end{aligned}
$$

Thus, from our sign assumptions on the data we have

$$
\begin{aligned}
&M\left(\int_{\mathcal{A}} \frac{1}{p(x)}\big(|\nabla u_0(x)|^{p(x)} + \alpha(x)|u_0(x)|^{p(x)}\big)dx\right)\left(\int_{\mathcal{A}} |\nabla u_0(x)|^{p(x)}dx\right. \\
&\left.+\int_{\mathcal{A}} \alpha(x)|u_0(x)|^{p(x)}dx\right) \\
&= \lambda\int_{\mathcal{A}} f(x,u_0(x))u_0(x)dx + \mu\int_{\partial\Omega} g(\gamma(u_0(x)))\gamma(u_0(x))d\sigma \le 0.
\end{aligned}
$$

Hence, that is, $\|u_0\|_{w^{1,p(x)}(\mathcal{A})} = 0$ which is an absurd. Hence, our claim is proved. Then, we observe $u_1(x) \ge 0$ and $u_2(x) \ge 0$ for every $x \in \Omega$. Thus, it follows that

$su_1 + (1-s)u_2 \geq 0$ for all $s \in [0,1]$, and that

$$(\lambda f + \mu g)(x, su_1 + (1-s)u_2) \geq 0,$$

and so, $\Psi(su_1 + (1-s)u_2) \geq 0$, for every $s \in [0,1]$. From Theorem 2.2, for every

$$\lambda \in \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points, which are solutions of the problem $(P^{f,g}_{\lambda,\mu})$ and the conclusion is achieved. □

Remark 3.3. If in Theorems 3.1 and 3.2, either $f(x,0) \neq 0$ for some $x \in \Omega$, then the ensured solutions are obviously non-trivial.

Remark 3.4. If in Theorem 3.1, $f(\cdot,t)$ and $g(t)$ are odd functions in t, then we can ensured the existence of at least five distinct weak solutions.

In fact, by Theorem 3.1 the problem $(P^{f,g}_{\lambda,\mu})$ possesses at least three weak solutions. If u is a nontrivial weak solution, then $-u$ is a weak solution since satisfies the equation

$$\begin{aligned}
&M\left(\int_\Omega \frac{1}{p(x)}\big(|\nabla(-u(x))|^{p(x)} + \alpha(x)|-u(x)|^{p(x)}\big)dx\right)\\
&\int_\Omega \big(|\nabla(-u(x))|^{p(x)-2}\nabla(-u(x))\nabla v(x) + \alpha(x)|-u(x)|^{p(x)-2}(-u(x))v(x)\big)dx\\
&-\lambda\int_\Omega f(x,-u(x))v(x)dx - \mu\int_{\partial\Omega} g(\gamma(-u(x)))\gamma(v(x))d\sigma = 0
\end{aligned}$$

for every $v \in W^{1,p(x)}(\Omega)$.

We now present the following examples to illustrate Theorems 3.1 and 3.2, respectively.

Example 3.5. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 10^{-4}\}$. We consider the problem

$$\begin{cases}
M(\int_\Omega \frac{1}{p(x,y)}\big(|\nabla u(x)|^{p(x,y)} + \alpha(x)|u(x)|^{p(x,y)}\big)dx)(-\Delta_{p(x,y)}u + \alpha(x)|u|^{p(x,y)-2}u)\\
= \lambda f(x,y,u), \qquad \text{in } \Omega,\\
|\nabla u|^{p(x,y)-2}\frac{\partial u}{\partial v} = \mu g(\gamma(u(x))), \text{ on } \partial\Omega
\end{cases} \tag{3.12}$$

where $M(t) = \frac{3}{2} + \frac{\sin(t)}{2}$ for every $t \in [0,+\infty)$, $p(x,y) = x^2 + y^2 + 4$ for every $x, y \in \Omega$, $\alpha(x,y) = 4(x^2+y^2) + 2$ for every $x, y \in \Omega$ and

$$f(x,y,t) = 8(x^2+y^2)t$$

for every $x, y \in \Omega$ and every $t \in \mathbb{R}$. By the expression of f, we have

$$F(x,y,t) = 4(x^2+y^2)t^2$$

for every $x, y \in \Omega$ and every $t \in \mathbb{R}$. By simple calculations, we obtain

$$c = \sqrt[4]{\frac{27}{2\pi}}\frac{4\times10^{-9} + 2\times10^{-5}}{10^{-8}+10^{-4}}(1+10^{-4}\pi),$$

