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POSITIVE SOLUTION FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH NONLOCAL MULTI-POINT CONDITION

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Abstract. In this paper, we study and consider the positive solution of fractional differential equation with nonlocal multi-point conditions of the from:

$$ {}_{RL}D^{q}_{0^+}u(t) + g(t)f(t,u(t)) = 0,\ t \in (0,1) $$

$$ u^{(k)}(0) = 0, \quad u(1) = \sum_{i=1}^{m} \beta_i \ {}_{RL}D^{p_i}_{0^+}u(\eta_i) $$

where $n-1 < q < n,\ n \geq 2,\ n-1 < p_i < n,\ q > p_i\ m, n \in \mathbb{N},\ k = 0,1,\cdots,n-2,\ 0 < \eta_1 < \eta_2 < \cdots < \kappa,\ \beta_i \leq 0,\ \kappa \in (0,1],\ {}_{RL}D^{q}_{0^+},\ {}_{RL}D^{p_i}_{0^+}$ are the Riemann-Liouville fractional derivative of order $q,\ p_i,\ f : [0,1] \times C([0,1],E) \to E,\ E$ be Banach space and $g : (0,1) \to \mathbb{R}^+$ are continuous functions. The main tools for finding positive solutions of the above problem are the fixed point theorems of Guo-Krasnoselskii and of Boyd and Wong. An example is included to illustrate the applicability of our results.

Key Words and Phrases: Boundary value problems, Riemann-Liouville fractional derivative, fixed point theorems.
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1. Introduction

Fractional differential equations show that a variety of interesting and important results concerning existence and uniqueness of solutions. The concerned area has been recently proved to be valuable tools in the modeling of many phenomena in biology, chemistry, physics, networking, dynamics, fluid mechanics, electromagnetic theory, electro-chemistry, control theory movement through porous media etc, see [10, 9, 8, 2, 5] for more details.

In [7], the authors obtained the existence of fractional boundary value problem:

$$\begin{cases} {}_{RL}D^q_{0^+}u(t) + a(t)f(t,u(t)) = 0, \ 0 < t < 1, \\ u(0) = 0, \ {}_{RL}D^p_{0^+}u(1) = 0, \end{cases} \tag{1.1}$$

where $q \in (1,2], p \in [0,1]$, ${}_{RL}D^q_{0^+}$, ${}_{RL}D^p_{0^+}$ are the Riemann-Liouville fractional derivative of order q, p, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous function and $a : [0,1] \to [0,\infty)$ is an integrable function on $[0,1]$.

In [11], the authors investigated the existence of positive solutions for the following singular fractional differential equations with infinite-point boundary value conditions:

$$\begin{cases} {}_{RL}D^q_{0^+}u(t) + q(t)f(t,u(t)) = 0, \ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ {}_{RL}D^p_{0^+}u(1) = \displaystyle\sum_{i=1}^{\infty} \beta_i u(\eta_i), \end{cases} \tag{1.2}$$

where $q > 2$, $n-1 < q < n$, $p \in [1, q-1]$ is a fixed number, $\beta_i \geq 0, (i = 1, 2, \ldots)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{i-1} < \eta_i < \cdots < 1$, $\frac{\Gamma(q)}{\Gamma(q-p)} - \sum_{i=1}^{\infty} \beta_i \eta_i^{q-1} > 0$ and ${}_{RL}D^q_{0^+}$, ${}_{RL}D^p_{0^+}$ are the Riemann-Liouvill fractional derivative of order q, p.

