Fixed Point Theory, 21(2020), No. 2, 465-474
DOI: 10.24193/fpt-ro.2020.2.33
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

BEST PROXIMITY POINTS THEOREM IN b-METRIC SPACES ENDOWED WITH A GRAPH

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Abstract. In this paper the existence and uniqueness of best proximity point for Reich-type contraction on b-metric spaces endowed with a graph is established. These results are significant, since we replace the condition of continuity of mapping with the condition of G-continuity of mapping and we consider b-metric spaces endowed with a graph instead of metric spaces, under which can be unified some theorems of the existing literature.

Key Words and Phrases: b-metric space, best proximity point, P-property, G-proximal.
2010 Mathematics Subject Classification: 41A52, 41A65, 05C40, 47H10.

1. Introduction and preliminaries

Let (X, d) be a metric space, A and B two non-empty subsets of X and $T : A \to B$ be a non-self mapping. The best proximity point of T are the set all points x in A such that $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y); x \in A,\ y \in B\}$. The purpose of best proximity point theory is to furnish sufficient conditions that assure the existence of such points. Hence, numerous works on best proximity point theory were studied by giving sufficient conditions assuring the existence and uniqueness of these points such that several authors have studied different contractions for having the best approximation point in metric spaces and partially ordered metric spaces in [4, 5, 13, 14, 15, 16, 17, 19] and references therein. In 2008, Jachymski [10] applied graphs in metric fixed point theory and generalized the Banach contraction principle in both metric and partially ordered metric spaces. In the sequel, many authors proved some fixed point theorems and best proximity point results in various metric spaces endowed with a graph (for example, see [1, 2, 3, 9, 18]). On the other hand, the concept of b-metric spaces was studied by Bakhtin [6] and Czerwik [8].

The purpose of this paper is to prove the existence and uniqueness of best proximity points for contractive mappings (specially, Reich-type contraction [12]) in b-metric spaces endowed with a graph. Our results are the extensions of some best proximity point theorems given in terms of metric spaces, partially ordered metric spaces and b-metric spaces to b-metric spaces equipped with a graph G.

We start by reviewing a few basic notions about b-metric spaces and graph theory.

Definition 1.1. Let X be a nonempty set and $s \geq 1$ be a real number. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies in the following conditions:

(d_1) $d(x, y) = 0$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

Then d is called a b-metric and (X, d) is called a b-metric space (or type metric space).

Obviously, for $s = 1$, a b-metric space is a metric space. Also, for notions such as convergent and Cauchy sequences, completeness, continuity and etc in b-metric spaces, we refer to [11].

In an arbitrary graph G, by a link, it is meant an edge of G with distinct ends and by a loop, an edge of G with identical ends. Two or more links of G with the same pairs of ends are called parallel edges of G. Let (X, d) be a metric space and G be a directed graph with vertex set $V(G) = X$ such that the edge set $E(G)$ contains all loops and G has no parallel edges. Under these hypotheses, the graph G can be easily denoted by a pair $(V(G), E(G))$ and it is said that the metric space (X, d) is endowed with the graph G. For more details on graphs, see [7].

Now, consider a pair (A, B) of nonempty subsets of (X, d). Then we will apply the following notations in the sequel.

$$d(A, B) = \inf \big\{ d(x, y) : x \in A,\ y \in B \big\},$$
$$A_0 = \big\{ x \in A :\ d(x, y) = d(A, B) \text{ for some } y \in B \big\},$$
$$B_0 = \big\{ y \in B :\ d(x, y) = d(A, B) \text{ for some } x \in A \big\}.$$

2. Main results

In this section, we assume that (X, d) is a b-metric space with parameter $s \geq 1$ endowed with a graph G and (A, B) is a pair of nonempty closed subsets of X with $A_0 \neq \emptyset$ unless otherwise stated.

Definition 2.1. Let (A, B) be a pair of nonempty subsets of a b-metric space (X, d) and $T : A \to B$ be a non-self mapping. An element $x \in A$ is said to be a best proximity point for T if $d(x, Tx) = d(A, B)$.

By the above notations, if x is a best proximity point for T, then we have $x \in A_0$ and $Tx \in B_0$.

Definition 2.2. A pair (A, B) of nonempty subsets of a b-metric space (X, d) is said to have the P-property if

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2)$$

for all $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

In Definition 2.1 and Definition 2.2, set $s = 1$. Then we have the same definitions in metric spaces introduced in [14, 16]. Also, with change in Jachymski's definition [10, Definition 2.3], we formulate the notion of G-continuity in b-metric spaces endowed with a graph for non-self mappings as follows.

