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A STRONG CONVERGENCE THEOREM FOR AN INERTIAL ALGORITHM FOR A COUNTABLE FAMILY OF GENERALIZED NONEXPANSIVE MAPS

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Abstract. Let E be a uniformly smooth and strictly convex real Banach space with dual space, E^*. In this paper, we present a Krasnoselkii-type inertial algorithm and prove a strong convergence theorem for approximating a common fixed point for a countable family of generalized nonexpansive maps. Furthermore, we apply our theorem and prove a strong convergence theorem for approximating a common fixed point for a countable family of generalized-J-nonexpansive maps. Our theorem is an improvement of the results of Klin-earn *et al.* (Taiwanese J. of Maths. Vol. 16, No. 6, pp. 1971-1989, Dec. 2012), Chidume *et al.* (Advances in Fixed Point Theory, Vol. 7, No. 3 (2017), 413-431) and Dong *et al.* (Optimization Letters, 2017, DOI: 10.1007/s11590-016-1102-9). Finally, we give a numerical experiment to illustrate the efficiency and advantage of the inertial algorithm over an algorithm without inertial term.

Key Words and Phrases: Generalized nonexpansive maps, NST-condition, inertial term, fixed point.

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1. Introduction

Let E be a real normed space with dual space, E^*. Let K be a nonempty closed and convex subset of E. An inertial algorithm was first introduced by Polyak [15], as a speeding up process in solving a smooth convex minimization problem. An *inertial-type algorithm* is a two-step iterative process in which the next iterate is defined by making use of the previous two iterates. It is well known that incorporating an inertial term in an algorithm speeds up the rate of convergence of the sequence generated by the algorithm. Therefore, a lot of research interest is now devoted to inertial-type algorithm (see e.g., Bot and Csetnek [3], Dong *et al.* [11] and the references contained in them). In 2008, Takahashi *et al.* [16] studied the following iteration method for approximating a common fixed point for a family of *nonexpansive maps* $\{T_n\}$ in a

Hilbert space, H:

$$\begin{cases} x_1 = x \in C, \\ u_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in C : ||z - u_n|| \leq ||z - x_n||\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \ \forall \ n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\} \subset [0,1]$. They proved a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.1) under the assumption that the sequence $\{T_n\}$ satisfies the so-called *NST-condition.*

In 2012, Klin-earn *et al.* [13] improved this result of Takahashi *et al.* [16] by studying the following *CQ algorithm* for a family of *generalized nonexpansive maps* in a uniformly smooth and uniformly convex real Banach space:

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = \alpha_n x_n + (1-\alpha_n)T_n x_n, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(u_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Jz\rangle \leq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \ \forall \ n \geq 1, \end{cases} \tag{1.2}$$

where J is the normalized duality map on E, $\{T_n\}$ satisfies NST-condition and $\{\alpha_n\} \subset [0,1]$ with $\liminf \alpha_n(1-\alpha_n) > 0$. They proved the following theorem:

Theorem 1.1. (Klin-earn *et al.* [13] *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed subset of E such that JC is closed and convex. Let $\{T_n\}$ be a countable family of generalized nonexpansive mappings from C into E and let Γ be a family of closed generalized nonexpansive mappings from C into E such that $\cap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset$. Suppose that $\{T_n\}$ satisfies the NST-condition with Γ. Then, the sequence $\{x_n\}$ generated by equation (1.2) converges strongly to $R_{F(\Gamma)}x$, where $R_{F(\Gamma)}$ denote the sunny generalized nonexpansive retraction from E onto $F(\Gamma)$.*

With the emergence of a new notion of fixed points called *J-fixed points* defined for maps from a normed space E to its dual E^*, (see e.g., Zegeye [17], Liu [14], Chidume and Idu, [5], Chidume *et al.* [9] and the references contained in them), Chidume *et al.* [7] in 2017, motivated by the result of Klin-earn *et al.* [13], proposed the following *CQ algorithm* for an infinite family of *generalized-J-nonexpansive maps* in a uniformly smooth and uniformly convex real Banach space:

$$\begin{cases} x_1 = x \in C, \ C_0 = Q_0 = C, \\ u_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)J(J_* o T_n)x_n\big), \\ C_n = \big\{u \in C_{n-1} : \phi(u, u_n) \leq \phi(u, x_n)\big\}, \\ Q_n = \{u \in C_{n-1} \cap Q_{n-1} : \langle x - x_n, Jx_n - Ju\rangle \leq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x, \ \forall \ n \geq 1, \end{cases} \tag{1.3}$$

where $\{T_n\}$ satisfies NST-condition and $\{\alpha_n\} \subset (0,1)$ with $\liminf \alpha_n(1-\alpha_n) > 0$. They proved the following theorem:

