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THE SPLIT FIXED POINT PROBLEM FOR DEMICONTRACTIVE MAPPINGS AND APPLICATIONS

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Abstract. In this paper, we propose a new algorithm to approximate a split common fixed point problem for two demicontractive mappings and prove strong convergence of the proposed method in real Hilbert spaces. As the application, we apply our main results to study the split common null point problem, split variational inequality problem, split convex minimization problem and split equilibrium problem in frame work of real Hilbert spaces. Some numerical example supporting our main result is also given.
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1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let I be the identity mapping. Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*.

The *split feasibility problem* (SFP) which was first introduced by Censor and Elfving [4] is to find

$$v^* \in C \quad \text{such that} \quad Av^* \in Q. \tag{1.1}$$

Let P_C and P_Q be the orthogonal projection onto the set C and Q respectively. Assume that (1.1) has a solution, it is known that $v^* \in H_1$ solves (1.1) if and only if it solves the fixed point equation

$$v^* = P_C(I + \gamma A^*(P_Q - I)A)v^*,$$

where $\gamma > 0$ is any positive constant. In 2002, Byrne [2] introduced and proved that CQ algorithm converges to a solution of (1.1) in finite dimensional Hilbert spaces.

The *split fixed point problem* (SFPP) for mappings T and S is to find

$$v^* \in F(T) \quad \text{such that} \quad Av^* \in F(S), \tag{1.2}$$

where $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are two mappings satisfying

$$F(T) = \{x \in H_1 \, : \, Tx = x\} \neq \emptyset \text{ and } F(S) = \{x \in H_2 \, : \, Sx = x\} \neq \emptyset,$$

respectively. Since each closed and convex subset may be considered as a fixed point set of a projection onto the subset, hence the SFPP is a generalization of the SFP. Recently, the SFPP and SFP have been studied by many authors such as [9, 10, 17, 19, 20, 8], *etc.*

In 2010, Moudafi [8] introduced the following algorithm for solving (1.2) for two demicontractive mappings:

$$\begin{cases} x_1 \in H_1 \text{ choose arbitrarily,} \\ u_n = x_n + \gamma\alpha A^*(S-I)Ax_n, \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

and he proved that $\{x_n\}$ converges weakly to some solution of SFPP.

Five years later, Shehu and Cholamjiak [12] modified algorithim (1.3) by adding some control sequence in the second step as follows:

$$\begin{cases} x_1 \in H_1 \text{ choose arbitrarily,} \\ u_n = x_n + \gamma A^*(S-I)Ax_n, \\ x_{n+1} = (1-\beta_n)(\lambda_n u_n) + \beta_n T u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

They proved under some control conditions that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a solution v^* of SFPP.

It is noted that the algorithm (1.3) introduced by Moudafi [8] obtained only weak convergence. However, strong convergence is more desirable than that of weak convergence. So, it is natural to ask, how can we modify or construct some new algorithms which give us strong convergence?. Recently, Shehu and Cholamjiak [12] modifies algorithm (1.3) to obtain strong convergence. In this work, by using the idea of viscosity approximation method and Mann iteration method, we propose a new algorithm to approximate a split common fixed point of two demicontractive mappings and prove strong convergence of the proposed method to a solution of the split common fixed point problem.

Throughout this paper, we adopt the following notations.

(i) "$\to$ "and "$\rightharpoonup$"denote the strong and weak convergence, respectively.
(ii) $\omega_\omega(x_n)$ denote the set of the cluster point of $\{x_n\}$ in the weak topology, that is, $\exists\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x$.
(iii) Γ is the solution set of the split common fixed point problems (1.2), that is,

$$\Gamma = \{v^* \in F(T) \, : \, Av^* \in F(S)\} = F(T) \cap A^{-1}(F(S)).$$

2. Preliminaries

A mapping P_C is said to be *metric projection* of H onto C, if for every $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - z\|, \quad \forall z \in C.$$

It is known that P_C is firmly nonexpansive mapping. Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, y \in C,$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \quad \forall x \in H, y \in C.$$

A bounded linear operator $B : H \to H$ is said to be *strongly positive* if there is a constant $\xi > 0$ such that

$$\langle Bx, x\rangle \geq \xi\|x\|^2 \quad \text{for all } x \in H.$$

Let M be the set-valued mapping of H into 2^H. The *effective domain* of M is denote by $D(M)$, that is, $D(M) = \{x \in H \ : \ Mx \neq \emptyset\}$. The mapping M is said to be *monotone* if

$$\langle x - y, u - v\rangle \geq 0 \quad \forall x, y \in D(M), u \in Mx, v \in My$$

A monotone mapping M is said to be *maximal* if the graph $G(M)$ is not property contained in the graph of any other monotone map, where

$$G(M) = \{(x, y) \in H \times H \ : \ y \in Mx\}.$$

It is known that M is maximal if and only if for $(x, u) \in H \times H, \langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in G(M)$ implies $u \in Mx$. For the maximal monotone operator M, we can associate its *resolvent* J_δ^M defined by

$$J_\delta^M \equiv (I + \delta M)^{-1} : H \to D(M), \quad \text{where } \delta > 0.$$

It is known that if M is a maximal monotone operator, then the resolvent J_δ^M is firmly nonexpansive and $F(J_\delta^M) = M^{-1}0 \equiv \{x \in H \ : \ 0 \in Mx\}$ for every $\delta > 0$.