Definition 2.3. [10] Let (X, d) be a b-metric space endowed with a graph G. A mapping $T : A \to B$ is said to be G-continuous on A if $x_n \to x$ in A implies $Tx_n \to Tx$ in B for all sequences $\{x_n\}$ in A with $(x_n, x_{n+1}) \in E(G)$ for $n = 1, 2, \cdots$.

Also, motivated form the idea of Sadiq Basha [14], we introduce the concept of a G-proximal mapping in a b-metric space endowed with a graph as follows.

Definition 2.4. A non-self mapping $T : A \to B$ is G-proximal if T satisfies

$$\left.\begin{array}{r} (y_1, y_2) \in E(G) \\ d(x_1, Ty_1) = d(A, B) \\ d(x_2, Ty_2) = d(A, B) \end{array}\right\} \Longrightarrow (x_1, x_2) \in E(G)$$

for all $x_1, x_2, y_1, y_2 \in A$.

Theorem 2.5. *Let (X, d) be a complete b-metric with parameter $s \geq 1$ endowed with a graph G and d be a continuous mapping in two variables. Suppose that there exists G-continuous mapping $T : A \to B$ such that the following conditions hold:*

- (i) *T is a G-proximal with $T(A_0) \subseteq B_0$ and the pair (A, B) satisfies the P-property;*
- (ii) *there exist elements $x_0, x_1 \in A_0$ such that $(x_0, x_1) \in E(G)$ and*

$$d(x_1, Tx_0) = d(A, B);$$

- (iii) *there exist non-negative constants α, β, γ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) - s(\beta + \gamma) d(A, B) \tag{2.1}$$

for all $x, y \in A$ with $(x, y) \in E(G)$, where $s(\alpha + \gamma) + s^2\beta < 1$.

Then T has a best proximity point in A. Furthermore, if for any two best proximity points $u, v \in A$ we have $(u, v) \in E(G)$, then T has a unique best proximity point in A.

Proof. From $x_1 \in A_0$ and $T(A_0) \subseteq B_0$, there exists an $x_2 \in A$ such that $d(x_2, Tx_1) = d(A, B)$. In particular, $x_2 \in A_0$. Since $d(x_1, Tx_0) = d(A, B)$, $(x_0, x_1) \in E(G)$ and T is G-proximal, then $(x_1, x_2) \in E(G)$. Continuing this process, we obtain a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{and} \quad (x_n, x_{n+1}) \in E(G) \qquad n = 0, 1, \cdots. \tag{2.2}$$

Since the pair (A, B) satisfies the P-property, it follows for all $n \in \mathbb{N}$ that

$$\left.\begin{array}{r} d(x_n, Tx_{n-1}) = d(A, B) \\ d(x_{n+1}, Tx_n) = d(A, B) \end{array}\right\} \Longrightarrow d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n). \tag{2.3}$$

Using (2.1)-(2.3) and triangle inequality. Since $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we get

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_n, Tx_n) - s(\beta+\gamma)d(A, B) \\
&\leq \alpha d(x_{n-1}, x_n) + \beta s[d(x_{n-1}, x_n) + d(x_n, Tx_{n-1})] \\
&\quad + \gamma s[d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n)] - s(\beta+\gamma)d(A, B) \\
&= \alpha d(x_{n-1}, x_n) + s\beta d(x_{n-1}, x_n) + s\beta[d(x_n, Tx_{n-1}) - d(A, B)] \\
&\quad + s\gamma d(x_n, x_{n+1}) + s\gamma[d(x_{n+1}, Tx_n) - d(A, B)] \\
&= \alpha d(x_{n-1}, x_n) + s\beta d(x_{n-1}, x_n) + s\gamma d(x_n, x_{n+1}),
\end{aligned}$$

which implies that

$$d(x_n, x_{n+1}) \leq \frac{\alpha + s\beta}{1 - s\gamma} d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. By repeating this process, we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \tag{2.4}$$

for all $n \in \mathbb{N}$, where

$$0 \leq \lambda = \frac{\alpha + s\beta}{1 - s\gamma} < \frac{1}{s}.$$

Now, let $m, n \in \mathbb{N}$ with $m \geq n \geq 1$. Using (2.4) and triangle inequality, we have

$$\begin{aligned}
d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
&\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\
&\ \ \vdots \\
&\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} d(x_{m-1}, x_m) \\
&\leq [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n}\lambda^{m-1}] d(x_0, x_1) \\
&\leq \frac{s\lambda^n}{1 - s\lambda} d(x_0, x_1) \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence, $\{x_n\}$ is Cauchy sequence. Since (X, d) is a complete space, there exists $x^* \in X$ (depending on x_0 and x_1) such that $x_n \to x^*$. Moreover, $x^* \in A$ (since A is closed).