Inspired by the above paper in previous works, [7, 11, 4], the objective of this paper is to derive the existence and uniqueness of positive solution of nonlinear fractional differential equation and nonlocal multi point condition:

$$\begin{cases} {}_{RL}D^q_{0^+}u(t) + g(t)f(t,u(t)) = 0, \ t \in (0,1), \\ u^{(k)}(0) = 0, \quad u(1) = \displaystyle\sum_{i=1}^{m} \beta_i \ {}_{RL}D^{p_i}_{0^+}u(\eta_i), \end{cases} \tag{1.3}$$

where $n-1 < q < n, n \geq 2$, $m, n \in \mathbb{N}$, $k = 0, 1, \cdots, n-2$, $0 < \eta_1 < \eta_2 < \cdots < \kappa$, $\beta_i \leq 0$, $\kappa \in (0,1]$, ${}_{RL}D^q_{0^+}$, ${}_{RL}D^{p_i}_{0^+}$ are the Riemann-Liouville fractional derivative of order q, p_i, $f : [0,1] \times C([0,1], E) \to E$, E be Banach space and $g : (0,1) \to \mathbb{R}^+$ are continuous functions. The structure of the paper is as follows. Section 2, contains some fundamental concepts of fractional derivative. In section 3, we present our main result by applying Guo-Krasnoselskii and Boyd-Wong fixed point theorems respectively. We will also present an example, illustrating the main results.

2. Preliminaries

In this section, we recall some basic notations, definitions, lemmas and theorem use to prove the main result.

Definition 2.1. [12] The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$_{RL}I^q_{0+} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

where $\Gamma(\cdot)$ denotes the Gamma function defined by $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds$.

Definition 2.2. [3] The Riemann-Liouville fractional derivative of order $q > 0$ of a function $f : (0, \infty) \to \mathbb{R}$, is defined by

$$\left(_{RL}D^q_{0+} f\right)(t) = \frac{1}{\Gamma(n-q)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-q-1} f(s) ds,$$

where n is the smallest integer greater than or equal to q.

Definition 2.3. [6] The Caputo fractional derivative of order $q > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$\left(^C D^q_{0+} f\right)(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds,$$

where n is the smallest integer greater than or equal to q.

Lemma 2.4. [14] *Let $b, q \geq 0$ and $x \in C(0,b) \cap L^1(0,b)$. Then*

$$_{RL}I^q_{0+}\, {}_{RL}D^q_{0+} x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, n is the smallest integer greater than or equal to q.

Proposition 2.5. [8]*If $p, q > 0$ then*

(1) *$f(t) = k \neq 0$, k is a constant, then $^C D^q_{0+} k = 0$ and $_{RL}D^q_{0+} k = \frac{t^{-q} k}{\Gamma(1-q)}$.*
(2) *$_{RL}D^q_{0+} t^{q-1} = 0$.*
(3) *For $p > 1$, we have $_{RL}I^q_{0+} t^p = \frac{\Gamma(p+1)}{\Gamma(q+p+1)} t^{q+p}$.*
(4) *For $p > -1 + q$, we have $_{RL}D^q_{0+} t^p = \frac{\Gamma(p+1)}{\Gamma(1+p-q)} t^{p-q}$.*
(5) *For $p > 0$, we have $^C D^q_{0+} t^p = \frac{\Gamma(p+1)}{\Gamma(1+p-q)} t^{p-q}$.*

Proposition 2.6. [15] *If $0 < r < q$ then for $f \in L^p([a,b], \mathbb{R})$, $(1 \leq p \leq \infty)$, the relation $_{RL}D^r_{a^+}(_{RL}I^q_{a^+} f(t)) =_{RL} I^{q-r}_{a^+} f(t)$ hold almost every on $[a,b]$.*

Definition 2.7. [13] Let E be a Banach space and let $A : E \to E$ be a mapping. A is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\epsilon) < \epsilon$ for all $\epsilon > 0$ with the property:

$$\|Ax - Ay\| \leq \Psi(\|x - y\|), \ \forall x, y \in E.$$

Theorem 2.8. [13] ***(Boyd and Wong fixed point theorem)***
Let E be a Banach space and let $A : E \to E$ be a nonlinear contraction. Then A has a unique fixed point in E.

