Fixed Point Theory, 21(2020), No. 2, 481-494
DOI: 10.24193/fpt-ro.2020.2.35
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

NONEXPANSIVE MAPPINGS AND CONTINUOUS s-POINT SPACES

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Abstract. Here, the concept of continuous s-point space is introduced in b-metric spaces. Under suitable assumptions, these spaces are absolute retracts and a generalization of the continuous midpoint spaces. Moreover, an important fixed point theorem is proved for nonexpansive mappings in continuous s-point spaces.
Key Words and Phrases: Continuous s-point spaces, b-metric spaces, nonexpansive mappings.
2010 Mathematics Subject Classification: 47H10, 51D99, 47H10.

1. Introduction

Let (X,d) be a metric space. A point m of X is a midpoint for the pair $(a,b) \in X \times X$ if $d(a,m) = \frac{1}{2}d(a,b) = d(m,b)$. For all pairs of points of a complete metric space (X,d) to have a midpoint it is necessary and sufficient that the metric be strictly intrinsic. That is, there exists a continuous rectifiable path $\gamma : [0,1] \to X$ from a to b whose length is $d(a,b)$ [3].
A continuous midpoint map on a metric space (X,d) is a continuous map $\mu : X \times X \to X$ such that, for all $(a,b) \in X \times X, d(a,\mu(a,b)) = \frac{1}{2}d(a,b) = d(\mu(a,b),b)$. If μ is a continuous midpoint then $\breve{\mu}(a,b) = \mu(b,a)$ is also a continuous midpoint map. The triple (X,d,μ) is a continuous midpoint space [8].
Now, we assume that the metric space (X,d) is a b-metric space, that is:
Definition 1.1. [4] Let X be a nonempty set and let $s \geqslant 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x,y,z \in X$ the following conditions are satisfied:

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \leqslant s[d(x,y) + d(y,z)]$.

A pair (X, d) is called a b-metric space with constant s. Observe that if $s = 1$, then the ordinary triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$. Thus the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. That is, every metric space is a b-metric space, but the converse need not be true.
In this paper, we introduced the continuous s-point space in Section 2. Section 3 characterizes complete continuous midpoint spaces as those spaces in which every two points can be joined by a geodesic path and the geodesic path can be chosen in such a way as to depend continuously on its endpoints. Finally, an important fixed point theorem is proved for nonexpansive mappings in continuous s-point spaces.

2. Continuous s-point space

In this section we introduce the concept of s-point space which is a generalization of the continuous midpoint space.
Definition 2.1. Let (X, d) be a b-metric space with constant s_b. A point m of X is called a s-point for the pair $(a, b) \in X \times X$, if

$$d(a, m) = \frac{1}{2s_b} d(a, b) = d(m, b).$$

Throughout this article, for convenience, we use s instead of s_b.
Definition 2.2. A continuous s-point map on a b-metric space (X, d) is a continuous map $\mu : X \times X \to X$ such that, for all $(a, b) \in X \times X$ we have

$$d(a, \mu(a, b)) = \frac{1}{2s} d(a, b) = d(b, \mu(a, b)).$$

The triple (X, d, μ) is called a continuous s-point space. A continuous s-point space (X, d, μ) is a unique continuous s-point space, if for all pair of points there exists a unique s-point. In a unique continuous s-point space the s-point map is symmetric, that is $\mu(a, b) = \mu(b, a)$. When $s = 1$, the continuous s-point space reduces to continuous midpoint space in [8].
Here, we say that a closed subset C of X is convex if $\mu(a, b) \in C$, for all $(a, b) \in C \times C$.
Example 2.3. Let $X = [0, 1]$ and $d : X \times X \to [0, +\infty)$ be defined by

$$d(x, y) = (x - y)^2,$$

for all $x, y \in X$. Clearly (X, d) is a b-metric space with constant 2. If

$$\mu(a, b) = \frac{(a + b)}{2}$$

then

$$d(a, \mu(a, b)) = \left(a - \frac{a + b}{2}\right)^2 = \frac{(a - b)^2}{4}$$

and

$$d(\mu(a, b), b) = \left(\frac{a + b}{2} - b\right)^2 = \frac{(a - b)^2}{4},$$

for all $a, b \in X$. Therefore (X, d, μ) is a complete continuous s-point space.

Lemma 2.4. *Let (X,d) be a b-metric space with constant s. If for each $(a,b)\in X\times X$, there exists a $z\in X$ such that for each $x\in X$ we have*

$$d(a,b)^2+4s^2(2s^2-1)d(x,z)^2=2s^2d(a,x)^2+2s^2d(x,b)^2, \tag{2.1}$$

then (X,d) is a unique continuous s-point spaces.