Theorem 1.2. (Chidume *et al.* [7] *Let C be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E with dual space E^* such that JC is closed and convex. Let $T_n : C \to E^*, n = 1, 2, \cdots$ be an infinite family of generalized J_*-nonexpansive maps and Γ be a family of J_*-closed and generalized J_*-nonexpansive maps from C to E such that $\cap_{n=1}^{\infty} F_J(T_n) = F_J(\Gamma) \neq \emptyset$. Assume that $\{T_n\}$ satisfies the NST-condition with Γ. Then, the sequence generated by equation (1.3) converges strongly to $R_{F_J(\Gamma)}$, where $R_{F_J(\Gamma)}$ denote the sunny generalized nonexpansive retraction from E onto $F_J(\Gamma)$.*

Recently, Dong *et al.* [11] studied the following CQ algorithm for a nonexpansive map in a real Hilbert space:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = (1-\beta_n)w_n + \beta_n T w_n, \\ C_n = \{z \in H : ||y_n - z|| \leq ||w_n - z||\}, \\ Q_n = \{z \in H : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \ \forall \ n \geq 0, \end{cases} \tag{1.4}$$

where $\{\alpha_n\} \subset [\alpha_1, \alpha_2], \alpha_1 \in (-\infty, 0], \alpha_2 \in [0, \infty), \{\beta_n\} \subset [\beta, 1]$ and $\beta \in (0,1]$. They proved the following theorem:

Theorem 1.3. (Dong *et al.* [11] *Let $T : H \to H$ be a nonexpansive map such that $F(T) \neq \emptyset$. Let $\{\alpha_n\} \subset [\alpha_1, \alpha_2], \alpha_1 \in (-\infty, 0], \alpha_2 \in [0, \infty), \{\beta_n\} \subset [\beta, 1]$ and $\beta \in (0,1]$. Set $x_0, \ x_1 \in H$ arbitrarily. Then, the sequence $\{x_n\}$ generated by equation (1.4) convergence in norm to $P_{F(T)}$.*

Remark 1. Algorithm (1.2) of Klin-earn *et al.* [13] is defined on a uniformly smooth and uniformly convex real Banach space. However, it does not include Algorithm (1.4) of Dong *et al.* [11] which is defined on a real Hilbert space because Algorithm (1.4) contains an *inertial term*, w_n, which is introduced to improve speed of convergence.

Motivated by the results of Klin-earn *et al.* [13], Dong *et al.* [11] and Chidume *et al.* [7], it is our purpose in this paper to present a *Krasnoselskii-type algorithm* with an *inertial term*, in a uniformly smooth and strictly convex real Banach space and prove a *strong convergence theorem* for approximating a *common fixed point for a countable family of generalized nonexpansive maps.* Furthermore, we apply our theorem to prove a strong convergence theorem for approximating a *common J-fixed point for a countable family of generalized-J-nonexpansive maps.* For more on the important concept of J-fixed points for a map from a real normed space E to its dual space E^*, and their applications, recently introduced, the reader is referred to Zegeye [17], Liu [14], Chidume and Idu, [5], Chidume *et al.* [9], Chidume *et. al* [8], and the references contained in them. Our theorem includes the results of Klin-earn *et al.*, [13], Dong *et al.* [11] and Chidume *et al.* [7], as special cases.

Finally, we give a numerical experiment to illustrate the efficiency and advantage of our inertial algorithm over the algorithm of Klin-earn *et al.*, [13] without inertial term.

2. Preliminaries

Let E be a real normed space with dual space E^*. We shall denote $x_k \rightharpoonup x^*$ and $x_k \to x^*$ to indicate that the sequence $\{x_k\}$ converges *weakly* to x^* and converges *strongly* to x^*, respectively. A map $J : E \to 2^{E^*}$ defined by

$$J(x) := \left\{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2\right\}$$

is called the *normalized duality map* on E. The following properties of the normalized duality map will be needed in the sequel (see e.g., Cioranescu [10] and the references contained in them).

(1) If E is a reflexive, strictly convex and smooth real Banach space, then, J is surjective, injective and single-valued.
(2) If E is uniformly smooth, then, J is uniformly continuous on bounded subset of E.
(3) If $E = H$, a real Hilbert space, then, J is the identity map on H.

Let E be a smooth real normed space and let $\phi : E \times E \to \mathbb{R}$ be a map defined by $\phi(x, y) := ||x||^2 - 2\langle x, Jy\rangle + ||y||^2$, for all $x, y \in E$. This map was introduced by Alber [1] and has been extensively studied by a host of other authors.