Theorem 2.9. [1] ***(Guo-Grasnoselskii fixed point theorem)***
Let E be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely comtinuous operator such that:

(1) $\|Au\|\geq\|u\|$, $u\in P\cap\partial\Omega_1$, *and* $\|Au\|\leq\|u\|$, $u\in P\cap\partial\Omega_2$; *or*
(2) $\|Au\|\leq\|u\|$, $u\in P\cap\partial\Omega_1$, *and* $\|Au\|\geq\|u\|$, $u\in P\cap\partial\Omega_2$.

Then A has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. Main results

In this section we consider the positive solutions of nonlinear fractional differential equation (1.3).

Definition 3.1. A function $u\in C([0,1],E)$ is said to be a solution of (1.3), if u satisfies the boundary value problem.

$$\begin{aligned}&{}_{RL}D^{q}_{0^+}u(t)+g(t)f(t,u(t))=0,\ t\in(0,1),\\ &u^{(k)}(0)=0,\quad u(1)=\sum_{i=1}^{m}\beta_i\ {}_{RL}D^{p_i}_{0^+}u(\eta_i).\end{aligned}$$

We prove now the following lemma, which establish the existence of solution to problem (1.3).

Lemma 3.2. *Let the function $h\in C([0,1],E)$. Suppose that the function $u\in C([0,1],E)$ is a solution of the following boundary value problem (BVP):*

$$\begin{aligned}&{}_{RL}D^{q}_{0^+}u(t)=-h(t),\ t\in(0,1),\\ &u^{(k)}(0)=0,\quad u(1)=\sum_{i=1}^{m}\beta_i\ {}_{RL}D^{p_i}_{0^+}u(\eta_i),\end{aligned}$$

where $n-1<q<n, n\geq 2,\ m,n\in\mathbb{N},\ k=0,1,\cdots,n-2,\ 0<\eta_1<\eta_2<\cdots<\kappa$, $\beta_i\leq 0,\ \kappa\in(0,1]$, ${}_{RL}D^{q}_{0^+}$, ${}_{RL}D^{p_i}_{0^+}$ are the Riemann-Liouville fractional derivative of order q, p_i. Then the solution of the above BVP is unique and it is given by

$$u(t)=\int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s)=\begin{cases}\frac{1}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(t-s)^{q-1}+\sigma(s)t^{q-1}(1-s)^{q-1}\Big], & 0<s<t<1.\\ \frac{1}{\Gamma(q)\sigma(0)}\Big[\sigma(s)t^{q-1}(1-s)^{q-1}\Big], & 0<t<s<1.\end{cases}\tag{3.1}$$

Proof. From Lemma 2.4, we get for certain constant vector $c_0,c_1,\cdots,c_{n-1}$ belong to E

$$u(t)=-{}_{RL}I^{q}_{0^+}h(t)+c_1t^{q-1}+c_2t^{q-2}+\cdots+c_nt^{q-n}.$$

From the first condition in BVP, we choose

$$c_2=\ c_3=,\ \cdots,\ c_{n-1}=0,$$

and so,

$$u(t)=-{}_{RL}I^{q}_{0^+}h(t)+c_1t^{q-1}.$$

The substitution $t=1$ yields,

$$u(1)=-{}_{RL}I^{q}_{0^+}h(1)+c_1,$$

and applying the operator ${}_{RL}D_{0^+}^{p_i}u(t)$ results and using Proposition 2.6, $q > p_i$ we get

$$ {}_{RL}D_{0^+}^{p_i}u(\eta_i) = -{}_{RL}I_{0^+}^{q-p_i}h(\eta_i) + c_1\frac{\Gamma(q)}{\Gamma(q-p_i)}\eta_i^{q-1-p_i}. $$