Proof. First, we show that z is a s-point. Taking $x=a$ in (2.1) we obtain

$$d(a,z)=\frac{1}{2s}d(a,b).$$

Similarly, with $x=b$ we obtain

$$d(z,b)=\frac{1}{2s}d(a,b).$$

Let us for each $(a,b)\in X\times X$, see that there is a unique point $z\in X$ for which the property (2.1) holds. If

$$d(a,b)^2+4s^2(2s^2-1)d(x,z_i)^2=2s^2d(a,x)^2+2s^2d(x,b)^2,$$

holds for all $x\in X$ with $i\in\{1,2\}$ then, put $x=z_1$, we have

$$d(a,b)^2+4s^2(2s^2-1)d(z_1,z_2)^2=2s^2d(a,z_1)^2+2s^2d(z_1,b)^2.$$

Since z_1 is a s-point, $d(a,z_1)=\dfrac{1}{2s}d(a,b)=d(b,z_1)$. Thus

$$4s^2(2s^2-1)d(z_1,z_2)^2=0,$$

that is $d(z_1,z_2)=0$.

Remark 2.5. The following inequality, which can be easily derived from Lemma 2.4 and the definition of a s-point, shows that the s-point map is continuous.

$$\begin{aligned}
&d(a,b)^2+4s^2(2s^2-1)d(\mu(a,b),\mu(a',b'))^2\\
=\;&2s^2d(\mu(a',b'),a)^2+2s^2d(\mu(a',b'),b)^2\\
\le\;&2s^2\big[s\big(d(a,a')+d(a',\mu(a',b'))\big)\big]^2+2s^2\big[s\big(d(b,b')+d(b',\mu(a',b'))\big)\big]^2\\
=\;&2s^2\big[sd(a,a')+\frac{1}{2}d(a',b')\big]^2+2s^2\big[sd(b,b')+\frac{1}{2}d(a',b')\big]^2.
\end{aligned}$$

3. Main results

Now, we want to determine the s-point map in b-metric space.

Lemma 3.1. *Let (X,d) be a b-metric space. If there exists a continuous map $\psi: X\times X\times[0,1]\to X$, such that for each $(a,b)\in X\times X$ and each $t\in[0,1]$:*

$$d(a,\psi(a,b,t))=\frac{t}{s}d(a,b)\quad\text{and}\quad d(b,\psi(a,b,t))=\frac{(1-t)}{s}d(a,b),$$

then there exists a continuous s-point map on $X\times X$.

Proof. Put $\mu(a,b)=\psi(a,b,\frac{1}{2})$. So we have $d(a,\mu(a,b))=\dfrac{1}{2s}d(a,b)=d(\mu(a,b),b)$.

All of the results of this paper are consequences Theorem 3.2.

Theorem 3.2. *If (X,d,μ) is a complete continuous s-point space then there exists a continuous map $\psi: X\times X\times[0,1]\to X$ such that*

(i) *$d(a,\psi(a,b,t))\le td(a,b)$ and $d(b,\psi(a,b,t))\le(1-t)d(a,b)$;*

(ii) $d(\psi(a,b,t),\psi(a,b,t')) \leq |t-t'|d(a,b)$,

for each $(a,b) \in X \times X$ *and for all* $(t,t') \in [0,1]\times[0,1]$. *Furthermore, if* (X,d) *is also a unique s-point space then there is a unique map* $\psi : X \times X \times [0,1] \to X$ *for which (i) and (ii) hold. It also has the following property*

(iii) $\psi(a,b,t) = \psi(b,a,1-t)$.

Proof. Let $E_m = \left\{\frac{k}{2^m} : 0 \leq k \leq 2^m\right\}$. For example

$$E_0 = \{0,1\},$$
$$E_1 = \left\{0,\frac{1}{2},1\right\},$$
$$E_2 = \left\{0,\frac{1}{4},\frac{2}{4},\frac{3}{4},1\right\},$$
$$E_3 = \left\{0,\frac{1}{8},\frac{2}{8},\frac{3}{8},\frac{4}{8},\frac{5}{8},\frac{6}{8},\frac{7}{8},1\right\},\cdots.$$

The set $E = \bigcup_{m\in\mathbb{N}} E_m$ is dense in $[0,1]$. If $t \in E_{m+1}\setminus E_m$, $t = \frac{k}{2^{m+1}}$ and k is odd. Let $t_r = \frac{k-1}{2^{m+1}}$, $t_d = \frac{k+1}{2^{m+1}}$, then t_r and t_d are both in E_m.