It is easy to see from the definition of ϕ that, if $E = H$, a real Hilbert space, then, $\phi(x, y) = ||x - y||^2$, for all $x, y \in H$. Furthermore, for any $x, y, z \in E$ and $\beta \in (0, 1)$, we have the following properties.

(P_1) $\left(||x|| - ||y||\right)^2 \leq \phi(x, y) \leq \left(||x|| + ||y||\right)^2, \ \forall\, x, y \in E,$

(P_2) $\phi(x, z) = \phi(x, y) + \phi(y, z) + 2\langle y - x, Jz - Jy\rangle,$

(P_3) $\phi(x, y) \leq ||x||||Jx - Jy|| + ||y||||x - y||.$

Definition 2.1. Let K be a nonempty closed and convex subset of a real Banach space E.

(i) A map T from K to E is called *generalized nonexpansive* if the fixed points set of T denoted by $F(T) \neq \emptyset$ and $\phi(Tx, p) \leq \phi(x, p)$, for all $x \in K,\ p \in F(T)$.

(ii) A map R from E onto K is called a *retraction* if $R^2 = R$. Furthermore, R is called *sunny* if $R(Rx + t(x - Rx)) = Rx$, for all $x \in E$ and $t \leq 0$.

(iii) A nonempty closed subset K of a smooth real Banach space E is called a *sunny generalized nonexpansive retract* of E if there exists a sunny generalized nonexpansive retraction R from E onto K.

(iv) A map T is called *closed* if for any $\{x_n\} \subset E$ such that $x_n \to x^*$ and $Tx_n \to y$, then, $y = Tx^*$.

Lemma 2.2. (Alber, [1]) *Let C be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then, the following are equivalent.*
(i) C is a sunny generalized nonexpansive retract of E;
(ii) C is a generalized nonexpansive retract of E; and
(iii) JC is closed and convex.

Lemma 2.3. (Alber, [1]) *Let C be a nonempty closed and convex subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto K. Then, the following hold:*
(i) $z = Rx$ iff $\langle y - z, Jz - Jx\rangle \geq 0$, for all $y \in C$,
(ii) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$, for all $z \in C$.

Lemma 2.4. (Chang *et al.*, [4]) *Let E be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property and C be a closed convex subset of E. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that $x_n \to p$ and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$. Then, $y_n \to p$.*

Remark 2. Using (P_3), it is easy to see that the converse of Lemma 2.4 is also true whenever $\{x_n\}$ and $\{y_n\}$ converge to the same limit point.

Lemma 2.6. *A real normed space E is said to have the Kadec-Klee property, if for any sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x \in E$ and $||x_n|| \to ||x||$, then, $||x_n - x|| \to 0$.*

Lemma 2.7. (Ibaraki and Takahashi, [12]) *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.8. (Klin-earn *et al.* [13]) *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive mapping from C into E. Then, $F(T)$ is closed and $J\big(F(T)\big)$ is closed and convex.*

Lemma 2.9. (Klin-earn *et al.* [13]) *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that JC is closed and convex. Let T be a generalized nonexpansive map from C into E. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

NST-Condition. (Klin-earn *et al.* [13]) Let $\{T_n\}$ and Γ be two families of generalized nonexpansive maps from C into E such that $\cap_{n=1}^{\infty} F(T_n) = F(\Gamma) \neq \emptyset$, where $F(T_n)$ is the set of fixed points of T_n and $F(\Gamma)$ is the set of fixed points of Γ. Then, $\{T_n\}$ is said to satisfy the *NST-condition with* Γ if for each bounded sequence $\{x_n\} \subset C$,

$$\lim_{n\to\infty} ||x_n - T_n x_n|| = 0 \implies \lim_{n\to\infty} ||x_n - T x_n|| = 0, \ \forall\, T \in \Gamma.$$

Remark 3. In particular, if $\Gamma = \{T\}$, i.e., Γ consists of one map T, then, $\{T_n\}$ is said to satisfy the *NST-condition* with T. It is obvious that $\{T_n\}$ with $T_n = T$, for all $n \in \mathbb{N}$ satisfies NST-condition with $\Gamma = \{T\}$. For more examples of sequences with *NTS-condition*, see e.g., Klin-earn *et al.* [13].