Definition 2.1. The mapping $T : H \to H$ is said to be

(i) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tu - v\| \leq \|u - v\| \qquad \text{for all } u \in H, v \in F(T);$$

(ii) *strictly quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tu - v\| < \|u - v\| \qquad \text{for all } u \notin F(T), v \in F(T);$$

(iii) *firmly nonexpansive* if

$$\|Tu - Tv\|^2 \leq \|u - v\|^2 - \|(u - v) - (Tu - Tv)\|^2 \qquad \text{for all } u, v \in H;$$

equivalently, for all $u, v \in H$,

$$\|Tu - Tv\|^2 \leq \langle Tu - Tv, u - v\rangle;$$

(iv) *k-demicontractive* if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$\|Tu - v\|^2 \leq \|u - v\|^2 + k\|u - Tu\|^2 \qquad \text{for all } u \in H, v \in F(T);$$

(v) λ-*inverse strongly monotone* if there exists $\lambda > 0$ such that

$$\langle u - v, Tu - Tv\rangle \geq \lambda\|Tu - Tv\|^2 \qquad \text{for all } u, v \in H.$$

Definition 2.2. Let H be a Hilbert space and C be a nonempty subset of H. The mapping $T : C \to H$ is said to (α, β)-*generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tu - Tv\|^2 + (1-\alpha)\|u - Tv\|^2 \leq \beta\|Tu - v\|^2 + (1-\beta)\|u - v\|^2, \text{ for all } u, v \in C.$$

Definition 2.3. The mapping $T : H \to H$ is said to be *demiclosed at zero* if for any sequence $\{u_n\} \subset H$ with $u_n \rightharpoonup u$ and $Tu_n \to 0$, then $Tu = 0$.

Lemma 2.4 ([7])**.** *Assume that B is a self-adjoint strongly positive bounded linear operator on a Hilbert space H with coefficient $\xi > 0$ and $0 < \mu \leq \|B\|^{-1}$. Then*

$$\|I - \mu B\| \leq 1 - \xi\mu.$$

We also note that if B is a self-adjoint strongly positive bounded linear operator on a Hilbert space H, then also is $B_1 = \frac{B}{||B||}$.

Lemma 2.5 ([14])**.** *Let H be a real Hilbert space. Then the following results hold:*

(i) for all $t \in [0, 1]$ and $u, v \in H$,

$$\|tu + (1-t)v\|^2 = t\|u\|^2 + (1-t)\|v\|^2 - t(1-t)\|u - v\|^2;$$

(ii) $\|u + v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \quad \forall u, v \in H;$

(iii) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle \quad \forall u, v \in H.$

Lemma 2.6 ([18])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \in \mathbb{N},$$

where

(i) $\{\gamma_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([6])**.** *Let $\{\kappa_n\}$ be a sequence of real numbers that dose not decrease at infinity, that is there exists at a subsequence $\{\kappa_{n_i}\}$ of $\{\kappa_n\}$ which satisfies $\kappa_{n_i} < \kappa_{n_i+1}$ for all $i \in \mathbb{N}$. For every $n \geq n_o$, define an integer sequence $\{\tau(n)\}$ as follow:*

$$\tau(n) = \max\{l \in \mathbb{N} : l \leq n, \kappa_l < \kappa_{l+1}\},$$

where $n_o \in \mathbb{N}$ such that $\{l \leq n_o : \kappa_l < \kappa_{l+1}\} \neq \emptyset$. Then the following hold:

(i) $\tau(n_o) \leq \tau(n_o + 1) \leq \ldots$ *and* $\tau(n) \to \infty$;

(ii) for all $n \geq n_o$, $\max\{\kappa_n, \kappa_{\tau(n)}\} \leq \kappa_{\tau(n)+1}$.

3. Main results

In this section, we first introduce a new algorithm for solving SFPP of two demicontraction mapping by using idea of viscosity approximation method and Mann iteration method.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $f : H_1 \to H_1$ be a ρ-contraction mapping and B be a self-adjoint strongly positive bounded linear operator on H_1 with coefficient $\xi > 2\rho$ and $||B|| = 1$. Let $S : H_2 \to H_2$ and $T : H_1 \to H_1$ be k_1 and k_2-demicontractive mappings such that $S-I$ and $T-I$ are demiclosed at zero, respectively. Suppose that $\Gamma \neq \emptyset$. For $x_1 \in H_1$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ be generated by:*

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(S - I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n T u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\delta_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1 - k_2$;
(C4) $0 < c \leq \delta_n \leq d < \frac{1-k_1}{\|A\|^2}$.

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Gamma(f + I - B)x^*$.

Proof. For any $u, v \in H_1$, by Lemma 2.4, we have

$$\begin{aligned} \|P_\Gamma(f + I - B)u - P_\Gamma(f + I - B)v\| &\leq \|(f + I - B)u - (f + I - B)v\| \\ &\leq \|f(u) - f(v)\| + \|I - B\|\|u - v\| \\ &\leq \rho\|u - v\| + (1 - \xi)\|u - v\| \\ &\leq (1 - \rho)\|u - v\|, \end{aligned}$$

that is the mapping $P_\Gamma(f + I - B)$ is contraction.

Let $x^* = P_\Gamma(f + I - B)x^*$, that is $x^* \in F(T) \cap A^{-1}(F(S))$. By (3.1) and Lemma 2.5(i), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1 - \beta_n)u_n + \beta_n T u_n - x^*\|^2 \\ &= \|(1 - \beta_n)(u_n - x^*) + \beta_n(Tu_n - x^*)\|^2 \\ &= (1 - \beta_n)\|u_n - x^*\|^2 + \beta_n\|Tu_n - x^*\|^2 - \beta_n(1 - \beta_n)\|u_n - Tu_n\|^2 \\ &\leq (1 - \beta_n)\|u_n - x^*\|^2 + \beta_n\Big[\|u_n - x^*\|^2 + k_2\|u_n - Tu_n\|^2\Big] \\ &\quad - \beta_n(1 - \beta_n)\|u_n - Tu_n\|^2 \\ &= \|u_n - x^*\|^2 - \beta_n(1 - k_2 - \beta_n)\|u_n - Tu_n\|^2 \\ &\leq \|u_n - x^*\|^2. \end{aligned} \tag{3.2}$$