Step 1. The construction below defines by induction a sequence of maps $\psi_m(a,b,.) : E_m \to X$ such that the restriction of $\psi_{m+1}(a,b,.)$ to E_m is $\psi_m(a,b,.)$ for a fixed pair $(a,b) \in X\times X$ and so a map $\psi_\omega(a,b,.) : E \to X$ whose restriction to E_m is $\psi_m(a,b,.)$. Put $\psi_m(a,b,0) = a$ and $\psi_m(a,b,1) = b$ for all $m \in \mathbb{N}$; this defines $\psi_0(a,b,.)$. Then

$$\psi_{m+1}(a,b,t) = \psi_m(a,b,t) \text{ if } t \in E_m,$$

and

$$\psi_{m+1}(a,b,t) = \mu\big(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\big) \text{ if } t \in E_{m+1}\setminus E_m.$$

Step 2. We show that when t_d and t_r are two consecutive elements of E_m then

$$d\big(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\big) = \frac{1}{(2s)^m}d(a,b).$$

For $m = 0,1$ this is a consequence of the definition of ψ_0 and μ. Since $t_d - t_r = \frac{1}{2^m}$ we have either $t_r \in E_{m-1}$ or $t_d \in E_{m-1}$. If $t_r \in E_{m-1}$, then $t_r + \frac{1}{2^{m-1}} \in E_{m-1}$ and

$$t_d = \frac{1}{2}\left[t_r + \left(t_r + \frac{1}{2^{m-1}}\right)\right] \in E_m \setminus E_{m-1}.$$

Also

$$\psi_m(a,b,t_r) = \psi_{m-1}(a,b,t_r),$$
$$\psi_m(a,b,t_d) = \mu\left(\psi_{m-1}(a,b,t_r),\psi_{m-1}\left(a,b,t_r+\frac{1}{2^{m-1}}\right)\right).$$

Hence by induction we have

$$\begin{aligned}
&d\big(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\big)\\
&= d\left(\psi_{m-1}(a,b,t_r),\mu\left(\psi_{m-1}(a,b,t_r),\psi_{m-1}(a,b,t_r+\frac{1}{2^{m-1}})\right)\right)\\
&= \frac{1}{2s}d\left(\psi_{m-1}(a,b,t_r),\psi_{m-1}\left(a,b,t_r+\frac{1}{2^{m-1}}\right)\right)\\
&= \frac{1}{2s}\left(\frac{1}{(2s)^{m-1}}d(a,b)\right) = \frac{1}{(2s)^m}d(a,b).
\end{aligned} \tag{3.1}$$

One proceeds similarly if $t_d \in E_{m-1}$.

Step 3. By induction on m we show that

$$d\big(a,\psi_\omega(a,b,t)\big) \le td(a,b) \text{ and } d\big(b,\psi_\omega(a,b,t)\big) \le (1-t)d(a,b),$$

for all $t \in E$. If $t \in E_0$ or $t \in E_1 \setminus E_0$ this is obvious from the definition of ψ_m and μ. Assume that $t \in E_{m+1} \setminus E_m$. By equation (3.1), definition of ψ and hypothesis of induction, we have

$$\begin{aligned}
&d\big(a,\psi_{m+1}(a,b,t)\big)\\
&= d\big(a,\mu\big(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\big)\big)\\
&\le s\big[d\big(a,\psi_m(a,b,t_r)\big) + d\big(\psi_m(a,b,t_r),\mu\big(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\big)\big)\big]\\
&\le s\left[d\big(a,\psi_{m-1}(a,b,t_r)\big) + \frac{1}{2s}d\left(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\right)\right]\\
&\le s\left[t_r d(a,b) + \frac{1}{(2s)(2s)^m}d(a,b)\right]\\
&= \left(st_r + \frac{s}{(2s)^{m+1}}\right)d(a,b) \leqslant td(a,b).
\end{aligned}$$

Therefore for all $t \in E$,

$$d\big(a,\psi_\omega(a,b,t)\big) \leqslant td(a,b).$$

Similarly, we can show that

$$\begin{aligned}
d\big(\psi_{m+1}(a,b,t),b\big) &= d\big(\mu\big(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\big),b\big)\\
&\le s\big[d\big(\mu\big(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\big),\psi_m(a,b,t_d)\big)\\
&+ d\big(\psi_m(a,b,t_d),b\big)\big]\\
&\le s\left[\frac{1}{(2s)}d\left(\psi_m(a,b,t_r),\psi_m(a,b,t_d)\right) + (1-t_d)d(a,b)\right]\\
&\le s\left[\frac{1}{(2s)(2s)^m}d(a,b) + (1-t_d)d(a,b)\right]\\
&= \left(\frac{s}{(2s)^{m+1}} + s(1-t_d)\right)d(a,b) \leqslant (1-t)d(a,b),
\end{aligned}$$

for all $t \in E$. Hence

$$d\big(\psi_\omega(a,b,t),b\big) \leqslant (1-t)d(a,b).$$

Step 4. We claim that $t \to \psi_\omega(a,b,t)$ is uniformly continuous on E. For $t,t' \in E$ we choose m such that $t,t' \in E_m$, let us $t = t_k = \frac{k}{2^m}$, $t' = t_{k+j} = \frac{k+j}{2^m}$. By (3.1) we have