We next show that x^* is a best proximity point for T. By the G-continuity of T on A, since $x_n \to x^*$ and $(x_n, x_{n+1}) \in E(G)$ for $n = 0, 1, \cdots$, we get $Tx_n \to Tx^*$. Also, the joint continuity of the metric function d implies that $d(x_n, Tx_n) \to d(x^*, Tx^*)$. On the other hand, (2.2) shows that the sequence $\{d(x_n, Tx_n)\}$ is a constant sequence converging to $d(A, B)$. Since the limit of a sequence is unique, we get $d(x^*, Tx^*) = d(A, B)$; that is, x^* is a best proximity point for T. Moreover $x^* \in A_0$ and $Tx^* \in B_0$.

To show uniqueness, suppose that x^{**} is another best proximity point of T such that $(x^*, x^{**}) \in E(G)$. Since the pair (A, B) satisfies the P-property, we have

$$\left.\begin{aligned} d(x^*, Tx^*) = d(A, B) \\ d(x^{**}, Tx^{**}) = d(A, B) \end{aligned}\right\} \Longrightarrow d(x^*, x^{**}) = d(Tx^*, Tx^{**})$$

for $x^*, x^{**} \in A_0$ and $Tx^*, Tx^{**} \in B_0$. Hence, by (2.1),

$$\begin{aligned}
d(x^*, x^{**}) &= d(Tx^*, Tx^{**}) \\
&\le \alpha d(x^*, x^{**}) + \beta d(x^*, Tx^*) + \gamma d(x^{**}, Tx^{**}) - s(\beta+\gamma)d(A,B) \\
&\le \alpha d(x^*, x^{**}) + \beta[d(x^*, Tx^*) - d(A,B)] \\
&\quad + \gamma[d(x^{**}, Tx^{**}) - d(A,B)] - (s-1)(\beta+\gamma)d(A,B) \\
&\le \alpha d(x^*, x^{**}).
\end{aligned}$$

Since $\alpha \le s\alpha$ and $s(\alpha+\gamma) + s^2\beta < 1$ we have $\alpha < 1$. Thus, we get $d(x^*, x^{**}) = 0$; that is, $x^* = x^{**}$. This completes the proof.

Example 2.6. Let $X = \mathbb{R}^2$. Define $d : X \times X \to [0, +\infty)$ by

$$d\big((x_1, y_1), (x_2, y_2)\big) = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Then (X, d) is a b-metric space with parameter $s = 2$. Let

$$\begin{aligned}
A &= \big\{(x,1) : x \in [0,1]\big\}, \\
B &= \big\{(y,0) : y \in [0,1]\big\}.
\end{aligned}$$

Clearly, $d(A,B) = 1$, $A = A_0$ and $B = B_0$. In particular A_0 is nonempty. Also, let $T : A \to B$ defined by

$$T(x,1) = \begin{cases} (0,0) & 0 \le x < 1, \\ \left(\dfrac{2}{3}, 0\right) & x = 1 \end{cases} \qquad \big(x \in [0,1]\big).$$

Observe that for elements $(1,1)$ and $(\frac{1}{2}, 1)$ and given any $\alpha, \beta, \gamma \in [0,1)$ with $2\alpha + 2\gamma + 4\beta < 1$ for $s = 2$, we have

$$\begin{aligned}
d\left(T(1,1), T\left(\frac{1}{2}, 1\right)\right) &= d\left(\left(\frac{2}{3}, 0\right), (0,0)\right) \\
&= \frac{4}{9} \\
&> \frac{4}{9}(2\alpha + 2\gamma + 4\beta) \\
&> \frac{\alpha}{4} + \frac{10\beta}{9} + \frac{5\gamma}{4} - 2(\beta+\gamma) \\
&= \alpha d\left((1,1), \left(\frac{1}{2}, 1\right)\right) + \beta d\left((1,1), \left(\frac{2}{3}, 0\right)\right) \\
&\quad + \gamma d\left(\left(\frac{1}{2}, 1\right), (0,0)\right) - 2(\beta+\gamma) \\
&= \alpha d\left((1,1), \left(\frac{1}{2}, 1\right)\right) + \beta d\big((1,1), T(1,1)\big) \\
&\quad + \gamma d\left(\left(\frac{1}{2}, 1\right), T\left(\frac{1}{2}, 1\right)\right) - 2(\beta+\gamma)d(A,B).
\end{aligned}$$

So T does not satisfy the usual version (non-graph version) of (2.1).