$a(\partial\Omega) = 2 \times 10^{-2}\pi$, $m_0 = 1$, $m_1 = 2$, $p^- = 4$ and $p^+ = 4 + 10^{-4}$. Taking $\theta = \frac{3}{10}$ and $\eta = 10$, we clearly see that all assumptions of Theorem 3.1 are satisfied. Therefore, it follows that for each

$$\lambda \in \left(\frac{(10^{-8} + 10^{-4})10^{4+10^{-4}}}{2 \times 10^{-6}}, \frac{9 \times 10^6}{2\pi c^4(4 + 10^{-4})} \right),$$

and:

1) for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\bar{\delta}_{\lambda,g} > 0$ such that for each $\mu \in [0, \bar{\delta}_{\lambda,g})$, the problem (3.12) admits at least three distinct weak solutions in $W^{1,p(x,y)}(\Omega)$.
2) for every odd continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\bar{\delta}_{\lambda,g} > 0$ such that for each $\mu \in [0, \bar{\delta}_{\lambda,g})$, the problem (3.12) admits at least five distinct weak solutions in $W^{1,p(x,y)}(\Omega)$.
3) for $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = t$, since
$$\limsup_{t \to +\infty} \frac{G(t)}{|t|^{p^-}} \leq 0,$$
then $\bar{\delta}_{\lambda,g} = +\infty$ and for each $\mu \in [0, +\infty)$, the problem (3.12) admits at least five distinct weak solutions in $W^{1,p(x,y)}(\Omega)$.

Example 3.6. We consider the problem

$$\begin{cases} M(\int_\Omega \frac{1}{p(x,y)}\big(|\nabla u(x)|^{p(x,y)} + \alpha(x)|u(x)|^{p(x,y)}\big)dx)(-\Delta_{p(x,y)}u + \alpha(x)|u|^{p(x,y)-2}u) \\ = \lambda f(x, y, u), \qquad \text{in } \Omega, \\ |\nabla u|^{p(x,y)-2}\frac{\partial u}{\partial v} = \mu g(\gamma(u(x))), \ \text{on } \partial\Omega \end{cases} \tag{3.13}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 10^{-4}\}$, $M(t) = 2 + \frac{2\arctan(t)}{\pi}$ for every $t \in [0, +\infty)$, $p(x, y) = x^2 + y^2 + 4$ for every $x, y \in \Omega$, $\alpha(x, y) = 4(x^2 + y^2) + 2$ for every $x, y \in \Omega$ and

$$f(x, y, t) = 26(x^2 + y^2)t^{12}$$

for every $x, y \in \Omega$ and every $t \in \mathbb{R}$. By the expression of f, we have

$$F(x, y, t) = 2(x^2 + y^2)t^{13}$$

for every $x, y \in \Omega$ and every $t \in \mathbb{R}$. Direct calculations, give

$$c = \sqrt[4]{\frac{27}{2\pi}} \frac{4 \times 10^{-9} + 2 \times 10^{-5}}{10^{-8} + 10^{-4}}(1 + 10^{-4}\pi),$$

$a(\partial\Omega) = 2 \times 10^{-2}\pi$, $m_0 = 1$, $m_1 = 3$, $p^- = 4$ and $p^+ = 4 + 10^{-4}$. Choosing $\theta_1 = \frac{3}{10}$, $\theta_2 = 10$ and $\eta = 10$, then all conditions in Theorem 3.2 are satisfied. Therefore, it follows that for each $\lambda \in \left(\frac{9(10^{-8}+10^{-4})10^{4+10^{-4}}}{4\times10^5}, \frac{1}{20\pi(4+10^{-4})c^4}\right)$ and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta_{\lambda,g} > 0$ such that for each $\mu \in [0, \delta_{\lambda,g})$, the problem (3.13) admits at least three distinct non-negative weak solutions u_i for $i = 1, 2, 3$, such that

$$0 \leq u_i(x) < 10, \quad \forall\, x \in \Omega, \ (i = 1, 2, 3).$$

Remark 3.7. If we consider the autonomous case of the problem $(P^{f,g}_{\lambda,\mu})$,

$$\begin{cases} T(u)=\lambda f(u(x)), & \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v}=\mu g(\gamma(u(x))), & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

where $f,g:\mathbb{R}\to\mathbb{R}$ are two non-negative continuous and nonzero functions, putting $F(t)=\int_0^t f(\xi)d\xi$ for each $t\in\mathbb{R}$, in Theorem 3.1 the assumptions (A_1) and (A_2) can be written as