The analytical representations of the duality map in these spaces where $p^{-1}+q^{-1}=1$, (see e.g., Theorem 3.1, Alber and Ryazantseva [2]; page 36) are:

$$\begin{aligned}
Jx &= ||x||_{l_p}^{2-p} y \in l_q,\ y=\{|x_1|^{p-2}x_1,\ |x_2|^{p-2}x_2,\cdots\},\ x=\{x_1,x_2,\cdots\},\\
J^{-1}x &= ||x||_{l_q}^{2-q} y \in l_p,\ y=\{|x_1|^{q-2}x_1,\ |x_2|^{q-2}x_2,\cdots\},\ x=\{x_1,x_2,\cdots\},\\
Jx &= ||x||_{L_q}^{2-q}|x(s)|^{q-2}x(s)\in L_q(G),\ s\in G,\\
J^{-1}x &= ||x||_{L_q}^{2-q}|x(s)|^{q-2}x(s)\in L_p(G),\ s\in G \text{ and}\\
Jx &= ||x||_{W_m^p}^{2-p}\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^{\alpha}\big(|D^{\alpha}x(s)|^{p-2}D^{\alpha}x(s)\big)\in W_{-m}^{q}(G),\ m>0,\ s\in G.
\end{aligned}$$

3. Main results

Theorem 3.1. *Let E be a uniformly smooth and strictly convex real Banach space with Kadec-Klee-property and dual space E^*. Let K be a nonempty closed and convex subset of E such that JK is closed and convex. Let $T_k : E \to E,\ k=1,2,\cdots$ be a countable family of generalized-nonexpansive maps and Γ be a family of closed and generalized-nonexpansive maps from E to E such that $\cap_{k=1}^{\infty}F(T_k)=F(\Gamma)\neq\emptyset$. Let $\{v_k\}$ be generated by:*

$$\begin{cases}
v_0, v_1\in E,\ K_1=E,\\
w_k = v_k+\beta_k(v_k-v_{k-1}),\\
y_k=\alpha w_k+(1-\alpha)T_k w_k,\\
K_{k+1}=\big\{v\in K_k:\phi(y_k,v)\le\phi(w_k,v)\big\},\\
v_{k+1}=R_{K_{k+1}}v_0,\ \forall\ k\ge 1.
\end{cases}\tag{3.1}$$

Assume that $\{T_k\}$ satisfies the NST-condition with Γ, $\{\beta_k\}\subset(0,1)$ and $\alpha\in(0,1)$. Then, $\{v_k\}$ converges strongly to $R_{F(\Gamma)}v_0$, where $R_{F(\Gamma)}$ is the sunny generalized-nonexpansive retraction of E onto $F(\Gamma)$.

Proof. We divide our proof into five steps.

Step 1: The sequence $\{v_k\}$ is well defined and $F(\Gamma)\subset K_k,\ \forall\ k\ge 1$.
First, we show that JK_k is closed and convex. Clearly, $K_1=E$ is closed and convex. Assume that K_k is closed and convex for some $k\ge1$, utilizing the definition of K_{k+1}, it is clear that $K_{k+1}=\{2\langle w_k-y_k,Jv\rangle\le ||w_k||^2-||y_k||^2\}$. Thus, K_{k+1} is closed and convex. Hence, we conclude that JK_k is closed and convex. Furthermore, by Lemma 2.2, K_k is a sunny generalized-nonexpansive retract of E. Hence, $\{v_k\}$ is well defined.

Next, we prove that $F(\Gamma)\subset K_k, \forall\, k\ge1$. Clearly, $F(\Gamma)$ is a subset of K_1. Assume that $F(\Gamma)\subset K_k$ for some $k\ge1$. Let $p\in F(\Gamma)$. Then, we have that

$$\begin{aligned}
\phi(y_k,p) &= ||\alpha w_k+(1-\alpha)T_kw_k||^2-2\big\langle\alpha w_k+(1-\alpha)T_kw_k,Jp\big\rangle+||p||^2\\
&\le \alpha\phi(w_k,p)+(1-\alpha)\phi(T_kw_k,p)\\
&\le \alpha\phi(w_k,p)+(1-\alpha)\phi(w_k,p)\phi(w_k,p),
\end{aligned}\tag{3.2}$$

which implies that $p\in K_{k+1}$. Hence, $F(\Gamma)\subset K_k,\ \forall\ k\ge1$.

Step 2: $\lim_{k\to\infty}\phi(v_k, v_0)$ exists and $\{v_k\}$ converges.

First, we prove that $\{v_k\}$ is bounded. From the definition of $\{v_k\}$ and Lemma 2.3, we have that

$$\phi(v_k, v_0) = \phi(R_{K_k}v_0, v_0) \le \phi(p, v_0) - \phi(p, R_{K_k}v_0) \le \phi(p, v_0),\ \forall\ p \in F(\Gamma) \subset K_k.$$

This implies that $\{\phi(v_k, v_0)\}$ is bounded. Furthermore, $\{v_k\}$ is bounded.
Since $v_{k+1} \in K_{k+1} \subset K_k$ and $v_k = R_{K_k}v_0$, we have that $\phi(v_k, v_0) \le \phi(v_{k+1}, v_0)$, and this implies that $\{\phi(v_k, v_0)\}$ is nondecreasing. Hence, $\lim_{n\to\infty}\phi(v_k, v_0)$ exists.