By employing the second boundary value condition, we get

$$ \begin{aligned} \sum_{i=1}^{m}\beta_i {}_{RL}D_{0^+}^{p_i}u(\eta_i) &= -\sum_{i=1}^{m}\beta_i {}_{RL}I_{0^+}^{q-p_i}h(\eta_i) + c_1\sum_{i=1}^{m}\frac{\beta_i\Gamma(q)}{\Gamma(q-p_i)}\eta_i^{q-p_i-1} \\ c_1 &= \frac{{}_{RL}I_{0^+}^{q}h(1) - \sum_{i=1}^{m}\beta_i {}_{RL}I_{0^+}^{q-p_i}h(\eta_i)}{1-\sum_{i=1}^{m}\frac{\beta_i\Gamma(q)}{\Gamma(q-p_i)}\eta_i^{q-p_i-1}}. \end{aligned} $$

Hence, the result

$$ \begin{aligned} u(t) &= -{}_{RL}I_{0^+}^{q}h(t) + t^{q-1}\left(\frac{{}_{RL}I_{0^+}^{q}h(1) - \sum_{i=1}^{m}\beta_i {}_{RL}I_{0^+}^{q-p_i}h(\eta_i)}{1-\sum_{i=1}^{m}\frac{\beta_i\Gamma(q)}{\Gamma(q-p_i)}\eta_i^{q-p_i-1}}\right) \\ &= \int_0^t\Bigg(-\frac{1}{\Gamma(q)}(t-s)^{q-1} + \frac{t^{q-1}}{1-\sum_{i=1}^{m}\frac{\beta_i\Gamma(q)}{\Gamma(q-p_i)}\eta_i^{q-p_i-1}}\Bigg\{\frac{1}{\Gamma(q)}(1-s)^{q-1} \\ &\quad -\sum_{i=1}^{m}\frac{\beta_i(\eta_i-s)^{q-p_i-1}}{\Gamma(q-p_i)}\Bigg\}\Bigg)h(s)ds \\ &\quad + \int_t^1\Bigg(\frac{t^{q-1}}{1-\sum_{i=1}^{m}\frac{\beta_i\Gamma(q)}{\Gamma(q-p_i)}\eta_i^{q-p_i-1}}\Bigg\{\frac{1}{\Gamma(q)}(1-s)^{q-1} \\ &\quad -\sum_{i=1}^{m}\frac{\beta_i(\eta_i-s)^{q-p_i-1}}{\Gamma(q-p_i)}\Bigg\}\Bigg)h(s)ds \end{aligned} $$

We get:

$$ u(t) = \int_0^1 G(t,s)h(s)ds, $$

where

$$ \begin{aligned} \sigma(s) &= 1-\sum_{s\le\eta_i}\frac{\beta_i\Gamma(q)(\eta_i-s)^{q-p_i-1}(1-s)^{1-q}}{\Gamma(q-p_i)}, \\ \sigma(0) &= 1-\sum_{i=1}^{m}\frac{\beta_i\Gamma(q)(\eta_i)^{q-p_i-1}}{\Gamma(q-p_i)}, \end{aligned} $$

$$G(t,s)=\begin{cases}\frac{1}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(t-s)^{q-1}+\sigma(s)t^{q-1}(1-s)^{q-1}\Big], & 0<s<t<1.\\ \frac{1}{\Gamma(q)\sigma(0)}\Big[\sigma(s)t^{q-1}(1-s)^{q-1}\Big], & 0<t<s<1.\end{cases}$$

□

Lemma 3.3. *Suppose that $\sigma(0)>0$ and $\beta_i\leq 0$ then the function $\sigma(s)>0,\ s\in[0,1]$ and σ is non-decreasing on $[0,1]$.*

