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PACKING, w_n^*- SEPARATION AND NORMAL STRUCTURE IN BANACH SPACES

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Abstract. Let X and X^* be a Banach space and its dual, and let $B(X)$ and $S(X)$ be the unit ball and unit sphere of X respectively. In this paper, we introduce a new parameter of w_n^*- Separation, $w_n^*(X^*)$, in X^* and study the relation between this parameter and normal structure in X, and the relation between packing constant $P(\alpha, X)$ introduced by Kottman and normal structure that implies the existence of fixed point for non-expansive mappings. Some new results about fixed points of non-expansive mapping are obtained.
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1. Introduction

Let X be a normed linear space. Let $B(X) = \{x \in X : \|x\| \leq 1\}$ and
$$S(X) = \{x \in X : \|x\| = 1\}$$
be the unit ball and the unit sphere of X, respectively. Let X^* be the dual space of X. Let $B(x_0, r) = \{x \in X : \|x - x_0\| \leq r\}$ be the ball with center at x_0, and radius r in X.

Brodskiĭ and Mil'man [3] introduced the following geometric concepts in 1948:

Definition 1.1. A bounded and convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that $\sup\{\|x_0 - y\| : y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H.

A Banach space X is said to have normal structure if every bounded and convex subset of X has normal structure.

A Banach space X is said to have weak normal structure if each weakly compact convex set K in X has normal structure.

A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any bounded closed convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\| : y \in K\} \leq c \cdot d(K)$.

For a reflexive Banach space, the normal structure and weak normal structure coincide.

Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is called to be non-expansive whenever $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A Banach space has the fixed point property if for every nonempty bounded closed and convex subset C of X and for each non-expansive mapping $T : C \to C$, there is a point $x \in C$ such that $x = Tx$. ([8]).

Kirk [8] proved that if a Banach space X has weak normal structure then it has the weak fixed point property, that is, every non-expansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

In 1970, Kottman [11] introduced the following concept:

Definition 1.2. Let X be a Banach space. For each cardinal number α let

$$P(\alpha, X) = \sup\{r : \text{there exist } \alpha \text{ disjoint balls of radius } r \text{ in } B(X)\}.$$

(In this setting we take $\sup \emptyset = 0$).

Definition 1.3. A Banach space X is called P-convex, if $P(n, X) < \frac{1}{2}$ for some positive integer n.

Kottman proved [11]:

Theorem 1.4. *Let X be an infinite dimensional normed space and α be a cardinal number greater than one but less than or equal to the density character of X. Then*

$$\frac{1}{3} \leq P(\alpha, X) \leq \frac{1}{2}.$$

Definition 1.5. ([4], [6]) Let X and Y be Banach spaces. We say that Y is finitely representable in X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to T(N)$ such that for any $y \in N$,

$$(1 - \varepsilon)\|y\| \leq \|Ty\| \leq (1 + \varepsilon)\|y\|.$$

The Banach space X is called super-reflexive if any space Y which is finitely representable in X is reflexive.

Theorem 1.6. *X is super-reflexive if and only if X^* is super-reflexive.*

Theorem 1.7. ([2], [11]) *If a Banach space X is P-convex, then X is super-reflexive.*

The following n-dimensional modulus was introduced by Jiménez-Melado [7] and Mazcuñán-Navarro [12]:

Definition 1.8. For a Banach space X, let

$$s_n(X) := \sup\{\varepsilon \in [0, 2] : \exists x_1, x_2, \ldots x_{n+1} \in B(X), \text{ such that } \min_{i \neq j} \|x_i - x_j\| \geq \varepsilon\}.$$

In this paper, we introduce a new parameter of w_n^*- Separation, $w_n^*(X^*)$, in X^* and study the relation between this parameter and normal structure in X, and the relation between packing constant $P(\alpha, X)$ introduced by Kottman above and normal structure that implies the existence of fixed point for non expansive mappings. Some new results about fixed points of non-expansive mapping are obtained. In section 3 we prove that if X is an infinite dimensional normed space with $P(\aleph_0, X^*) < \frac{1}{2}$ then X has uniformly normal structure, where $\aleph_0$ is the cardinal number of all natural numbers.