$$\begin{aligned}
d\big(\psi_m(a,b,t),\psi_m(a,b,t')\big) &\le s\big[d\big(\psi_m(a,b,t_k),\psi_m(a,b,t_{k+1})\big)\\
&+ d\big(\psi_m(a,b,t_{k+1}),\psi_m(a,b,t_{k+j})\big)\big]\\
&\le sd\big(\psi_m(a,b,t_k)),\psi_m(a,b,t_{k+1})\big)\\
&+ s^2 d\big(\psi_m(a,b,t_{k+1})),\psi_m(a,b,t_{k+2})\big)\\
&+ \cdots + s^{j-1}d\big(\psi_m(a,b,t_{k+j-2})),\psi_m(a,b,t_{k+j-1})\big)\\
&+ s^{j-1}d\big(\psi_m(a,b,t_{k+j-1})),\psi_m(a,b,t_{k+j})\big)\\
&\le \frac{1}{(2s)^m}(s+s^2+\cdots+s^{j-2}+s^{j-1}+s^{j-1})d(a,b)\\
&= \left(\frac{j}{2^m}\right)\left(\frac{1}{js^m}\right)(s+s^2+\cdots+s^{j-2}+s^{j-1}+s^{j-1})d(a,b)\\
&\le |t-t'|\frac{1}{js^m}(js^{j-1})d(a,b) = |t-t'|\frac{1}{s^{m-j+1}}d(a,b)\\
&\le |t-t'|d(a,b).
\end{aligned}$$

According to above steps and since E is dense in $[0,1]$ and X is complete, there exists a unique uniformly continuous map $\psi(a,b,.) : E \to X$ such that

(1) $\psi(a,b,0) = a$ and $\psi(a,b,1) = b$;
(2) for all $t \in [0,1]$, $d(a,\psi(a,b,t)) \le td(a,b)$ and $d(b,\psi(a,b,t)) \le (1-t)d(a,b)$;
(3) for all $t \in [0,1]$, $d(\psi(a,b,t),\psi(a,b,t')) \le |t-t'|d(a,b)$.

Step 5. Now, we prove that ψ is continuous on $X \times X \times [0,1]$.

(a) From the continuity of the s-point map μ, the definition of $\psi_{m+1}(a,b,t)$ and an induction on m, we can see that $(a,b) \to \psi(a,b,t) = \psi_m(a,b,t)$ is continuous for all $t \in E_m$.
(b) Let us show that $(a,b) \to \psi(a,b,t)$ is continuous for all $t \in [0,1]$.
Put $\Delta = d\big(\psi(a,b,t),\psi(a',b',t)\big)$. From (3) we have, for arbitrary $t' \in [0,1]$,

$$\begin{aligned}
\Delta &\le sd\big(\psi(a,b,t),\psi(a,b,t')\big) + sd\big(\psi(a,b,t'),\psi(a',b',t)\big)\\
&\le sd\big(\psi(a,b,t),\psi(a,b,t')\big) + s^2 d\big(\psi(a,b,t'),\psi(a',b',t')\big)\\
&+ s^2 d\big(\psi(a',b',t'),\psi(a',b',t)\big)\\
&\le |t-t'|\big(sd(a,b)+s^2d(a',b')\big) + s^2 d\big(\psi(a,b,t'),\psi(a',b',t')\big).
\end{aligned}$$

Since

$$d(a',b') \le sd(a',a) + s^2 d(a,b) + s^2 d(b,b'),$$

if $sd(a',a) + s^2 d(b,b') = 1$, then $d(a',b') \le 1 + s^2 d(a,b)$, then

$$sd(a,b) + s^2 d(a',b')) \le s^2 + (s+s^4)d(a,b),$$

and therefore

$$\Delta \le |t-t'|\big(s^2 + (s+s^4)d(a,b)\big) + s^2 d\big(\psi(a,b,t'),\psi(a',b',t')\big).$$

We can choose $t' \in E$ such that $|t - t'| < \dfrac{\varepsilon}{s^2 + (s + s^4)d(a,b)}$ and conclude from the continuity of the map $\psi(.,.,t')$ at (a,b).

(c) Put $\Delta = d\big(\psi(a,b,t), \psi(a',b',t')\big)$ and notice that

$$\begin{aligned}\Delta &\le sd\big(\psi(a,b,t), \psi(a',b',t)\big) + sd\big(\psi(a',b',t), \psi(a',b',t'))\big)\\ &\le sd\big(\psi(a,b,t), \psi(a',b',t)\big) + s|t - t'|d(a',b').\end{aligned}$$

Therefore ψ is continuous at (a,b,t).