(A$_3$) $\frac{F(t)}{\theta^{p^-}}<\frac{p^-m_0}{m_1c^{p^-}p^+\|\alpha\|_1}\dfrac{F(\eta)}{\eta^{p^+}}$;

(A$_4$) $\operatorname{meas}(\Omega)c^{p^-}\limsup\limits_{t\to+\infty}\dfrac{F(t)}{|t|^{p^-}}\le\Theta$ where

$$\Theta=\frac{c^{p^-}p^+\operatorname{meas}(\Omega)F(\theta)}{m_0\theta^{p^-}},$$

respectively, as well as

$$\Lambda:=\left]\frac{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1}{\operatorname{meas}(\Omega)F(\eta)},\frac{m_0\theta^{p^-}}{c^{p^-}p^+\operatorname{meas}(\Omega)F(\theta)}\right[$$

and

$$\min\left\{\frac{m_0\theta^{p^-}-\lambda c^{p^-}p^+\operatorname{meas}(\Omega)F(\theta)}{c^{p^-}p^+G^\theta},\left|\frac{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1-\lambda\operatorname{meas}(\Omega)F(\eta)}{\min\{0,G_\eta\}}\right|\right\}.$$

In this case, in Theorem 3.2 the assumption (B$_2$) assumes the form

(B$_3$) $\max\left\{\dfrac{F(\theta_1)}{\theta_1^{p^-}},\dfrac{2F(\theta_2)}{\theta_2^{p^-}}\right\}<\frac{2}{3}\frac{p^-m_0}{m_1c^{p^-}p^+\|\alpha\|_1}\dfrac{F(\eta)}{\eta^{p^+}}$,

as well as

$$\Lambda':=\left]\frac{3}{2}\frac{\frac{m_1\eta^{p^+}}{p^-}\|\alpha\|_1}{\operatorname{meas}(\Omega)F(\eta)},\frac{m_0}{p^+c^{p^-}\operatorname{meas}(\Omega)}\min\left\{\frac{\theta_1^{p^-}}{F(\theta_1)},\ \frac{\theta_2^{p^-}}{2F(\theta_2)}\right\}\right[$$

and

$$\delta^*_{\lambda,g}=\min\left\{\frac{m_0\theta_1^{p^-}-\lambda c^{p^-}p^+\operatorname{meas}(\Omega)F(\theta_1)}{c^{p^-}p^+a(\partial\Omega)G(\theta_1)},\frac{m_0\theta_2^{p^-}-2\lambda c^{p^-}p^+\operatorname{meas}(\Omega)F(\theta_2)}{2c^{p^-}p^+a(\partial\Omega)G(\theta_2)}\right\}.$$

A special case of Theorem 3.1 is the following theorem.

Theorem 3.8. *Assume that*

$$\liminf_{t\to0}\frac{F(t)}{t^{p^-}}=\limsup_{t\to+\infty}\frac{F(t)}{t^{p^-}}=0.$$

Then, there is $\lambda^*>0$ such that for each $\lambda>\lambda^*$ and for every non-negative continuous function $g:\mathbb{R}\to\mathbb{R}$ satisfying the condition

$$\limsup_{t\to+\infty}\frac{G(t)}{|t|^{p^-}}<+\infty,$$

there exists $\delta > 0$ such that, for each $\mu \in [0, \delta[$, the problem (3.14) *admits at least three distinct solutions.*

Proof. Fix $\lambda > \lambda^* := \frac{\frac{m_1 \eta^{p^+}}{p^-} \|\alpha\|_1}{\text{meas}(\Omega) F(\eta)}$ for some $\eta \geq 1$. From the condition $\liminf_{\xi \to 0} \frac{F(\xi)}{\xi^{p^-}} = 0$, there is a sequence $\{\theta_n\} \subset]0, +\infty[$ such that $\lim_{n\to\infty} \theta_n = 0$ and $\lim_{n\to+\infty} \frac{\sup_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^{p^-}} = 0$. Indeed, one has

$$\lim_{n\to\infty} \frac{\sup_{|\xi|\leq\theta_n} F(\xi)}{\theta_n^{p^-}} = \lim_{n\to\infty} \frac{F(\xi_{\theta_n})}{\xi_{\theta_n}^{p^-}} \frac{\xi_{\theta_n}^{p^-}}{\theta_n^{p^-}} = 0$$