Since E is reflexive and $\{v_k\}$ is bounded, then, there exists a subsequence $\{v_{k_j}\}$ of $\{v_k\}$ such that $v_{k_j} \rightharpoonup x^* \in K_{K_j}$. In view of $v_{k_j} = R_{K_j}v_0$ and Lemma 3.3, we get that $\phi(v_{k_j}, v_0) \le \phi(p, v_0)$, $\forall\ j \ge 1$. Applying the weak lower semi-continuity of norm $||\cdot||$, we obtain that

$$\phi(x^*, v_0) \le \liminf_{j\to\infty}\phi(v_{k_j}, v_0) \le \limsup_{j\to\infty}\phi(v_{k_j}, v_0) \le \phi(x^*, v_0), \tag{3.3}$$

which implies that $\lim_{j\to\infty}\phi(v_{k_j}, v_0) = \phi(x^*, v_0)$. Furthermore, $\lim_{j\to\infty}||v_{k_j}|| = ||x^*||$. By Lemma 2.5, we obtain that $\lim_{j\to\infty}v_{k_j} = x^*$. Since $\lim_{k\to\infty}\phi(v_k, v_0)$ exists and $\lim_{j\to\infty}\phi(v_{k_j}, v_0) = \phi(x^*, v_0)$, then, $\lim_{k\to\infty}\phi(v_k, v_0) = \phi(x^*, v_0)$.

Next we show that $\lim_{k\to\infty}v_k = x^*$. Suppose for contraction that there exists a subsequence $\{v_{k_i}\}$ of $\{v_k\}$ such that $\lim_{i\to\infty}v_{k_i} = x$ with $x^* \ne x$, then, by Lemma 2.3, we have that

$$\begin{aligned}\phi(x^*, x) &= \lim_{i,j\to\infty}\phi(v_{k_j}, v_{k_i}) = \lim_{i,j\to\infty}\phi(v_{k_j}, R_{K_{k_i}}v_0)\\ &\le \lim_{i,j\to\infty}\Big(\phi(v_{k_j}, v_0) - \phi(v_{k_i}, v_0)\Big)\\ &= \phi(x^*, v_0) - \phi(x^*, v_0) = 0.\end{aligned}$$

This implies that $x^* = x$, which is a contraction. Hence, $\lim_{k\to\infty}v_k = x^*$.

Step 3: $\lim_{k\to\infty}||w_k - v_k|| = \lim_{k\to\infty}||v_k - y_k|| = \lim_{k\to\infty}||w_k - y_k|| = 0$.

Using the definition of w_k in equation (3.1) and convergence of $\{v_k\}$ in step **2**, we obtain that

$$\lim_{k\to\infty}||w_k - v_k|| = 0.$$

By Lemma 2.4 and Remark 2, we obtain that $\lim_{k\to\infty}\phi(w_k, v_k) = 0$. Since $v_k \in K_k$, and by inequality (3.2), we have that $\phi(y_k, v_k) \le \phi(w_k, v_k)$, and this implies that $\lim_{k\to\infty}\phi(y_k, v_k) = 0$. Hence, by Lemma 2.4, we obtain that $\lim_{k\to\infty}||y_k - v_k|| = 0$. By triangle inequality, we obtain that $||y_k - w_k|| \le ||y_k - v_k|| + ||v_k - w_k|| \to 0$ as $k \to \infty$. This completes the proof of Step **3**.

Step 4: $\lim_{k\to\infty}||w_k - Tw_k|| = 0$ and $x^* \in F(T)$.

From equation (3.1) and step **3**, we obtain that

$$\begin{aligned}||v_k - y_k|| &= ||v_k - \alpha w_k - (1-\alpha)T_k w_k|| \\ &= ||(1-\alpha)(v_k - T_k w_k) - \alpha(w_k - v_k)|| \\ &\geq (1-\alpha)||v_k - T_k w_k|| - \alpha||w_k - v_k||,\end{aligned}$$

which implies that

$$||v_k - T_k w_k|| \leq \frac{1}{1-\alpha}\Big(||v_k - y_k|| + \alpha||w_k - v_k||\Big) \to 0 \text{ as } k \to \infty. \tag{3.4}$$

From inequality (3.4) and step **3**, we obtain that

$$||w_k - T_k w_k|| \leq ||w_k - v_k|| + ||v_k - T_k w_k|| \to 0 \text{ as } k \to \infty. \tag{3.5}$$

Since T_k satisfies the NST-condition with Γ, we obtain from inequality (3.5) that

$$\lim_{k\to\infty} ||w_k - Tw_k|| = 0, \ \forall\, T \in \Gamma.$$

Furthermore, since $\lim_{k\to\infty} w_k = x^*$ in step **3** and T is closed by assumption, then, $x^* \in F(T)$.