Step 6. Suppose that (X,d) is a unique continuous s-point space. We claim that $\psi : X \times X \times [0,1] \to X$ is the unique continuous map for which (1) and (2) hold. If φ is such a map then $\varphi(a,b,0) = \psi(a,b,0), \varphi(a,b,1) = \psi(a,b,1)$ and $\varphi\left(a,b,\dfrac{1}{2}\right) = \mu(a,b)$. On the other hand

$$\varphi(a,b,\frac{t_1+t_2}{2}) = \mu\big(\varphi(a,b,t_1), \varphi(a,b,t_2)\big).$$

If $t_1 < t_2$ and $M = \varphi\left(a,b,\dfrac{t_1+t_2}{2}\right)$ then by (3) we have

$$d\big(M, \varphi(a,b,t_1)\big) \le \left(\frac{t_1+t_2}{2} - t_1\right) d(a,b) = \left(\frac{t_2-t_1}{2}\right) d(a,b),$$

and similarly

$$d(M, \varphi(a,b,t_2)) \le \left(t_2 - \frac{t_1+t_2}{2}\right) d(a,b) = \left(\frac{t_2-t_1}{2}\right) d(a,b).$$

Also by step (4)

$$d\big(\varphi(a,b,t_1), \varphi(a,b,t_2)\big) \le (t_2 - t_1)d(a,b).$$

Hence M less than (or equal) of midpoint of $\varphi(a,b,t_1)$ and $\varphi(a,b,t_2)$.
If $t \in E \setminus \{0, \frac{1}{2}, 1\}$. Let m be the smallest integer for which $t \in E_m$, from $t = \dfrac{t_r + t_d}{2}$ we have

$$\varphi(a,b,t) = \mu\big(\varphi(a,b,t_r), \varphi(a,b,t_d)\big).$$

and by induction on m shows that $\varphi = \psi$ on $X \times X \times E$ and therefore $\varphi = \psi$.
Finally, notice that (i) and (ii) hold for $\psi(a,b,t) = \psi(b,a,1-t)$; this proves (iii).

Remark 3.3. In proof of Theorem 3.2, we observe that

(1) We have the following before step 2: $\psi_0(a,b,0) = a,\ \psi_0(a,b,1) = b$.

$\psi_1(a,b,0) = a,\ \psi_1\left(a,b,\dfrac{1}{2}\right) = \mu(a,b),\ \psi_1(a,b,1) = b.$

$\psi_2(a,b,0) = a,\ \psi_2\left(a,b,\dfrac{1}{4}\right) = \mu(a,\mu(a,b)),\ \psi_2\left(a,b,\dfrac{2}{4}\right) = \mu(a,b),$

$\psi_2\left(a,b,\dfrac{3}{4}\right) = \mu(\mu(a,b),b),\ \psi_2(a,b,1) = b.$

$\psi_3(a,b,0) = a,\ \psi_3\left(a,b,\dfrac{1}{8}\right) = \mu(a,\mu(a,\mu(a,b))),$

$\psi_3\left(a,b,\dfrac{2}{8}\right) = \mu(a,\mu(a,b)),$

$$\psi_3\left(a,b,\frac{3}{8}\right)=\mu(\mu(a,\mu(a,\mu(a,b))),\mu(a,b)),\psi_3\left(a,b,\frac{4}{8}\right)=\mu(a,b),$$
$$\psi_3\left(a,b,\frac{5}{8}\right)=\mu(\mu(a,b),\mu(\mu(a,b),b)),\ \psi_3\left(a,b,\frac{6}{8}\right)=\mu(\mu(a,b),b),$$
$$\psi_3\left(a,b,\frac{7}{8}\right)=\mu(\mu(\mu(a,b),b),b),\ \psi_3(a,b,1)=b.$$

(2) For more details you can see for all $m\in\mathbb{N}$, in step 3:

$$d\big(a,\psi_m(a,b,0)\big)=0, d\big(a,\psi_m(a,b,1)\big)=d(a,b),$$
$$d\big(\psi_m(a,b,0),b\big)=d(a,b), d\big(\psi_m(a,b,1),b\big)=0.$$
$$d\left(a,\psi_m\left(a,b,\frac{1}{2}\right)\right)=\frac{1}{2s}d(a,b)\le\frac{1}{2}d(a,b),$$
$$d\left(\psi_m\left(a,b,\frac{1}{2}\right),b\right)=\frac{1}{2s}d(a,b)\le\frac{1}{2}d(a,b).$$
$$d\left(a,\psi_2\left(a,b,\frac{1}{4}\right)\right)=\frac{1}{4s^2}d(a,b)\le\frac{1}{4}d(a,b),$$
$$d\left(a,\psi_2\left(a,b,\frac{3}{4}\right)\right)\le\frac{2s^2+s}{4s^2}d(a,b)\le\frac{3}{4}d(a,b),$$
$$d\left(\psi_2\left(a,b,\frac{1}{4}\right),b\right)\le\frac{2s^2+s}{4s^2}d(a,b)\le\frac{3}{4}d(a,b),$$
$$d\left(\psi_2\left(a,b,\frac{3}{4}\right),b\right)\le\frac{1}{4s^2}d(a,b)\le\frac{1}{4}d(a,b).$$
$$d\left(a,\psi_3\left(a,b,\frac{1}{8}\right)\right)=\frac{1}{8s^3}d(a,b)\le\frac{1}{8}d(a,b),$$
$$d\left(a,\psi_3\left(a,b,\frac{3}{8}\right)\right)\le\frac{2s^2+s}{8s^3}d(a,b)\le\frac{3}{8}d(a,b),$$
$$d\left(\psi_3\left(a,b,\frac{5}{8}\right),b\right)\le\frac{4s^3+s}{8s^3}d(a,b)\le\frac{5}{8}d(a,b),$$
$$d\left(\psi_3\left(a,b,\frac{7}{8}\right),b\right)\le\frac{4s^3+2s^2+s}{8s^3}d(a,b)\le\frac{7}{8}d(a,b).$$

A metric space X is locally equiconnected [5] if there exists a neighborhood U of the diagonal $\Delta\subset X\times X$ and continuous map $\varphi:U\times[0,1]\to X$ such that $\varphi(a,b,0)=a$, $\varphi(a,b,1)=b$ and $\varphi(a,a,t)=a$ for all $(a,b)\in U$ and $t\in[0,1]$.

Given a continuous s-point space (X,d,μ) the map ψ constructed in Theorem 3.2 is the locally equiconnected mapping associated to μ. At times we will also write $\psi_{(a,b)}(t)$ for $\psi(a,b,t)$. Since $d\left(a,\psi\left(a,b,\frac{1}{2}\right)\right)=\frac{1}{2s}d(a,b)$ and $d\left(b,\psi\left(a,b,\frac{1}{2}\right)\right)=\frac{1}{2s}d(a,b)$ we have the following characterization of complete spaces which carry a continuous s-point map.

Proposition 3.4. *If (X,d) is a complete metric space then there exists a continuous s-point map $\mu:X\times X\to X$ if and only if there exists a continuous map $\psi:X\times X\times[0,1]\to X$ such that*

(i) $d(a,\psi(a,b,t))\le td(a,b)$ *and* $d(b,\psi(a,b,t))\le(1-t)d(a,b)$;
(ii) $d(\psi(a,b,t),\psi(a,b,t'))\le|t-t'|d(a,b)$,

for each $(a,b) \in X \times X$ *and for all* $(t,t') \in [0,1] \times [0,1]$.

The map $\psi_{(a,b)}$ defines a rectifiable path with length $d(a,b)$, for all pair (a,b).

Proposition 3.5. *In a symmetric and continuous s-point space* (X,d,μ) *the map* $t \to d\big(\psi(x,u,t), \psi(x,v,t)\big)$ *is convex on* $[0,1]$*, if for each* $(x,u,v) \in X^3$*, we have*

$$d(\mu(x,u), \mu(x,v)) \leq \frac{1}{2s} d(u,v). \tag{3.2}$$

Proof. Let $\delta_{(x,u,v)}(t) = d\big(\psi(x,u,t), \psi(x,v,t)\big)$. Note that

$$d\big(\mu(a,b), \mu(a',b')\big) \leq sd\big(\mu(a,b), \mu(a,b')\big) + sd\big(\mu(a,b'), \mu(a',b')\big),$$

therefore by (3.2) we have

$$d\big(\mu(a,b), \mu(a',b')\big) \leq \frac{1}{2}\big(d(a,a') + d(b,b')\big),$$

for all $a, b, a', b' \in X$. For $t, t' \in E_0 \times E_0$, put $t'' = \dfrac{1}{2}(t + t')$ and so

$$d\big(\psi_{(a,b)}(t''), \psi_{(a',b')}(t'')\big) \leq \frac{1}{2}\Big(d\big(\psi_{(a,b)}(t), \psi_{(a',b')}(t)\big) + d\big(\psi_{(a,b)}(t'), \psi_{(a',b')}(t')\big)\Big).$$