2. Preliminary and main results

We define the following two n-dimensional modules:

Definition 2.1. Let X and X^* be a Banach space and its dual, and $B(X)$ and $B(X^*)$ be the unit ball of X and X^* respectively. We define

$$w_n(X) := \sup\{\varepsilon \in [0,2] : \exists x_1, x_2, \dots x_{n+1} \in B(X), \text{ such that}$$
$$\min_{i \neq j}(sup_{f \in B(X^*)}\langle x_i - x_j, f\rangle) \geq \varepsilon\};$$

and

$$w_n^*(X^*) := \sup\{\varepsilon \in [0,2] : \exists f_1, f_2, \dots f_{n+1} \in B(X^*), \text{ such that}$$
$$\min_{i \neq j}(sup_{x \in B(X)}\langle x, f_i - f_j\rangle) \geq \varepsilon\}.$$

It is easy to show that:

Corollary 2.2. *For a Banach space X, $w_n(X) \leq s_n(X) \leq 2$, and for X^*,*

$$w_n^*(X^*) \leq s_n(X^*) \leq 2.$$

Example 2.3. Let $X = c_0$, and $X^* = l_1$, then $w_n^*(l_1) = s_n(l_1) = 2$.
Proof. Let $x = (0,0,0,...,0,1,0,...,0,-1,0...,0,0,0,...) \in S(c_0)$, where i-th position of x is 1, j-th position of x is -1, others are 0 and $i < j$.

Let $f_i = (0,0,0,...,0,1,0,...,0,0,0,...) \in S(l_1)$, where i-th position of f_1 is 1 and others are 0.

And $f_j = (0,0,0,...,0,0,0,...,0,1,0,...) \in S(l_1)$, where j-th position of f_j is 1 and others are 0. We have $< x, f_i - f_j >= 2$.

From the definition of $w_n^*(X^*)$, $w_n^*(l_1) = s_n(l_1) = 2$.

The following three results refer to a Banach space with weak* sequentially compact unit ball of the dual. Notice that this property is satisfied by reflexive or separable Banach spaces, and by those that admit an equivalent smooth norm (see [5], Ch. XIII).

Lemma 2.4. [13] *If X is a Banach space with $B(X^*)$ weak* sequentially compact and fails to have weak normal structure, then for any $\varepsilon > 0$ there are a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$, *whenever* $i \neq j$;
(b) $\langle x_i, f_i\rangle = 1$, *whenever* $1 \leq i \leq \infty$;
(c) $|\langle x_j, f_i\rangle| < \varepsilon$, *whenever* $i \neq j$; *and*
(d) $\|f_i - f_j\| > 2 - \varepsilon$, *whenever* $i \neq j$.

Theorem 2.5. *If X is a Banach space with $B(X^*)$ weak* sequentially compact and $w_n^*(X^*) < 2$, then X have weak normal structure.*
Proof. From the proof of Lemma 2.27 of [13], if X fails to have weak normal structure, then for any $\eta > 0$ there are a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that $|\|x_i - x_j\| - 1| < \eta$ and $|\langle x_j, f_i\rangle| < \eta$, if $i \neq j$.

So, $|\langle x_i - x_j, f_i - f_j\rangle| = |2 - \langle x_j, f_i\rangle - \langle x_i, f_j\rangle| \geq 2 - 2\eta$ for any $i \neq j$.

Since η can be arbitrarily small, we have $w_n^*(X^*) = 2$.

Theorem 2.6. *If X is a Banach space with $B(X^*)$ weak* sequentially compact and $P(\aleph_0, X^*) < \frac{1}{2}$, then X has weak normal structure.*
Proof. Suppose X does not have weak normal structure. From Lemma 2.4, for any $\varepsilon > 0$ there is a sequence $\{f_n\} \subseteq S(X^*)$ such that $\|f_i - f_j\| > 2 - \varepsilon$, whenever $i \neq j$.

Considering the sequence $\{\frac{f_n}{2}\}$, we have $B\left(\frac{f_m}{2}, \frac{1}{2} - \frac{\varepsilon}{4}\right) \cap B\left(\frac{f_n}{2}, \frac{1}{2} - \frac{\varepsilon}{4}\right) = \emptyset$ if $m \neq n$, and $B\left(\frac{f_n}{2}, \frac{1}{2} - \frac{\varepsilon}{4}\right) \subseteq B(X^*)$ for all n.
We have $P(\aleph_0, X^*) > \frac{1}{2} - \frac{\varepsilon}{4}$. Since ε can be arbitrarily small, we have $P(\aleph_0, X^*) = \frac{1}{2}$.

Definition 2.7. A Banach space X is called G-convex, if $P(\aleph_0, X) < \frac{1}{2}$.

Proposition 2.8. If the Banach space X is P-convex, then X is G-convex.