By the condition (C1) and Lemma 2.4, we get

$$\begin{aligned}
\|u_n - x^*\| &= \|\alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(S-I)Ax_n) - x^*\| \\
&= \|\alpha_n (f(x_n) - Bx^*) + (I - \alpha_n B)(x_n + \delta_n A^*(S-I)Ax_n - x^*)\| \\
&\le \alpha_n \Big[\|f(x_n) - f(x^*)\| + \|f(x^*) - Bx^*\| \Big] \\
&\quad + \|I - \alpha_n B\| \|x_n + \delta_n A^*(S-I)Ax_n - x^*\| \\
&\le \alpha_n \rho \|x_n - x^*\| + \alpha_n \|f(x^*) - Bx^*\| \\
&\quad + (1 - \alpha_n \xi)\|x_n + \delta_n A^*(S-I)Ax_n - x^*\|, \qquad (3.3)
\end{aligned}$$

for sufficient large n. Using Lemma 2.5(ii), we obtain

$$\begin{aligned}
\|x_n - x^* + \delta_n A^*(S-I)Ax_n\|^2 &= \|x_n - x^*\|^2 + 2\langle x_n - x^*, \delta_n A^*(S-I)Ax_n\rangle \\
&\quad + \|\delta_n A^*(S-I)Ax_n\|^2 \\
&\le \|x_n - x^*\|^2 + 2\delta_n \langle x_n - x^*, A^*(S-I)Ax_n\rangle \\
&\quad + \delta_n^2 \|A\|^2 \|(S-I)Ax_n\|^2. \qquad (3.4)
\end{aligned}$$

Since A is a bounded linear operator with its adjoint operator A^* and S is a k_1-demicontractive mapping, by Lemma 2.5(ii), we deduce that

$$\begin{aligned}
\langle x_n - x^*, A^*(S-I)Ax_n\rangle &= \langle Ax_n - Ax^*, (S-I)Ax_n\rangle \\
&= \langle SAx_n - Ax^*, SAx_n - Ax_n\rangle - \|(S-I)Ax_n\|^2 \\
&= \frac{1}{2}\Big[\|SAx_n - Ax^*\|^2 + \|SAx_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2 \Big] \\
&\quad - \|(S-I)Ax_n\|^2 \\
&\le \frac{1}{2}\Big[\|Ax_n - Ax^*\|^2 + k_1 \|SAx_n - Ax_n\|^2 \\
&\quad + \|SAx_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2 \Big] - \|(S-I)Ax_n\|^2 \\
&= \frac{k_1 - 1}{2} \|(S-I)Ax_n\|^2. \qquad (3.5)
\end{aligned}$$

From (3.4) and (3.5), we obtain

$$\begin{aligned}
\|x_n - x^* + \delta_n A^*(S-I)Ax_n\|^2 &\le \|x_n - x^*\|^2 - \delta_n (1 - k_1 - \delta_n \|A\|^2) \|(S-I)Ax_n\|^2 \\
&\le \|x_n - x^*\|^2. \qquad (3.6)
\end{aligned}$$

By (3.2),(3.3) and (3.6), we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|u_n - x^*\|^2 \\
&\le \alpha_n \rho \|x_n - x^*\| + \alpha_n \|f(x^*) - Bx^*\| + (1 - \alpha_n \xi)\|x_n - x^*\| \\
&= [1 - \alpha_n(\xi - \rho)]\|x_n - x^*\| + \alpha_n \|f(x^*) - Bx^*\| \\
&\le \max\left\{ \|x_n - x^*\|, \frac{\|f(x^*) - Bx^*\|}{\xi - \rho} \right\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is a bounded sequence.

Next, we show that $x_n \to x^*$. To this end, we consider the following two cases.
Case 1. Suppose that $\{\|x_n - x^*\|\}_{n=n_o}^{\infty}$ is non-increasing for some $n_o \in \mathbb{N}$. Then we get $\lim_{n\to\infty} \|x_n - x^*\|$ exists. By (3.2) and (3.3), we get

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \|u_n - x^*\|^2 \\
&\le \Big[\alpha_n(\rho\|x_n - x^*\| + \|f(x^*) - Bx^*\|) \\
&\quad + (1-\alpha_n\xi)\|x_n + \delta_n A^*(S-I)Ax_n - x^*\|\Big]^2 \\
&\le \alpha_n(\rho\|x_n - x^*\| + \|f(x^*) - Bx^*\|)^2 \\
&\quad + (1-\alpha_n\xi)\|x_n + \delta_n A^*(S-I)Ax_n - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n\xi)(\rho\|x_n - x^*\| + \|f(x^*) - Bx^*\|)\|x_n - x^*\| \\
&\le \alpha_n\Big[(2+\rho)\|x_n - x^*\| + \|f(x^*) - Bx^*\|\Big]\Big[\rho\|x_n - x^*\| + \|f(x^*) - Bx^*\|\Big] \\
&\quad + (1-\alpha_n\xi)\Big[\|x_n - x^*\|^2 - \delta_n(1-k_1-\delta_n\|A\|^2)\|(S-I)Ax_n\|^2\Big] \\
&\le \alpha_n M + (1-\alpha_n\xi)\Big[\|x_n - x^*\|^2 - \delta_n(1-k_1-\delta_n\|A\|^2)\|(S-I)Ax_n\|^2\Big],
\end{aligned}$$

where

$$M = \sup_n \left\{ \Big[3\|x_n - x^*\| + \|f(x^*) - Bx^*\|\Big]^2 \right\}.$$

This implies

$$(1-\alpha_n\xi)\delta_n(1-k_1-\delta_n\|A\|^2)\|(S-I)Ax_n\|^2 \le \alpha_n M + (1-\alpha_n\xi)\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

By condition (C1), we obtain

$$\lim_{n\to\infty} \|(S-I)Ax_n\| = 0. \tag{3.7}$$

By (3.1), we get

$$\begin{aligned}
\|u_n - x_n\| &= \|\alpha_n f(x_n) + (I-\alpha_n B)(x_n + \delta_n A^*(S-I)Ax_n) - x_n\| \\
&= \|\delta_n A^*(S-I)Ax_n + \alpha_n(f(x_n) - Bx_n - \delta_n BA^*(S-I)Ax_n)\| \\
&\le \delta_n\|A\|\|(S-I)Ax_n\| + \alpha_n\|f(x_n) - Bx_n - \delta_n BA^*(S-I)Ax_n\|.
\end{aligned}$$