By induction, the above inequality is hold on $E \times E$ and therefore on $[0,1] \times [0,1]$. One can take $a = a' = x$, $b = u$ and $b' = v$ to see that $\delta_{x,u,v}(t)$ convex on $[0,1]$, indeed

$$d\big(\psi_{(x,u)}(t''), \psi_{(x,v)}(t'')\big) \leq \frac{1}{2}\Big(d\big(\psi_{(x,u)}(t), \psi_{(x,v)}(t)\big) + d\big(\psi_{(x,u)}(t'), \psi_{(x,v)}(t')\big)\Big),$$

that is

$$\delta_{(x,u,v)}\left(\frac{t+t'}{2}\right) \leq \frac{\delta_{(x,u,v)}(t) + \delta_{(x,u,v)}(t')}{2}.$$

Note that if the map $\delta_{(x,a,b)}$ in Lemma 3.5 is convex on $[0,1]$, then for $t = 0$, $t' = 1$ we have $\delta_{(x,u,v)}(0) = 0$, $\delta_{(x,u,v)}(1) = d(u,v)$ and $\delta_{(x,u,v)}\left(\dfrac{1}{2}\right) = d(\mu(x,u), \mu(x,v))$, so

$$d(\mu(x,u), \mu(x,v)) \leq \frac{1}{2} d(u,v).$$

4. Fixed point for nonexpansive maps

We need the following lemma for main result of this section.

Lemma 4.1. *If* (X,d) *be a complete b-metric space with constant s and satisfies in* (2.1) *and* (3.2)*, then the following properties hold*

(1) *for each* $x \in X$, $(a,b) \in X \times X$ *and* $r, R \geq 0$ *if* $R \geq \max\{d(x,a), d(x,b)\}$ *and* $r \leq d\big(x, \mu(a,b)\big)$ *then*

$$\frac{1}{2s} d(a,b) \leq \sqrt{R^2 - (2s^2 - 1)r^2};$$

(2) *the map* $u \to \psi(x,u,t)$ *is contractive for all* $(x,t) \in X \times [0,1]$.

Proof. From (2.1) we can easily see that

$$d^2(a,b) = 2s^2\big[d^2(a,x)+d^2(x,b)\big] - 4s^2(2s^2-1)d^2(x,\mu(a,b)) \leq 4s^2(R^2-(2s^2-1)r^2),$$

hence $d(a,b) \leq 2s\sqrt{R^2-(2s^2-1)r^2}$. This proves (1).

By Proposition 3.5 the map $t \to d\big(\psi(x,u,t),\psi(x,v,t)\big)$ is convex, so for all $t \in (0,1)$

$$d\big(\psi(x,u,t),\psi(x,v,t)\big) \leq (1-t)d\big(\psi(x,u,0),\psi(x,v,0)\big) + td\big(\psi(x,u,1),\psi(x,v,1)\big).$$

From $\psi(a,b,0)=a$, $\psi(a,b,1)=b$ we obtain

$$d\big(\psi(x,u,t),\psi(x,v,t)\big) \leq td(u,v).$$

Using Lemma 4.1 we can prove the following statement:

Lemma 4.2. *Let C be a convex and bounded subset of complete continuous s-point space (X,d,μ) for which (1) and (2) of Lemma 4.1 hold and let $F: C \to C$ be nonexpansive. Then for each $u,v \in C$ and for all $R \geq 0$ such that $d(u,F(u)) \leq R$ and $d(v,F(v)) \leq R$ we have*

$$d\big(\mu(u,v),F(\mu(u,v))\big)^2 \leq 8s^3R(\text{diam } C).$$

Proof. Since F is nonexpansive and μ is continuous s-point we have

$$\begin{aligned} d\big(u,F(\mu(u,v))\big) &\leq sd\big(u,F(u)\big) + sd\big(F(u),F(\mu(u,v))\big) \\ &\leq sR + sd\big(u,\mu(u,v)\big) \\ &= sR + \frac{1}{2}d(u,v), \end{aligned}$$

and $d\big(u,\mu(u,v)\big) = \dfrac{1}{2s}d(u,v)$, for all $u,v \in C$ and $R \geq 0$. Then

$$sR + \frac{1}{2}d(u,v) \geq \max\{d\big(u,\mu(u,v)\big), d\big(u,F(\mu(u,v))\big)\},$$

and put $r = \dfrac{1}{2\sqrt{2s^2-1}}d(u,v)$. On the other hand, by (1) of Lemma 4.1 we have

$$\begin{aligned} d\big(\mu(u,v),F(\mu(u,v))\big)^2 &\leq 4s^2\left(\left(sR+\frac{1}{2}d(u,v)\right)^2 - (2s^2-1)r^2\right) \\ &\leq 4s^2\left(\left(sR+\frac{1}{2}d(u,v)\right)^2 - \frac{1}{4}d(u,v)^2\right) \\ &= 4s^2(sR)\big(R+d(u,v)\big) \\ &\leq 4s^3R\big(2\text{diam } C\big) = 8s^3R\big(\text{diam } C\big). \end{aligned}$$

Now, we prove the fixed point property for nonexpansive mappings under following conditions.