Now, define a graph G_4 by $V(G_4) = \mathbb{R}^2$ and

$$E(G_4) = \big\{\big((x_1, x_2), (x_1, x_2)\big) : (x_1, x_2) \in \mathbb{R}^2\big\} \cup \big\{\big((0, 1), (1, 1)\big), \big((1, 1), (0, 1)\big)\big\},$$

Suppose that $(\mathbb{R}^2, d)$ is endowed with G_4. Moreover, one can be simply show that the pair (A, B) satisfies the P-property, T is G_4-proximal and $T(A_0) \subseteq B_0$ and T is a G_4-continuous.

Now, take $\alpha = \frac{17}{36}$, $\beta = 0$ and $\gamma = 0$, so $2\alpha + 2\gamma + 4\beta < 1$. Assume that $x \in [0, 1]$. Then we have

$$\begin{aligned} d\big(T(x, 1), T(x, 1)\big) &= 0 \\ &\leq \alpha d\big((x, 1), (x, 1)\big) + \beta d\big((x, 1), T(x, 1)\big) + \gamma d\big((x, 1), T(x, 1)\big) \\ &\quad - 2(\beta + \gamma) d(A, B). \end{aligned}$$

Also,

$$\begin{aligned} d\big(T(0, 1), T(1, 1)\big) &= d\left((0, 0), \left(\frac{2}{3}, 0\right)\right) \\ &= \frac{4}{9} \\ &\leq \alpha d\big((0, 1), (1, 1)\big) + \beta d\big((0, 1), T(0, 1)\big) + \gamma d\big((1, 1), T(1, 1)\big) \\ &\quad - 2(\beta + \gamma) d(A, B). \end{aligned}$$

Thus, T is satisfy the (2.1). Moreover, all hypotheses of Theorem 2.5 is satisfied. Therefore, T has a best proximity point $x^* = (0, 1)$.
Now, let $x^{**} = (x, 1) \in A$ with $x \in (0, 1]$ be another best proximity point of T.
If $x \in (0, 1)$, then

$$d\big((x, 1), T(x, 1)\big) = d\big((x, 1), (0, 0)\big) = x^2 + 1 > d(A, B).$$

Otherwise, if $x = 1$, then

$$d\big((1, 1), T(1, 1)\big) = d\left((1, 1), \left(\frac{2}{3}, 0\right)\right) = \frac{10}{9} > d(A, B),$$

which is a contradiction. Hence $(0, 1)$ is the unique best proximity point of T.

Several consequences of Theorem 2.5 follow for particular choices of the graph. First, consider the b-metric space (X, d) endowed with the complete graph G_0. If we set $G = G_0$ in Theorem 2.5, then it is clear that $T : A \to B$ is a G_0-proximally on A. Thus, we get the following corollary.

Corollary 2.7. *Let (X, d) be a complete b-metric with parameter $s \geq 1$ and d be a continuous mapping in two variables. Suppose that there exists continuous mapping $T : A \to B$ such that the following conditions hold:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*
(ii) *there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$;*
(iii) *there exist non-negative constants α, β, γ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) - s(\beta + \gamma) d(A, B)$$

for all $x, y \in A$, where $s(\alpha + \gamma) + s^2\beta < 1$.

Then T has a unique best proximity point in X.

Now, suppose that $(X, \preceq)$ is a poset. Consider on the poset X the graph G_1 given by $V(G_1) = X$ and $E(G_1) = \{(x, y) \in X \times X : x \preceq y\}$. If we set $G = G_1$ in Theorem 2.5, then we obtain following best proximity point result in complete b-metric spaces endowed with a partial order.