Proof. By simple computation, we get that

$$\begin{aligned}\sigma(s) &= 1-\sum_{s\leq\eta_i}\frac{\beta_i\Gamma(q)(\eta_i-s)^{q-p_i-1}(1-s)^{1-q}}{\Gamma(q-p_i)}.\\ \sigma'(s) &= -\sum_{s\leq\eta_i}\frac{\beta_i\Gamma(q)}{\Gamma(q-p_i)}\Big((\eta_i-s)^{q-p_i-1}(q-1)(1-s)^{-q}\\ &\quad +(1-s)^{1-q}(1+p_i-q)(\eta_i-s)^{q-p_i-2}\Big).\end{aligned}$$

This implies that σ is non-decreasing on $[0,1]$ and $\sigma(s)\geq\sigma(0)>0,\ s\in[0,1]$. □

Lemma 3.4. *The function $G(t,s)$, defined by (3.1) admits the following properties*

(1) $G(t,s)\geq 0,\ \frac{\partial}{\partial t}G(t,s)\geq 0$,

(2) $\max\limits_{t\in[0,\kappa]}G(t,s)=\dfrac{1}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(\kappa-s)^{q-1}+\sigma(s)\kappa^{q-1}(1-s)^{q-1}\Big]$,

(3) $G(t,s)\geq\frac{t^{q-1}G(\kappa,s)}{\kappa^{q-1}}$,

where $0\leq s\leq 1$, $0<t<\kappa$ and $\kappa=\frac{s}{1-(1-s)^{\frac{q-1}{q-2}}}$.

Proof. (1) For $0<s\leq t<1$, noticing that

$$\sigma(0)=1-\sum_{s\leq\eta_i}\frac{\beta_i\Gamma(q)(\eta_i)^{q-p_i-1}}{\Gamma(q-p_i)}>0.$$

In view of Lemma 3.3, gives

$$G(t,s)\geq\frac{t^{q-1}\sigma(s)}{\Gamma(q)\sigma(0)}\Big[-(1-\frac{s}{t})^{q-1}+(1-s)^{q-1}\Big]\geq 0.$$

So, for $0<t\leq s<1$, we have $G(t,s)>0$.
By direct computation, we obtain

$$\frac{\partial}{\partial t}G(t,s)=\begin{cases}\frac{1}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(q-1)(t-s)^{q-2}\\ +\sigma(s)(q-1)t^{q-2}(1-s)^{q-1}\Big], & 0<s<t<1\\ \frac{1}{\Gamma(q)\sigma(0)}\Big[\sigma(s)(q-1)t^{q-2}(1-s)^{q-1}\Big], & 0<t<s<1.\end{cases}$$

Obviously, $\frac{\partial}{\partial t}G(t,s)$ is continuous on $[0,1]\times[0,1]$. Let $\kappa=\frac{s}{1-(1-s)^{\frac{q-1}{q-2}}}$ and $0<s\le t\le\kappa\le 1$, we get that

$$\frac{\partial}{\partial t}G(t,s)\ge\frac{\sigma(s)(q-1)t^{q-2}}{\Gamma(q)\sigma(0)}\Big[-(1-\frac{s}{t})^{q-2}+(1-s)^{q-1}\Big]\ge 0. \tag{3.2}$$

Hence, $G(t,s)$ is non-decreasing in the interval $0<s\le t\le\kappa$. In is clear that for $0<t\le s<1$, $\frac{\partial G(t,s)}{\partial t}\ge 0$.

(2) By (3.2), we know that $G(t,s)$ is increasing with respect to t where $0<t<\kappa$. Hence

$$\max_{t\in[0,\kappa]}G(t,s)=G(\kappa,s)=\frac{1}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(\kappa-s)^{q-1}+\sigma(s)\kappa^{q-1}(1-s)^{q-1}\Big],\ 0\le s\le 1.$$

(3) For $0\le s\le t\le\kappa\le 1$, we get that

$$\begin{aligned}G(t,s)&=\frac{1}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(t-s)^{q-1}+\sigma(s)t^{q-1}(1-s)^{q-1}\Big]\\&\ge\frac{t^{q-1}}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(1-\frac{s}{\kappa})^{q-1}+\sigma(s)(1-s)^{q-1}\Big]\\&=\frac{t^{q-1}}{\kappa^{q-1}}G(\kappa,s).\end{aligned}$$