Step 5: $\lim_{k\to\infty} v_k = R_{F(\Gamma)}v_0$.

From Lemma 2.3, we obtain that

$$\phi(R_{F(\Gamma)}v_0, v_0) \leq \phi(x^*, v_0). \tag{3.6}$$

For $x^* \in F(\Gamma) \subset K_k$, $v_k = R_{K_k}v_0$ and by Lemma 2.3, we have that

$\phi(v_k, v_0) \leq \phi(R_{F(\Gamma)}v_0, v_0)$. Taking limit of both sides of this inequality, we obtain that

$$\phi(x^*, v_0) \leq \phi(R_{F(\Gamma)}v_0, v_0). \tag{3.7}$$

Combining inequality (3.6) and (3.7), we obtain that $\phi(x^*, v_0) = \phi(R_{F(\Gamma)v_0}, v_0)$.
By uniqueness of $R_{F(\Gamma)}v_0$, we conclude that $x^* = R_{F(\Gamma)}v_0$.

Remark 4. Theorem 3.1 extends the result of Klin-earn *et al.* [13] from a uniformly smooth and *uniformly convex* real Banach space to a uniformly smooth and *strictly convex* real Banach space. Furthermore, an *inertial term* is incorporated in the algorithm of Theorem 3.1, whereas the algorithm of Klin-earn *et al.* [13] does not involve this term. Moreover, the computation at each iteration process of two subsets C_n *and* Q_n *of* C*, their intersection* $C_n \cap Q_n$*,* and the retraction of the initial vector onto the intersection which is required in the theorem of Klin-earn *et al.* [13] has been dispensed with and replaced with a single retraction onto the subset K_{k+1} of E.

In Theorem 1.3 of Dong *et al.* [11], the authors proved a strong convergence theorem in a real *Hilbert space* for a *nonexpansive map.* Our Theorem 3.1 extends this result to a *uniformly smooth and strictly convex* real Banach space and to a *countable family of generalized nonexpansive maps* that satisfy the *NST-condition.* We observe that the NST-condition imposed in our Theorem 3.1 is trivially satisfied for a single operator T as in Theorem 1.3 of Dong *et al.* [11]. Furthermore, the control parameter in our algorithm is one arbitrarily fixed constant $\alpha \in (0,1)$ which is to be computed once and then used at each step of the iteration process, whereas the parameter in the algorithm studied by Dong *et al.* [11] is $\beta_n \in (0,1)$ which is to be computed at each step of the iteration process. Consequently, the sequence of equation (3.1) is of *Krasnoselskii-type* and the sequence defined by equations (1.2), (1.3) and (1.4) are of *Mann-type*.

Example 1. Let $E = l_p,\ 1 < p < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$ and $K = \overline{B_{l_p}}(0,1)$.

Let $T : l_p \to l_p$ be defined by $T(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, \cdots)$. Let $T_k : l_p \to l_p$ be defined by $T_k x = \alpha_k x + (1-\alpha_k)Tx,\ \forall\ k \geq 1,\ x \in l_p$ and $\alpha_k \in (0,1)$. Let $\Gamma = T$.

Clearly, $E,\ K,\ T_k$ and Γ satisfy all the conditions of Theorem 3.1 Hence, the sequence $\{v_k\}$ generated by equation (3.1) converges to $\bar{0}$, the unique element of $\cap_{k=1}^{\infty}(T_k) = F(\Gamma)$.

Corollary 3.2. *Let K be a nonempty closed and convex subset of $L_p,\ l_p$ or $W_p^m(\Omega)$ spaces, $1 < p < \infty$, such that JK is closed and convex. Let $T_k : E \to E,\ k = 1, 2, \cdots$ be a countable family of generalized nonexpansive maps and Γ be a family of closed and generalized nonexpansive maps from E to E such that $\cap_{k=1}^{\infty} F(T_k) = F(\Gamma) \neq \emptyset$. Let $\{v_k\}$ be generated by:*

$$\begin{cases} v_0, v_1 = v \in E,\ K_1 = E, \\ w_k = v_k + \beta_k(v_k - v_{k-1}), \\ y_k = \alpha w_k + (1-\alpha)T_k w_k, \\ K_{k+1} = \big\{v \in K_k : \phi(y_k, v) \leq \phi(w_k, v)\big\}, \\ v_{k+1} = R_{K_{k+1}} v_0,\ \forall\ k \geq 1. \end{cases} \tag{3.8}$$

Assume that $\{T_k\}$ satisfies the NST-condition with Γ, $\{\beta_k\} \subset (0,1)$ and $\alpha \in (0,1)$. Then, $\{v_k\}$ converges strongly to $R_{F(\Gamma)}v_0$, where $R_{F(\Gamma)}$ is the sunny generalized-nonexpansive retraction of E onto $F(\Gamma)$.