Corollary 2.8. *Let $(X, \preceq)$ be a poset, (X, d) be a complete b-metric with parameter $s \geq 1$ and endowed with a graph G_1 and d be a continuous mapping in two variables. Suppose that there exists G_1-continuous mapping $T : A \to B$ such that the following conditions hold:*

(i) *T is a G_1-proximal with $T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*
(ii) *there exist elements $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ and $d(x_1, Tx_0) = d(A, B)$;*
(iii) *there exist non-negative constants α, β, γ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(Ty, y) - s(\beta + \gamma)d(A, B)$$

for all $x, y \in A$ with $x \preceq y$, where $s(\alpha + \gamma) + s^2\beta < 1$.

Then T has a best proximity point in A. Furthermore, if $u \preceq v$ for any two best proximity point $u, v \in A$, then T has a unique best proximity point in A.

For our next consequence, suppose again that $(X, \preceq)$ is a poset and consider the graph G_2 defined by $V(G_2) = X$ and $E(G_2) = \{(x, y) \in X \times X : x \preceq y \ \vee \ y \preceq x\}$. Then an ordered pair $(x, y) \in X \times X$ is an edge of G_2 if and only if x and y are comparable elements of $(X, \preceq)$. If we set $G = G_2$ in Theorem 2.5, then we obtain another best proximity point theorem in complete b-metric spaces endowed with a partial order.

Corollary 2.9. *Let $(X, \preceq)$ be a poset, (X, d) be a complete b-metric space with parameter $s \geq 1$ and endowed with a graph G_2, and d be a continuous mapping in two variables. Suppose that there exists G_2-continuous mapping $T : A \to B$ such that the following conditions hold:*

(i) *T is a G_2-proximal with $T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*
(ii) *there exist comparable elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$;*
(iii) *there exist non-negative constants α, β, γ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(Ty, y) - s(\beta + \gamma)d(A, B)$$

for all comparable $x, y \in A$, where $s(\alpha + \gamma) + s^2\beta < 1$.

Then T has a best proximity point in A. Furthermore, if each two best proximity point are comparable, then T has a unique best proximity point in A.

Let $\varepsilon > 0$ be a fixed. Recall that two elements $x, y \in X$ are said to be ε-close if $d(x, y) < \varepsilon$. Finally, let a number $\varepsilon > 0$ be a fixed and consider the graph G_ε given by $V(G_\varepsilon) = X$ and $E(G_\varepsilon) = \{(x, y) \in X \times X : \ d(x, y) < \varepsilon\}$. If we set $G = G_\varepsilon$ in Theorem 2.5, then we get the following consequence of our best proximity point theorem in a complete b-metric spaces.

Corollary 2.10. *Let $\varepsilon > 0$ be a fixed, (X, d) be a complete b-metric space with parameter $s \geq 1$ and endowed with a graph G_ε, and d be a continuous mapping in two* *variables. Suppose that there exists G_ε-continuous mapping $T : A \to B$ such that the following conditions hold:*

(i) *T is a G_ε-proximal with $T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*

(ii) *there exist ε-close elements $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$;*

(iii) *there exist non-negative constants α, β, γ such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(Ty, y) - s(\beta + \gamma)d(A, B)$$

for all ε-close elements $x, y \in A$, where $s(\alpha + \gamma) + s^2\beta < 1$.

Then T has a best proximity point in A. Furthermore, if each two best proximity point are ε-close, then T has a unique best proximity point in A.

In Theorem 2.5 and his corollaries, set $\beta = \gamma = 0$. Then we obtain best proximity point result for Banach-type contraction in complete b-metric spaces endowed with the graph G.

Corollary 2.11. *Let (X, d) be a complete b-metric with parameter $s \geq 1$ endowed with a graph G and d be a continuous mapping in two variables. Suppose that there exists G-continuous mapping $T : A \to B$ such that the following conditions hold:*

(i) *T is a G-proximal such that $T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*

(ii) *there exist elements x_0 and x_1 in A_0 such that $(x_0, x_1) \in E(G)$ and $d(x_1, Tx_0) = d(A, B)$;*

(iii) *there exists $\alpha \in [0, \frac{1}{s})$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in A$ with $(x, y) \in E(G)$.*

Then T has a best proximity point in X. Furthermore, if $(u, v) \in E(G)$ for any two best proximity point $u, v \in A$, then T has a unique best proximity point in X.

In Theorem 2.5 and his corollaries, set $\alpha = 0$ and $\beta = \gamma = K$. Then we obtain best proximity point result for Kanan-type contraction in complete b-metric spaces endowed with the graph G.