By (3.7) and condition (C1), we obtain

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.8}$$

From (3.2), we have

$$\begin{aligned}
\beta_n(1-k_2-\beta_n)\|u_n - Tu_n\|^2 &\le \|u_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&= \|u_n - x_n\|^2 + 2\langle u_n - x_n, x_n - x^*\rangle \\
&\quad + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.
\end{aligned}$$

This together with (3.8) and condition (C3) imply

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0. \tag{3.9}$$

We now claim

$$\limsup_{n\to\infty}\langle f(x^*) - Bx^*, u_n - x^*\rangle \leq 0, \quad \text{where } x^* = P_\Gamma(f + I - B)x^*.$$

To see this, choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - Bx^*, u_n - x^*\rangle = \lim_{i\to\infty}\langle f(x^*) - Bx^*, u_{n_i} - x^*\rangle.$$

Since the sequence $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ such that $u_{n_{i_j}} \rightharpoonup z \in H_1$. Without loss of generality, we may assume that $u_{n_i} \rightharpoonup z \in H_1$. By the demiclosedness principle of $T - I$ at zero and (3.9), we get $z \in F(T)$. Using the fact that A is a bounded linear operator, $u_{n_i} \rightharpoonup z \in H_1$ and (3.8), we can conclude that $x_{n_i} \rightharpoonup z$ and $Ax_{n_i} \rightharpoonup Az$. Since $S - I$ is demiclosed at zero and (3.7), we also have $Az \in F(S)$. So $z \in F(T) \cap A^{-1}F(S)$. Thus, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - Bx^*, u_n - x^*\rangle &= \lim_{i\to\infty}\langle f(x^*) - Bx^*, u_{n_i} - x^*\rangle \\ &= \langle f(x^*) - Bx^*, z - x^*\rangle \\ &\leq 0,\end{aligned}$$

so we have the claim. Using Lemma 2.5(iii), we have

$$\begin{aligned}\|u_n - x^*\|^2 &= \|\alpha_n(f(x_n) - Bx^*) + (I - \alpha_n B)(x_n + \delta_n A^*(S - I)Ax_n - x^*)\|^2 \\ &\leq \|I - \alpha_n B\|^2 \|x_n + \delta_n A^*(S - I)Ax_n - x^*\|^2 \\ &\quad + 2\alpha_n\langle f(x_n) - Bx^*, u_n - x^*\rangle \\ &\leq (1 - \alpha_n\xi)\|x_n - x^*\|^2 + 2\alpha_n\langle f(x_n) - f(x^*), u_n - x^*\rangle \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &\leq (1 - \alpha_n\xi)\|x_n - x^*\|^2 + 2\alpha_n\rho\|x_n - x^*\|\|u_n - x^*\| \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &\leq (1 - \alpha_n\xi)\|x_n - x^*\|^2 + \alpha_n\rho\|x_n - x^*\|^2 + \alpha_n\rho\|u_n - x^*\|^2 \\ &\quad + 2\alpha_n\langle f(x^*) - Bx^*, u_n - x^*\rangle. \end{aligned} \tag{3.10}$$

From (3.2) and (3.10), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|u_n - x^*\|^2 \\ &\leq \left[\frac{1 - \alpha_n\xi + \alpha_n\rho}{1 - \alpha_n\rho}\right]\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n\rho}\langle f(x^*) - Bx^*, u_n - x^*\rangle \\ &= \left[1 - \frac{\alpha_n(\xi - 2\rho)}{1 - \alpha_n\rho}\right]\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n\rho}\langle f(x^*) - Bx^*, u_n - x^*\rangle\end{aligned} \tag{3.11}$$

By (3.11) and Lemma 2.6, we can conclude that $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists an integer m_o such that

$$\|x_{m_o} - x^*\| \leq \|x_{m_o+1} - x^*\|.$$

Put $\kappa_n = \|x_n - x^*\|$ for all $n \geq m_o$. Then we have $\kappa_{m_o} \leq \kappa_{m_o+1}$. Let $\{\tau(n)\}$ be a sequence defined by

$$\tau(n) = \max\{l \in \mathbb{N} : l \leq n, \kappa_l \leq \kappa_{l+1}\},$$

for all $n \geq m_o$. By Lemma 2.7, we obtain that $\{\tau(n)\}$ is a nondecreasing sequence such that

$$\lim_{n\to\infty} \tau(n) = \infty \quad \text{and} \quad \kappa_{\tau(n)} \leq \kappa_{\tau(n)+1}, \quad \text{for all} \quad n \geq m_o.$$

Similarly of Case 1, we also have

$$\lim_{n\to\infty} \|(S-I)Ax_{\tau(n)}\| = 0, \quad \text{and} \quad \lim_{n\to\infty} \|u_{\tau(n)} - Tu_{\tau(n)}\| = 0.$$

By the demiclosedness principle of $S - I$ and $T - I$ at zero, we obtain

$$\omega_\omega(u_{\tau(n)}) \subset \Gamma.$$

This implies that

$$\limsup_{n\to\infty} \langle f(x^*) - Bx^*, u_{\tau(n)} - x^* \rangle \leq 0. \tag{3.12}$$

It follows from (3.11) that

$$\kappa^2_{\tau(n)+1} \leq \left[1 - \frac{\alpha_{\tau(n)}(\xi - 2\rho)}{1 - \alpha_{\tau(n)}\rho}\right] \kappa^2_{\tau(n)} + \frac{2\alpha_{\tau(n)}}{1 - \alpha_{\tau(n)}\rho} \langle f(x^*) - Bx^*, u_{\tau(n)} - x^* \rangle. \tag{3.13}$$