Proof. Since these spaces are uniformly smooth and strictly convex, the result follows from Theorem 3.1.

4. Applications

In section 3, we considered a countable family of self maps on E. For application, we consider a countable family of maps from a space E to its dual space E^*. In this case, the usual notion of fixed points obviously does not make sense. However, a new notion of fixed points called *J-fixed points* has been defined for maps from a normed space E to its dual E^*, (see e.g., Chidume and Idu, [5]) for motivation and definition. This notion turns out to be very useful in proving convergence theorems for several important classes of nonlinear maps (see e.g., Chidume and Idu, [5]). We shall employ this concept here.

Let E be a uniformly smooth and strictly convex real Banach space with dual space E^*. Obviously, $J^{-1} = J_*$ exists under this setting.

Definition 4.1. (Chidume *et al.*, [5]) Let $T : E \to E^*$ be any map. A point $p \in E$ is called a *J-fixed point* of T if $Tp = Jp$, where $J : E \to E^*$ is the *normalized duality map.* The set of J-fixed points of T will be denoted by $F_J(T)$.

Definition 4.2. (Chidume *et al.*, [6]) A map $T : E \to E^*$ is called *J-nonexpansive* if for each $x, y \in E$, the following inequality holds:

$$||Tx - Ty||||x - y|| \leq \langle Jx - Jy, x - y\rangle.$$

Definition 4.3. (Chidume *et al.*, [7]) A map $T : K \to E^*$ is called *generalized-J-nonexpansive* if $F_J(T) \neq \emptyset$ and $\phi((J_*oT)p, x) \leq \phi(p, x), \forall\ x \in K,\ p \in F_J(T)$.

NST-Condition. (Chidume *et al.* [7]) Let $\{T_n\}$ and Γ be two families of generalized-J-nonexpansive maps from K into E^* such that $\cap_{k=1}^{\infty} F(T_k) = F(\Gamma) \neq \emptyset$, where $F(T_k)$ is the set of J-fixed points of T_k and $F(\Gamma)$ is the set of J-fixed points of Γ. Then, $\{T_k\}$ is said to satisfy the *NST-condition with* Γ if for each bounded sequence $\{v_k\} \subset K$,

$$\lim_{k\to\infty} ||Jv_k - T_k v_k|| = 0 \implies \lim_{n\to\infty} ||Jv_k - Tv_k|| = 0,\ \forall\ T \in \Gamma.$$

Theorem 4.4. *Let E be a uniformly smooth and strictly convex real Banach space with Kadec-Klee-property and dual space E^*. Let K be a nonempty closed and convex subset of E such that JK is closed and convex. Let $S_k : E \to E^*,\ k = 1, 2, \cdots$ be a countable family of generalized-J-nonexpansive maps and Γ be a family of closed and generalized-J-nonexpansive maps from E to E^* such that $\cap_{k=1}^{\infty} F_J(S_k) = F_J(\Gamma) \neq \emptyset$. Let $\{v_k\}$ be generated by:*

$$\begin{cases} v_0, v_1 \in E,\ K_1 = E, \\ w_k = v_k + \beta_k(v_k - v_{k-1}), \\ y_n = \alpha w_k + (1-\alpha)(J_*oS_k)w_k, \\ K_{k+1} = \big\{v \in K_k : \phi(y_k, v) \leq \phi(w_k, v)\big\}, \\ v_{k+1} = R_{K_{k+1}} v_0,\ \forall\ k \geq 1. \end{cases} \tag{4.1}$$

Assume that $\{S_k\}$ satisfies the NST-condition with Γ, $\{\beta_k\} \subset (0,1)$ and $\alpha \in (0,1)$. Then, $\{v_k\}$ converges strongly to $R_{F_J(\Gamma)} v_0$, where $R_{F_J(\Gamma)}$ is the sunny generalized-J-nonexpansive retraction of E onto $F_J(\Gamma)$.