where $F(\xi_{\theta_n}) = \sup_{|\xi|\leq\theta_n} F(\xi)$. Hence, there exists $\overline{\theta} > 0$ such that

$$\frac{\sup_{|t|\leq\overline{\theta}} F(t)}{\overline{\theta}^{p^-}} < \min\left\{ \frac{p^- m_0}{m_1 c^{p^-} p^+ \|\alpha\|_1} \frac{F(\eta)}{\eta^{p^+}}, \ \frac{m_0}{\lambda c^{p^-} p^+ \text{meas}(\Omega)} \right\}$$

and

$$\overline{\theta} < \sqrt[p^-]{\|\alpha\|_1} c\eta.$$

Applying Theorem 3.1 we have the conclusion. $\square$

Moreover, the following result is a consequence of Theorem 3.2.

Theorem 3.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$\lim_{t\to 0^+} \frac{f(t)}{t^2} = 0$$

and

$$\int_0^{100} f(\xi)d\xi < \frac{10^6 \times \int_0^1 f(\xi)d\xi}{18\pi c^3}.$$

Then, for every $\lambda \in \left] \frac{9}{4\int_0^1 f(\xi)d\xi}, \frac{10^6}{8\pi c^3 \int_0^{100} f(\xi)d\xi} \right[$ and for every nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta[$, the problem (3.14) *admits at least three distinct non-negative solutions.*

Proof. Our aim is to employ Theorem 3.2 by choosing

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}, \ p(x, y) = x^2 + y^2 + 3$$

for all $x, y \in \Omega$, $M(t) = 2 + \cos(t)$ for $t \in [0, +\infty)$, $\alpha(x, y) = x^2 + y^2 + 1$ for all $x, y \in \Omega$, $\eta = 1$ and $\theta_2 = 10^2$. Simple calculations show that $m_0 = 1$, $m_1 = 3$, $p^- = 3$, $p^+ = 4$ and $c = \frac{16}{3\sqrt[3]{\pi}}(1 + \pi)$.

$$\frac{3}{2} \frac{\frac{m_1 \eta^{p^+}}{p^-} \|\alpha\|_1}{\int_\Omega F(x, \eta)dx} = \frac{9}{4\int_0^1 f(\xi)d\xi}$$

and

$$\frac{m_0}{p^+ c^{p^-}} \frac{\theta_2^{p^-}}{2\int_\Omega \sup_{|t|\le\theta_2} F(x,t)dx} = \frac{10^6}{8\pi c^3 \int_0^{100} f(\xi)d\xi}.$$

Moreover, since $\lim_{t\to 0^+} \dfrac{f(t)}{t^2} = 0$, one has

$$\lim_{t\to 0^+} \frac{\displaystyle\int_0^t f(\xi)d\xi}{t^3} = 0.$$

Then, there exists a positive constant $\theta_1 < \sqrt[3]{\frac{3\pi}{4}}c$ such that

$$\frac{\displaystyle\int_0^{\theta_1} f(\xi)d\xi}{\theta_1^3} < \frac{\int_0^1 f(\xi)d\xi}{9\pi c^3}$$

and

$$\frac{\theta_1^3}{\displaystyle\int_0^{\theta_1} f(\xi)d\xi} > \frac{10^6}{2\displaystyle\int_0^{100} f(\xi)d\xi}.$$

Finally, we easily observe that all assumptions of Theorem 3.2 are satisfied, and it follows the result. □

We end this paper by presenting the following versions of Theorems 3.1 and 3.2, in the case $p(x) = p$ for every $x \in \Omega$, respectively.

Theorem 3.10. *Assume that there exist two positive constants $\theta \ge c$ and $\eta \ge 1$ with*

$$\theta < \sqrt[p]{\|\alpha\|_1} c\eta$$

such that

(A$_5$) $$\frac{\int_\Omega \sup_{|t|\le\theta} F(x,t)dx}{\theta^p} < \frac{m_0}{m_1 c^p \|\alpha\|_1} \frac{\int_\Omega F(x,\eta)dx}{\eta^p};$$

(A$_6$) $\mathrm{meas}(\Omega) c^p \limsup_{t\to+\infty} \dfrac{\sup_{x\in\Omega} F(x,t)}{|t|^p} \le \Theta$ *where*

$$\Theta = \frac{c^p p \displaystyle\int_\Omega \sup_{|t|\le\theta} F(x,t)dx}{m_0 \theta^p}.$$