From Theorem 1.7 and Theorem 2.6, we have:

Theorem 2.9. *If the Banach space X^* is P-convex, then X has normal structure.*
Proof. From Theorem 1.7, X^* is P-convex implies that X^* and therefore X is super-reflexive, so weak normal structure and normal structure coincide. Then from Theorem 2.6, X^* is P-convex implies X has weak normal structure, therefore normal structure.

3. Uniform normal structure

Let $\mathcal{F}$ be a filter on an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denote by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathcal{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$;
(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 3.1. ([1], [10], [14]) Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultra-product of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote an element of the ultra-product. It follows from the assertion (ii) above, and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$ for all $i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the corresponding ultra-product, called an ultra-power of X.

Lemma 3.2. ([1], [10] [14]) *Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and X is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is super-reflexive; and in this case, the mapping J defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}} \langle x_i, f_i\rangle, \qquad \text{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}}$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Theorem 3.3. *Let X be a Banach space. Then for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we have $P(n, X_{\mathcal{U}}) = P(n, X)$ for all $n \in \mathbb{N}$.*

Proof. Since X can be embedded into $X_{\mathcal{U}}$ isometrically, we may consider X as a subspace of $X_{\mathcal{U}}$. From the definition of $P(n, X)$, we have
$P(n, X_{\mathcal{U}}) \leq P(n, X)$ for all $n \in \mathbb{N}$.

We prove the reverse inequality.

Suppose $P(n, X_{\mathcal{U}}) = a$ and $\varepsilon > 0$, then for any set of n balls in $B(X_{\mathcal{U}})$ centering inside of $B(X_{\mathcal{U}})$ with radius $a + \varepsilon$, there must be at least two balls $B((f_i)_{\mathcal{U}}, a + \varepsilon)$ and $B((g_i)_{\mathcal{U}}, a + \varepsilon)$, such that $B((f_i)_{\mathcal{U}}, a + \varepsilon) \cap B((g_i)_{\mathcal{U}}, a + \varepsilon) \neq \emptyset$.

Let $(h_i)_{\mathcal{U}} \in B((f_i)_{\mathcal{U}}, a + \varepsilon) \cap B((g_i)_{\mathcal{U}}, a + \varepsilon)$.

We have $\|((f_i) - (h_i))_{\mathcal{U}}\| < a + \varepsilon$ and $\|((g_i) - (h_i))_{\mathcal{U}}\| < a + \varepsilon$.

Without of generality, from definition of ultra-product, we may assume the following sets:

$$\begin{aligned}
A &= \{i : \|f_i\| < 1 + \varepsilon\},\\
B &= \{i : \|g_i\| < 1 + \varepsilon\},\\
C &= \{i : \|h_i\| < 1 + \varepsilon\},\\
P &= \{i : \|f_i - h_i\| < a + \varepsilon\}, \text{ and}\\
Q &= \{i : \|g_i - h_i\| < a + \varepsilon\}
\end{aligned}$$

are all in $\mathcal{U}$.

So the intersection $A \cap B \cap C \cap P \cap Q$ is in $\mathcal{U}$ too, and is hence not empty.

Let $i \in A \cap B \cap C \cap P \cap Q$. For this fixed i, we have
$\|f_i\| < 1 + \varepsilon$, $\|g_i\| < 1 + \varepsilon$, $\|h_i\| < 1 + \varepsilon$, $\|f_i - h_i\| < a + \varepsilon$, and $\|g_i - h_i\| < a + \varepsilon$.

These imply that $f_i, g_i \in (1 + \varepsilon)B(X)$, and $B(f_i, a + \varepsilon) \cap B(g_i, a + \varepsilon) \neq \emptyset$.

Since ε can be arbitrarily small, we have $P(n, X_{\mathcal{U}}) \geq P(n, X)$.

Lemma 3.4. [9] *If X is a super-reflexive Banach space, then X has uniform normal structure if and only if $X_{\mathcal{U}}$ has normal structure.*

Theorem 3.5. *For a Banach space X, if $P(n, X^*) < \frac{1}{2}$ for some positive integer n, then X has uniform normal structure.*

Proof. From Definition 1.3 and Theorem 1.7, $P(n, X^*) < \frac{1}{2}$ implies X is super-reflexive. From Theorem 3.3, $X_{\mathcal{U}}$ has normal structure. Then from Theorem 2.9 and Theorem 3.4, X has uniform normal structure.

We proved that if X^* is P-convex, then X has uniform normal structure.

Since n is an arbitrary integer, we proved that:

Theorem 3.6. *If X is an infinite dimensional normed space with $P(\aleph_0, X^*) < \frac{1}{2}$ then X has uniformly normal structure, where $\aleph_0$ is the cardinal number of all natural numbers.*

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