Proof. Define $T_k := J_*oS_k$. Then, $T_k : E \to E,\ k = 1, 2, \cdots$. Furthermore, T_k is a generalised nonexpansive map and $\cap_{k=1}^{\infty} F(T_k) = \cap_{k=1}^{\infty} F_J(S_k) = F(\Gamma)$. Hence, by Theorem 3.1, $\{v_k\}$ converges strongly to some $x^* \in R_{F_J(\Gamma)}$.

Remark 5. Theorem 4.4 extends Theorem 1.3 of Chidume *et al.* [7] from a uniformly smooth and *uniformly convex* real Banach space to a uniformly smooth and *strictly convex* real Banach space. Furthermore, an *inertial term* is incorporated in the algorithm of Theorem 4.4, whereas the algorithm of Theorem 1.3 of Chidume *et al.* [7] does not involve this term. Moreover, the computation at each iteration process of two subsets C_n *and* Q_n *of* C, *their intersection* $C_n \cap Q_n$, and the retraction of the initial vector onto the intersection in Theorem 1.3 of Chidume *et al.* [7] has been dispensed with and replaced with a single retraction onto the subset K_{k+1} *of* E. In addition, the condition that T be J_*-closed in Theorem 1.3 of Chidume *et al.* [7] has also been dispensed with in our Theorem 4.4.

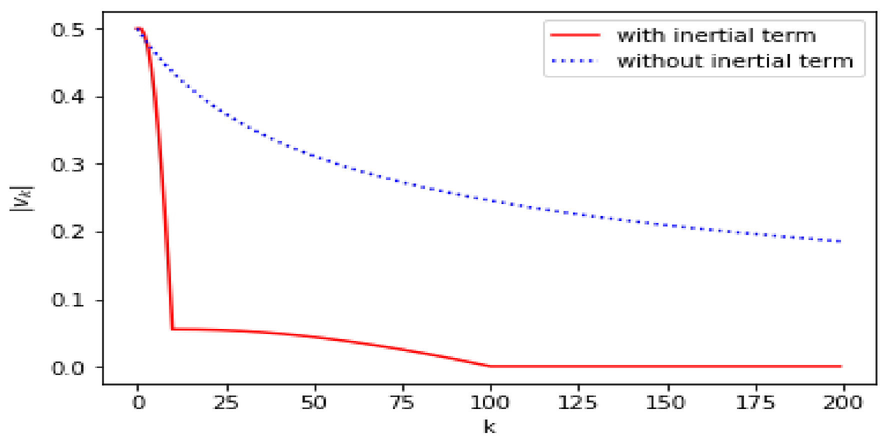
5. Numerical experiment

Here, we present a numerical example to compare the speed of convergence of inertial algorithm and the algorithm without inertial term.

Example. Let $E = \mathbb{R}$, $K = [\gamma, \eta]$, $\gamma, \eta \in \mathbb{R}$. Clearly, $v \in \mathbb{R}$,

$$P_K v \begin{cases} \gamma, & if \quad v < \gamma, \\ v, & if \quad v \in K, \\ \eta, & if \quad v > \eta. \end{cases} \tag{5.1}$$

Now, set $Tv = \sin v$, $K = [-1, 1]$ in Theorem 3.1. Clearly, S is generalized nonexpansive with 0 as its unique fixed point. Set $\beta_k = \frac{k}{k+\gamma-1}$, $\gamma = \eta = 4$, $v_0 = v_1 = \frac{1}{2}$. Then, by Theorem 3.1, the sequence generated by algorithm (3.1) converges to zero. The numerical results are sketched in the figure below, where the *y-axis* represents the value of $|v_k - 0|$ while the *x-axis* represents the number of iterations (k).



All computations and graph were done using spyder 3.2.6 on Hp Intel CORE DUO 2gb Ram. We observe from the figure above that the algorithm with inertial term converges much faster than algorithm without inertial term.

Conclusion: In this paper, we considered a Krasnoselskii-type inertial algorithm for approximating a common fixed point for a countable family of generalized nonexpansive maps in a uniformly smooth and strictly convex real Banach space. Strong convergence of the sequence generated by our algorithm is proved. Furthermore, we applied our theorem and proved a strong convergence theorem for approximating a *common J-fixed point for a countable family of generalized-J-nonexpansive maps.* Consequently, our theorem is applicable in L_p (l_p or $W_p^m(\Omega)$ spaces, $1 < p < \infty$, where $W_p^m(\Omega)$ denotes the usual Sobolev space. It is well known that a Krasnoselskii-type sequence converges as fast as a geometric progression which is slightly faster than the convergence rate obtained from any Mann-type sequence. Finally, a numerical example is presented which illustrates that the algorithm with inertial term converges much faster than that without inertial term.

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