Theorem 4.3. *Let C be a convex and bounded subset of complete continuous s-point space (X,d,μ) and let $F: C \to C$ be a nonexpansive map for which (1) and (2) of Lemma 4.1 hold. Then C have the fixed point property for continuous nonexpansive maps.*

Proof. We claim that F has a fixed point. Fix an arbitrary point $x \in C$ and for all $n > 0$ let

$$F_n(u) = \psi\left(x, F(u), 1 - \frac{1}{n}\right).$$

Clearly, F_n is contractive, because by the proof of second part of Lemma 4.1 we can write

$$\begin{aligned} d\big(F_n(u), F_n(v)\big) &= d\left(\psi\left(x, F(u), 1 - \frac{1}{n}\right), \psi\left(x, F(v), 1 - \frac{1}{n}\right)\right) \\ &\leq \frac{1}{n} d\big(F(u), F(v)\big) \\ &\leq \frac{1}{n} d(u, v), \end{aligned}$$

let u_n be its fixed point of F_n. From Theorem 3.2 we have

$$\begin{aligned} d\big(u_n, F(u_n)\big) = d\big(F_n(u_n), F(u_n)\big) &= d\left(\psi\left(x, F(u_n), 1 - \frac{1}{n}\right), F(u_n)\right) \\ &\leq \left(1 - \left(1 - \frac{1}{n}\right)\right) d\big(x, F(u_n)\big) \\ &\leq \frac{\text{diam } C}{n}. \end{aligned}$$

Let

$$A_n = \left\{u \in C \,:\, d\big(u, F(u)\big) \leq \frac{\text{diam } C}{n}\right\}$$

for $n \geq 2$. The set A_n is closed and nonempty for all $n \geq 2$. Let $d_n = \inf_{u \in A_n} d(x, u)$ and so for all $n \geq 2$, $d_n \leq d_{n+1} \leq \text{diam } C$. Let $d = \lim_{n \to \infty} d_n$ and

$$G_n = A_{8s^3n^2} \cap B\big(x, d + \frac{1}{n}\big).$$

Using Lemma 4.2, for all $u, v \in A_{8s^3n^2}$, $\mu(u, v) \in A_n$.
Therefore if $u, v \in G_n$, we must have

$$d(x, u) \leq d + \frac{1}{n},\ d(x, v) \leq d + \frac{1}{n}$$

as well as

$$d(x, \mu(u, v)) \geq \frac{1}{\sqrt{2s^2 - 1}} d_n.$$

Therefore by the first part of Lemma 4.1, we have the following estimation for the diameter of G_n:

$$\text{diam } G_n \leq 2s\sqrt{\frac{2d}{n} + \frac{1}{n^2} + d^2 - d_n^2}.$$

On the other hand $G_{n+1} \subset G_n$ and so $\bigcap_{n \geq 2} G_n$ reduces to a single point which is clearly a fixed point of F.

Example 4.4. In Example 2.3, put $F(x) = 1 - \frac{x}{2}$. Since

$$(a-b)^2 + 4s^2(2s^2-1)\left(x - \frac{a+b}{2}\right)^2 = 2s^2(a-x)^2 + 2s^2(x-b)^2$$

and

$$\left(\frac{x+u}{2} - \frac{x+v}{2}\right)^2 \leq \frac{1}{2s}(u-v)^2,$$

for all $a, b, x \in X$ and $s = 2$, then the conditions (2.1) and (3.2) are established. Now by Theorem 4.3 F has a fixed point. Clearly $x = \frac{2}{3}$ is a fixed point of F.

5. Discussion

Note that the comparison between a cone b-metric and a b-metric is likely the relation between a cone metric, and a metric, see [1]. Some authors have proved that fixed point theorems on cone metric spaces are, essentially, fixed point theorems on metric space. Recently, Du used a method to introduce a b-metric on a cone b-metric space and stated some relations between fixed point theorems on cone b-metric spaces and on b-metric spaces, (for more details see references in [9]).

In the article [9], authors used a method to introduce another b-metric on the cone b-metric space and then proved some equivalences between them. As applications, they show that fixed point theorems on cone b-metric spaces can be obtained from fixed point theorems on b-metric spaces, (see [7, 9, 10]).

We note that, much more papers published in the relations between cone metric spaces and metric spaces between years 2007-2011. Cone metric divided to three parts: normal cone, non-normal cone and cones with empty interior. Important cases were without normal form. And equivalency of contractive conditions was another important question as well. Some of papers published in this way, (such as [1, 6]). And final big problem was for the total case in the contractive condition $d(Tx, Ty) \leq \varphi(d(x, y))$. For the certain answers refer to [7, 9, 10].

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Received: December 20, 2017; Accepted: September 30, 2018.