Since $\kappa_{\tau(n)} \leq \kappa_{\tau(n)+1}$, and by (3.13), we obtain

$$\kappa^2_{\tau(n)} \leq \frac{2}{\xi - 2\rho} \langle f(x^*) - Bx^*, u_{\tau(n)} - x^* \rangle. \tag{3.14}$$

This together with (3.12), we get

$$\limsup_{n\to\infty} \kappa_{\tau(n)} \leq 0,$$

and hence $\lim_{n\to\infty} \kappa_{\tau(n)} = 0$. Using again (3.13), we get

$$\limsup_{n\to\infty} \kappa^2_{\tau(n)+1} \leq \limsup_{n\to\infty} \kappa^2_{\tau(n)},$$

which implies $\lim_{n\to\infty} \kappa_{\tau(n)+1} = 0$. Applying Lemma 2.7, we get

$$0 \leq \kappa_n \leq \max\{\kappa_{\tau(n)}, \kappa_{\tau(n)+1}\}.$$

It follows that $\lim_{n\to\infty} \kappa_n = 0$, i.e., $\lim_{n\to\infty} x_n = x^*$. This completes the proof. □

It is known that the resolvent operator J^M_δ of a maximal monotone mapping M is firmly nonexpansive for all $\delta > 0$ and $F(J^M_\delta) = M^{-1}0$, so J^M_δ is 0-demicontractive mapping and $J^M_\delta - I$ is demiclosed at zero. Hence we obtain the following result directly from Theorem 3.1.

Theorem 3.2. *Let H_1, H_2, A, A^*, f, B and S be the same as Theorem 3.1. Let $M : H_1 \to 2^{H_1}$ be a maximal monotone mapping. Suppose that $\Omega = M^{-1}0 \cap A^{-1}(F(S)) \neq \emptyset$. For $x_1 \in H_1$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ be generated by:*

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(S-I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n J^M_\delta u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.15}$$

where $\{\delta_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying following conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$;
(C4) $0 < c \leq \delta_n \leq d < \frac{1-k_1}{\|A\|^2}$.
Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(f + I - B)x^*$.

We are now interested to apply our main result(Theorem 3.1) to the problem of finding

$$x \in M^{-1}0 \cap F(U) \quad \text{such that} \quad Ax \in F(S), \tag{3.16}$$

where $M : H_1 \to 2^{H_1}$ is a maximal monotone mapping, $U : H_1 \to H_1$ is a quasi-nonexpansive mapping, $S : H_2 \to H_2$ is a k_1-demicontractive mapping and $A : H_2 \to H_2$ is a bounded linear operator. To do this, we need the following lemmas:

Lemma 3.3 ([3])**.** *Let $S : X \to X$ be quasi-nonexpansive mapping, $T : X \to X$ strictly quasi-nonexpansive mapping and $F(S) \cap F(T) \neq \emptyset$. Then*

$$F(ST) = F(TS) = F(S) \cap F(T).$$

Furthermore, ST is quasi-nonexpansive mapping and TS is strictly quasi-nonexpansive mapping.

Remark 3.4. Every a firmly nonexpansive mapping is a strictly quasi-nonexpansive mapping.

Lemma 3.5. *Let H be a real Hilbert space. Let $J : H \to H$ be a firmly nonexpansive and $V : H \to H$ be a quasi-nonexpansive mapping such that $V - I$ is demi-closed at zero. Asume that $F(J) \cap F(V) \neq \emptyset$, then*
(i) $VJ - I$ is demiclosed at zero.
(ii) $JV - I$ is demiclosed at zero.

Proof. Let $p \in F(J) \cap F(V)$.
(i) Let $\{x_n\} \subset H$ be such that $x_n \rightharpoonup x^*$ and $VJx_n - x_n \to 0$. We will show that $x^* \in F(VJ) = F(V) \cap F(J)$. By quasi-nonexpansiveness of V and firmly nonexpansiveness of J, we get

$$\begin{aligned}\|VJx_n - p\|^2 &\leq \|Jx_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \|Jx_n - x_n\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|Jx_n - x_n\|^2 &\leq \|x_n - p\|^2 - \|VJx_n - p\|^2 \\ &= [\|x_n - p\| + \|VJx_n - p\|]\,[\|x_n - p\| - \|VJx_n - p\|] \\ &\leq M\|x_n - VJx_n\|,\end{aligned}$$

where $N = \sup_n\{\|x_n - p\| + \|VJx_n - p\|\}$.
Thus, $Jx_n - x_n \to 0$. It follows that

$$\begin{aligned}\|Vx_n - x_n\| &\leq \|Vx_n - VJx_n\| + \|VJx_n - x_n\| \\ &\leq \|x_n - Jx_n\| + \|VJx_n - x_n\|.\end{aligned}$$

Hence $Vx_n - x_n \to 0$. By demiclosedness of $V - I$ and $J - I$ at zero, we have $x^* \in F(V) \cap F(J) = F(VJ)$ (by Lemma 3.3).
(ii) Let $\{x_n\} \subset H$ be such that $x_n \rightharpoonup x^*$ and $JVx_n - x_n \to 0$. We will show that $x^* \in F(V) \cap F(J)$. By quasi-nonexpansiveness of V and firmly nonexpansiveness of J, we get

$$\begin{aligned}\|JVx_n - p\|^2 &\leq \|Vx_n - p\|^2 - \|JVx_n - Vx_n\|^2 \\ &\leq \|x_n - p\|^2 - \|JVx_n - Vx_n\|^2.\end{aligned}$$