Then, for each

$$\lambda \in \left] \frac{\frac{m_1 \eta^p}{p}\|\alpha\|_1}{\displaystyle\int_\Omega F(x,\eta)dx}, \frac{m_0\theta^p}{c^p p \displaystyle\int_\Omega \sup_{|t|\le\theta} F(x,t)dx} \right[$$

and for every continuous function $g:\mathbb{R}\to\mathbb{R}$ satisfying the condition

$$\limsup_{t\to+\infty}\frac{G(t)}{|t|^p}<+\infty,$$

for each

$$\mu\in\min\left[0,\left\{\min\left\{\frac{m_0\theta^p-\lambda c^p p\int_\Omega\sup_{|t|\leq\theta}F(x,t)dx}{c^p pG^\theta},\left|\frac{\frac{m_1\eta^p}{p}\|\alpha\|_1-\lambda\int_\Omega F(x,\eta)dx}{\min\{0,G_\eta\}}\right|\right\}\right.\right.$$
$$\left.\left.,\frac{1}{\max\left\{0,\dfrac{c^p pa(\partial\Omega)}{m_0}\limsup\limits_{t\to+\infty}\dfrac{G(t)}{|t|^p}\right\}}\right\}\right),$$

the problem

$$\begin{cases}T(u)=\lambda f(x,u(x)), & \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial v}=\mu g(\gamma(u(x))), & \text{on } \partial\Omega\end{cases}\tag{3.15}$$

where

$$T(u)=M\left(\int_\Omega\frac{1}{p}\big(|\nabla u(x)|^p+\alpha(x)|u(x)|^p\big)dx\right)\big(-\Delta_p u+\alpha(x)|u|^{p-2}u\big),$$

admits at least three distinct weak solutions in $W^{1,p}(\Omega)$.

Theorem 3.11. *Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a non-negative L^1-Carathéodory function. Assume that there exist three positive constants $\theta_1\geq c$, θ_2 and $\eta\geq 1$, with $\theta_1<\sqrt[p]{\frac{\|\alpha\|_1}{2}}c\eta$ and $\sqrt[p]{\frac{2m_1\|\alpha\|_1}{m_0}}c\eta<\theta_2$ such that*

(B$_4$) *$f(x,t)\geq 0$ for each $(x,t)\in\Omega\times[-\theta_2,\theta_2]$;*

(B$_5$)

$$\max\left\{\frac{\int_\Omega\sup\limits_{|t|\leq\theta_1}F(x,t)dx}{\theta_1^p},\ \frac{2\int_\Omega\sup\limits_{|t|\leq\theta_2}F(x,t)dx}{\theta_2^p}\right\}<\frac{2}{3}\frac{m_0}{m_1c^p\|\alpha\|_1}\frac{\int_\Omega F(x,\eta)dx}{\eta^p}.$$

Then, for each

$$\lambda\in\left]\frac{3}{2}\frac{\frac{m_1\eta^p}{p}\|\alpha\|_1}{\int_\Omega F(x,\eta)dx},\frac{m_0}{pc^p}\min\left\{\frac{\theta_1^p}{\int_\Omega\sup\limits_{|t|\leq\theta_1}F(x,t)dx},\ \frac{\theta_2^p}{2\int_\Omega\sup\limits_{|t|\leq\theta_2}F(x,t)dx}\right\}\right[$$

and for every non-negative continuous function $g:\mathbb{R}\to\mathbb{R}$, there exists $\delta^*_{\lambda,g}>0$ given by

$$\min\left\{\frac{m_0\theta_1^p-c^p p\lambda\int_\Omega\sup_{|t|\leq\theta_1}F(x,t)dx}{c^p pa(\partial\Omega)G(\theta_1)},\frac{m_0\theta_2^p-2c^p p\lambda\int_\Omega\sup_{|t|\leq\theta_2}F(x,t)dx}{2c^p pa(\partial\Omega)G(\theta_2)}\right\}$$

such that, for each $\mu \in [0, \delta^*_{\lambda,g})$*, the problem* (3.15) *admits at least three distinct non-negative weak solutions* u_i *for* $i = 1, 2, 3$*, such that*

$$0 \leq u_i(x) < \theta_2, \quad \forall\, x \in \Omega, \ (i = 1, 2, 3).$$

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