Corollary 2.12. *Let (X, d) be a complete b-metric with parameter $s \geq 1$ endowed with a graph G and d be a continuous mapping in two variables. Suppose that there exists G-continuous mapping $T : A \to B$ such that the following conditions hold:*

(i) *T is a G-proximal such that $T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*

(ii) *there exist elements x_0 and x_1 in A_0 such that $(x_0, x_1) \in E(G)$ and $d(x_1, Tx_0) = d(A, B)$;*

(iii) *there exists $K \in [0, \frac{1}{s^2+s})$ such that*

$$d(Tx, Ty) \leq K[d(x, Tx) + d(y, Ty)] - 2Ksd(A, B)$$

for all $x, y \in A$ with $(x, y) \in E(G)$.

Then T has a best proximity point in X. Furthermore, if $(u, v) \in E(G)$ for any two best proximity point $u, v \in A$, then T has a unique best proximity point in X.

In Theorem 2.5, set $s = 1$. Then we have the following theorem in the framework of complete metric spaces endowed with a graph G.

Theorem 2.13. *Let (X, d) be a complete metric space endowed with a graph G. Also, let $T : A \to B$ be a G-continuous mapping such that following conditions hold:*

(i) *T is a G-proximal such that $T(A_0) \subseteq B_0$ and (A, B) satisfies the P-property;*

(*ii*) *there exist elements* x_0 *and* x_1 *in* A_0 *such that* $(x_0, x_1) \in E(G)$ *and* $d(x_1, Tx_0) = d(A, B)$;

(*iii*) *there exist non-negative constants* α, β, γ *such that*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) - (\beta + \gamma) d(A, B)$$

for all $x, y \in A$ *with* $(x, y) \in E(G)$, *where* $\alpha + \gamma + \beta < 1$.

Then T *has a best proximity point in* X. *Furthermore, if* $(u, v) \in E(G)$ *for any two best proximity point* $u, v \in A$, *then* T *has a unique best proximity point in* X.

Remark 2.14. In Theorem 2.13, consider G_0, G_1, G_2 and G_ε instead of G. Then we obtain same corollaries 2.7-2.10 in the framework of complete metric spaces endowed with a graph G.

Also, in two following corollaries, we introduce best proximity point result for Banach-type contraction and Kanan-type contraction in complete metric spaces endowed with the graph G.

Corollary 2.15. *Let* (X, d) *be a complete metric space endowed with a graph* G. *Also, let* $T : A \to B$ *be a* G*-continuous mapping such that following conditions hold:*

(*i*) T *is a* G*-proximal such that* $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the* P*-property;*

(*ii*) *there exist elements* x_0 *and* x_1 *in* A_0 *such that* $(x_0, x_1) \in E(G)$ *and* $d(x_1, Tx_0) = d(A, B)$;

(*iii*) *there exists* $\alpha \in [0, 1)$ *such that* $d(Tx, Ty) \leq \alpha d(x, y)$ *for all* $x, y \in A$ *with* $(x, y) \in E(G)$.

Then T *has a best proximity point in* X. *Furthermore, if* $(u, v) \in E(G)$ *for any two best proximity point* $u, v \in A$, *then* T *has a unique best proximity point in* X.

Corollary 2.16. *Let* (X, d) *be a complete metric space endowed with a graph* G. *Also, let* $T : A \to B$ *be a* G*-continuous mapping such that following conditions hold:*

(*i*) T *is a* G*-proximal such that* $T(A_0) \subseteq B_0$ *and* (A, B) *satisfies the* P*-property;*

(*ii*) *there exist elements* x_0 *and* x_1 *in* A_0 *such that* $(x_0, x_1) \in E(G)$ *and* $d(x_1, Tx_0) = d(A, B)$;

(*iii*) *there exists* $K \in [0, \frac{1}{2})$ *such that*

$$d(Tx, Ty) \leq K[d(x, Tx) + d(y, Ty)] - 2K d(A, B)$$

for all $x, y \in A$ *with* $(x, y) \in E(G)$.

Then T *has a best proximity point in* X. *Furthermore, if* $(u, v) \in E(G)$ *for any two best proximity point* $u, v \in A$, *then* T *has a unique best proximity point in* X.

Acknowledgments. The authors are thankful to the editor and the referees for their valuable comments to improve the paper.

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Received: June 21, 2018; Accepted: October 18, 2018.