Observe that

$$\begin{aligned}\|JVx_n - Vx_n\|^2 &\leq \|x_n - p\|^2 - \|JVx_n - p\|^2 \\ &\leq N\|x_n - JVx_n\|,\end{aligned}$$

where $N = \sup_n\{\|x_n - p\| + \|JVx_n - p\|\}$.
This implies $JVx_n - Vx_n \to 0$. From

$$\begin{aligned}\|Jx_n - x_n\| &\leq \|Jx_n - JVx_n\| + \|JVx_n - x_n\| \\ &\leq \|x_n - Vx_n\| + \|JVx_n - x_n\|,\end{aligned}$$

and

$$\|Vx_n - x_n\| \leq \|Vx_n - JVx_n\| + \|JVx_n - x_n\|,$$

it follows that $Vx_n - x_n \to 0$ and $Jx_n - x_n \to 0$. By demiclosedness of $V - I$ and $J - I$ at zero, we have $x^* \in F(V) \cap F(J) = F(JV)$ (by Lemma 3.3). □

Theorem 3.6. *Let H_1, H_2, A, A^*, f, B and S be the same as Theorem 3.1. Let $M : H_1 \to 2^{H_1}$ be a maximal monotone mapping. Let $U : H_1 \to H_1$ be a quasi-nonexpansive mapping such that $U - I$ is demiclosed at zero. Suppose that*

$$\Omega = M^{-1}0 \cap F(U) \cap A^{-1}(F(S)) \neq \emptyset.$$

For $x_1 \in H_1$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ be generated by:

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(S - I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n U J_\delta^M u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.17}$$

where $\{\delta_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying following conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$;
(C4) $0 < c \leq \delta_n \leq d < \frac{1-k_1}{\|A\|^2}$.
Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(f + I - B)x^*$.

Proof. If we set $T = UJ_\delta^M$, then T is 0-demicontractive mapping. By Lemma 3.3 and Lemma 3.5, Theorem 3.6 is directly obtained by Theorem 3.1. □

It is known that every (α, β)-generalized hybrid is quasi-nonexpansive mapping, so the following result is directly obtained by Theorem 3.6.

Theorem 3.7. *Let H_1, H_2, A, A^*, f, B and S be the same as Theorem 3.1. Let $M : H_1 \to 2^{H_1}$ be a maximal monotone mapping such that $D(M) \subset C$. Let $U : C \to C$ be a (α, β)-generalized hybrid mapping such that $U - I$ is demiclosed at zero. Suppose that $\Omega = M^{-1}0 \cap F(U) \cap A^{-1}(F(S)) \neq \emptyset$. For $x_1 \in C$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ be generated by:*

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(S - I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n U J_\delta^M u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{3.18}$$

where $\{\delta_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$;
(C4) $0 < c \leq \delta_n \leq d < \frac{1-k_1}{\|A\|^2}$.
Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(f + I - B)x^*$.

4. Applications

Now, we apply our main results to study the following problems:

4.1. **The split common null point problem.** In this section, we apply Theorem 3.1 to solve the split common null point problem in Hilbert spaces. Let H_1 and H_2 be two real Hilbert spaces. Let $M_i : H_1 \to 2^{H_1}$ $(1 \leq i \leq p)$ and $U_j : H_2 \to 2^{H_2}$ $(1 \leq j \leq q)$. The *split common null point problem* (SCNPP) is to find a point $u^* \in H_1$ such that

$$0 \in \bigcap_{i=1}^{p} M_i u^*, \tag{4.1}$$

and the point $v_j^* = A_j u^* \in H_2$ satisfy

$$0 \in \bigcap_{j=1}^{q} U_j v_j^*, \tag{4.2}$$

where $A_j : H_1 \to H_2$ $(1 \leq j \leq q)$ are bounded linear operators.

When $p = q = 1$ above SCNPP is reduced to find a point $u^* \in H_1$ such that

$$0 \in Mu^* \quad \text{and} \quad 0 \in U(Au^*). \tag{4.3}$$

We denote the solution set of (4.3) by Ω.

The following result is a strong convergence theorem for the split common null point problem (4.3).

Theorem 4.1. *Let H_1, H_2, A, A^*, f and B be the same as Theorem 3.1. Let $M : H_1 \to 2^{H_1}$ and $U : H_2 \to 2^{H_2}$ be two maximal monotone mappings. Suppose that $\Omega \neq \emptyset$. For $x_1 \in H_1$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ be generated by:*

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(J_\delta^U - I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n J_\delta^M u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{4.4}$$

where $\{\delta_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$;
(C4) $0 < c \leq \delta_n \leq d < \frac{1}{\|A\|^2}$.
Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_\Omega(f + I - B)x^*$.

Proof. Set $S = J_\delta^U$ and $T = J_\delta^M$ for all $\delta > 0$. Then S and T are 0-demicontractive mappings. Then Theorem 4.1 is directly obtained by Theorem 3.1. □

4.2. **The split variational inequality problem.** Let C and Q be nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, $g : H_1 \to H_1$ and $h : H_2 \to H_2$. The *split variational inequality problem* (SVIP) is to find a point $u^* \in C$ such that

$$\langle g(u^*), x - u^* \rangle \geq 0, \quad \forall x \in C, \tag{4.5}$$

and the point $v^* = Au^* \in Q$ satisfy

$$\langle h(v^*), y - v^* \rangle \geq 0, \quad \forall y \in Q. \tag{4.6}$$

We denote the solution set of the SVIP by $\varOmega = SVIP(C, Q, g, h, A)$. The set of all solutions of *variational inequality problem* (4.5) is denoted by $VIP(C, g)$ and it is known that $VIP(C, g) = F(P_C(I - \lambda g))$ for all $\lambda > 0$.

We now prove a strong convergence theorem for split variational inequality problem (4.5) and (4.6).

Theorem 4.2. *Let* H_1 *and* H_2 *be two real Hilbert spaces,* C *and* Q *be nonempty closed convex subsets of* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator with its adjoint operator* A^* *and* $f : H_1 \to H_1$ *be a* ρ*-contraction mapping and* B *be a self-adjoint strongly positive bounded linear operator on* H_1 *with coefficient* $\xi > 2\rho$ *and* $||B|| = 1$*. Let* $g : H_1 \to H_1$ *and* $h : H_2 \to H_2$ *be* η_1 *and* η_2*-inverse strongly monotone mappings, respectively. Let* $S := P_Q(I - \lambda h)$ *and* $T := P_C(I - \lambda g)$*, where* $\lambda \in (0, 2\eta]$ *and* $\eta = \min\{\eta_1, \eta_2\}$*. Suppose that* $\varOmega \neq \emptyset$*. For* $x_1 \in H_1$ *arbitrarily, let* $\{u_n\}$ *and* $\{x_n\}$ *be generated by:*

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(S - I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n T u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{4.7}$$

where $\{\delta_n\}, \{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$;
(C4) $0 < c \leq \delta_n \leq d < \frac{1}{\|A\|^2}$.
Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_\varOmega(f + I - B)x^*$.

Proof. It is known that $S := P_Q(I - \lambda h)$ and $T := P_C(I - \lambda g)$ are nonexpensive mappings and hence they are 0-demicontractive mappings. We obtain the desired result from Theorem 3.1. □

4.3. **The split convex minimization problem.** Let $G : C \to \mathbb{R}$ be a real-valued convex function. The constraint *minimization problem* is to find $z \in C$ such that

$$G(z) = \min\{G(x) : x \in C\}. \tag{4.8}$$

We denote Ω by the solution set of minimization problem. If G is *Fréchet* differentiable, then $u^* \in \Omega$ if and only if

$$\langle \nabla G(u^*), x - u^* \rangle \geq 0, \quad \forall x \in C, \tag{4.9}$$

where ∇G is the gradient of G. It is known that the solution set of (4.9) is the fixed point set of $P_C(I - \lambda \nabla G)$ for all $\lambda > 0$. Let C and Q be nonempty closed convex subsets of two real Hilbert spaces H_1 and H_2, respectively. Observe that if $G : H_1 \to H_1$ and $H : H_2 \to H_2$ are *Fréchet* differentiable convex functions on C and Q, respectively, and take $g = \nabla G$ and $h = \nabla H$, then the *split convex minimization problem* (SMP) is to find a point $u^* \in C$ such that

$$u^* = \arg\min\{g(x) : x \in C\}, \tag{4.10}$$

and the point $v^* = Au^* \in Q$ satisfy

$$v^* = \arg\min\{h(x) : x \in Q\}. \tag{4.11}$$

We denote the solution set of the SMP by Ω. The following result is obtained directly by Theorem 3.1.

Theorem 4.3. *Let $H_1, H_2, C, Q, A, A^*, f$ and B be the same Theorem 4.2. Let $G : H_1 \to H_1$ and $H : H_2 \to H_2$ be Fréchet differentiable convex functions on C and Q, respectively. Suppose that ∇G and ∇H be η_1 and η_2-inverse strongly monotone mappings, respectively and $\Omega \neq \emptyset$. Let $S := P_Q(I - \lambda \nabla H)$ and $T := P_C(I - \lambda \nabla G)$, where where $\lambda \in (0, 2\eta]$ and $\eta = \min\{\eta_1, \eta_2\}$. For $x_1 \in H_1$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ be generated by:*

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(S - I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n T u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{4.12}$$

where $\{\delta_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$;
(C4) $0 < c \leq \delta_n \leq d < \frac{1}{\|A\|^2}$.
Then the sequence $\{x_n\}$ converges strongly to $x^* = P_\Omega(f + I - B)x^*$.

4.4. **The split equilibrium problem.** Let H_1 and H_2 be two real Hilbert spaces and let C and Q be nonempty closed convex subset of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, $g : C \times C \to \mathbb{R}$ and $h : Q \times Q \to \mathbb{R}$ be two bifunctions. The *split equilibrium problem*(SEQP) is to find a point $u^* \in C$ such that

$$g(u^*, x) \geq 0, \quad \forall x \in C, \tag{4.13}$$

and $Au^* \in Q$ satisfy

$$h(Au^*, y) \geq 0, \quad \forall y \in Q. \tag{4.14}$$

We denote the solution set of the SEQP by Ω. The following lemmas are useful for our main result.

Lemma 4.4 ([1]). *Let C be a nonempty closed convex subset of H and g be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the following condition:*

(A1) $g(x,x) = 0$ for all $x \in C$;
(A2) g is monotone, i.e., $g(x,y) + g(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} g(tz + (1-t)x, y) \leq g(x,y);$$

(A4) $g(x,\cdot)$ is convex and lower semicontinuous for all $x \in C$.

If $g : C \times C \to \mathbb{R}$ is a bifunction satisfying the condition $(A1)-(A4)$ and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that

$$g(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \textit{for all } y \in C.$$

Lemma 4.5 ([5]). *Let C be a nonempty closed convex subset of H and g be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the condition $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ of g by*

$$T_r x = \left\{ z \in C \ : \ g(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}, \quad \forall x \in H.$$

Then the following hold:

(i) T_r is single-valued;
(ii) T_r is firmly nonexpansive;
(iii) $F(T_r) = EP(g)$;
(iv) $EP(g)$ is closed and convex.

Lemma 4.6 ([15]). *Let C be a nonempty closed convex subset of H and g be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the condition $(A1)-(A4)$. Define A_g as follows:*

$$A_g(x) = \begin{cases} \{z \in H \ : \ g(x,y) \geq \langle y - x, z\rangle \ \forall y \in C\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases} \tag{4.15}$$

Then $EP(g) = A_g^{-1}(0)$ and A_g is maximal monotone with the domain of A_g in C. Furthermore,

$$T_r(x) = (I + rA_g)^{-1}(x), \quad \forall r > 0.$$

Since the resolvent of the maximal monotone operators are firmly nonexpansive, the following result is immediately obtained by Theorem 3.1.

Theorem 4.7. *Let H_1, H_2, C, Q, f and B be the same Theorem 4.2. Let $g : C \times C \to \mathbb{R}$ and $h : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying the condition $(A1)-(A4)$. Let T_δ and T_r be the resolvent of A_g and A_h, (as defined in (4.15)) for $\delta, r > 0$, respectively. Suppose that $\Omega \neq \emptyset$. For $x_1 \in H_1$ arbitrarily, let $\{u_n\}$ and $\{x_n\}$ be generated by:*

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(T_r - I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n T_\delta u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{4.16}$$

where $\{\delta_n\}, \{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a \leq \beta_n \leq b < 1$;
(C4) $0 < c \leq \delta_n \leq d < \frac{1}{\|A\|^2}$.

Then the sequence $\{x_n\}$ *converges strongly to* $x^* = P_\Omega(f + I - B)x^*$.

Remark 4.8. We make the following remarks concerning our results.

(i) In the results of Moudafi [9], he obtained the weak convergence for the SFPP of two demicontractive mappings, but in this paper we obtain the strong convergence for the SFPP of two demicontractive mappings.
(ii) In 2014, Takahashi, Xu and Yao [16] proposed an iterative method for solving problem (3.16), where $U : H_1 \to H_1$ is a generalized hybrid mapping and $S : H_2 \to H_2$ is a nonexpansive mapping. They obtained only weak convergence but Theorem 3.7 a strong convergence for this problem.
(iii) Some authors, (see [12, 11, 13]), introduced iterative methods for solving SFPP for two demicontractive mapping and also obtained strong convergence. However, our iterative method is different from those works.

5. Numerical example for the main result

Let $H_1 = H_2 = (\mathbb{R}^5, ||\cdot||_2)$. Define mappings $f, S, T : \mathbb{R}^5 \to \mathbb{R}^5$ by

$$f(x) = \frac{1}{16}x, \quad S(x) = -\frac{5}{2}x, \quad T(x) = -2x,$$

where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \in \mathbb{R}^5$ and let $B : \mathbb{R}^5 \to \mathbb{R}^5$ be defined by $B\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} (2/3)x_1 \\ (1/3)x_2 \\ x_3 \\ (1/3)x_4 \\ (1/3)x_5 \end{pmatrix}$.

Then f, B, S, T are $\frac{1}{16}$-contraction, self-adjoint strongly positive linear bounded operator with coefficient $\xi = \frac{1}{3}$, $\frac{3}{7}$-demicontractive mapping and $\frac{1}{3}$-demicontractive mapping, respectively. Note that S and T are not quasi-nonexpansive mapping. Choose $\alpha_n = \dfrac{1}{11n-1}$, $\delta_n = \dfrac{n}{1,000n-1}$, and $\beta_n = \dfrac{1}{2} - \dfrac{1}{50n}$ for all $n \geq 1$. Let $A = \begin{pmatrix} 0 & 4 & 5 & 4 & 0 \\ 1 & 5 & -5 & 0 & 2 \\ 1 & -3 & -5 & 0 & 0 \\ 1 & 0 & 1 & 3 & -9 \\ 2 & 1 & 1 & 4 & 1 \end{pmatrix}$. We start with the initial point $x_1 = \begin{pmatrix} 3 \\ 5 \\ -4 \\ 2 \\ -3 \end{pmatrix}$ and let $\{x_n\}$ be the sequence generated by

$$\begin{cases} u_n = \alpha_n f(x_n) + (I - \alpha_n B)(x_n + \delta_n A^*(S - I)Ax_n), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n T u_n, \qquad n \in \mathbb{N}, \end{cases}$$

Suppose that x_n is in the form $x_n = \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \end{pmatrix}$, where $a_n, b_n, c_n, d_n, e_n \in \mathbb{R}$. The criterion for stopping our testing method is taken as: $\|x_{n-1} - x_n\|_2 < 10^{-6}$. The value of x_n and $\|x_{n-1} - x_n\|_2$ are shown in the following table:

n	a_n	b_n	c_n	d_n	e_n	$\|x_{n-1} - x_n\|_2$
1	3.00000000	5.00000000	-4.00000000	2.00000000	-3.00000000	-
2	-1.08774016	-1.79768060	1.36469369	-0.59389907	0.99813989	10.6963290
3	0.43935004	0.70132834	-0.52041873	0.18097608	-0.36492268	3.8195641
4	-0.18495636	-0.28155872	0.20344520	-0.05183887	0.13932102	1.4792799
5	0.07989845	0.11486268	-0.08000925	0.01234365	-0.05477975	0.5911336
6	-0.03519913	-0.04738171	0.03142980	-0.00127037	0.02203001	0.2409864
⋮	⋮	⋮	⋮	⋮	⋮	⋮
21	0.00000053	0.00000018	0.00000009	-0.00000035	-0.00000009	0.0000020
22	-0.00000026	-0.00000008	-0.00000005	0.00000017	0.00000004	0.0000010
23	0.00000013	0.00000004	0.00000003	-0.00000009	-0.00000002	0.0000005

We observe from the table that $x_n \to \bar{0} \in F(T) \cap A^{-1}(F(S))$, where $\bar{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

We also note that the error bounded of $\|x_{22} - x_{23}\|_2 < 10^{-6}$ and we can use

$$x_{23} = \begin{pmatrix} 0.00000013 \\ 0.00000004 \\ 0.00000003 \\ -0.00000009 \\ -0.00000002 \end{pmatrix},$$

to approximate the solution of SFP with accuracy at least 6 D.P.

6. Conclusion

In this work, by using the concept of viscosity approximate method and Mann iteration in a Hilbert space we introduce a new algorithm for solving the split fixed point problem for two demicontractive mappings, we obtain strong convergence result under some suitable control conditions and apply our main results to study split common null point problems, split variational inequality problems, split convex minimization problems and split equilibrium problems. Moreover, we give some numerical experiment to support our main results. The novelty of this work are the following:

(1) We obtain a new algorithm for solving the split fixed point problem for two demicontractive mappings.
(2) We obtain strong convergence of our proposed algorithm which is more desirable than that Moudafi [8].

(3) We can apply our main results to study split common null point problems, split variational inequality problems, split convex minimization problems and split equilibrium problems.

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