For $0\le t\le s\le 1$, we have that

$$\begin{aligned}G(t,s)&=\frac{1}{\Gamma(q)\sigma(0)}\Big[\sigma(s)t^{q-1}(1-s)^{q-1}\Big]\\&\ge\frac{t^{q-1}}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(1-\frac{s}{\kappa})^{q-1}+\sigma(s)(1-s)^{q-1}\Big]\\&=\frac{t^{q-1}}{\kappa^{q-1}}G(\kappa,s).\end{aligned}$$

Hence, we consider $G(t,s)$ where $0\le t<\kappa$, it follows that $G(t,s)$ is non-decreasing in that interval. □

Let $E=C([0,1],\mathbb{R})$, so that E is a Banach space with norm $\|u\|=\max_{t\in[0,1]}|u(t)|$. Set $P=\{u\in E: u(t)\ge\frac{t^{q-1}}{\kappa^{q-1}}\|u\|,\ t\in[0,\kappa]\}$. Then P is cone in E. Denote $\Omega(d)=\{u\in P:\|u\|<d\}$ and $\partial\Omega(d)=\{u\in P:\|u\|=d\}$. Define the operator $A:P\setminus\{\theta\}\to E$ by

$$(Au)(t)=\int_0^{\kappa}G(t,s)g(t)f(s,u(s))ds,\ t\in[0,\kappa]. \tag{3.3}$$

then the problem (1.3) has solution if and only if the operator A has fixed point. In the next theorem, we present the existence and uniqueness of solutions for problem (1.3) via the Guo-Krasnoselskii fixed point theorem and Boyd and Wong fixed point theorem.

Lemma 3.5. *Suppose that*

$(H1)$ $f\in C([0,\kappa]\times\mathbb{R}^+,\mathbb{R}^+)$ *is continuous.*

$(H2)$ $g\in C((0,\kappa),[0,+\infty))$ *and* g *does not vanish identically on any subinterval*

of $(0, \kappa)$.
(H3) For any positive number $d_1 < d_2$ *there exists a continuous function* $P_{d_1,d_2}(t) : (0,\kappa) \to [0,+\infty)$ *such that*

$$\int_0^\kappa g(t)P_{d_1,d_2}(t)dt < +\infty,\ f(t,u) \leq P_{d_1,d_2}(t),\ 0 < t < \kappa,\ \frac{t^{q-1}}{\kappa^{q-1}}d_1 \leq u \leq d_2,$$

hold and $0 < d_1 < d_2$. *Thus* $A : \overline{\Omega(d_2)} \setminus \Omega(d_1) \to P$ *is completely continuous.*

Proof. For any $u \in \overline{\Omega(d_2)} \setminus \Omega(d_1)$ it follow from (3.3) and Lemma 3.4

$$\begin{aligned}(Au)(t) &= \int_0^\kappa G(t,s)g(s)f(s,u(s))ds \\ &\leq \int_0^\kappa \frac{1}{\Gamma(q)\sigma(0)}\Big[-\sigma(0)(\kappa-s)^{q-1} + \sigma(s)\kappa^{q-1}(1-s)^{q-1}\Big]g(s)f(s,u(s))ds, \\ &\quad t \in [0,\kappa].\end{aligned}$$

So,

$$(Au)(t) \leq \int_0^\kappa G(\kappa,s)g(s)f(s,u(s))ds, \tag{3.4}$$

and

$$\begin{aligned}(Au)(t) &= \int_0^\kappa G(t,s)g(s)f(s,u(s))ds \\ &\geq \int_0^\kappa \frac{t^{q-1}}{\Gamma(q)\sigma(0)\kappa^{q-1}}\Big[-\sigma(0)(\kappa-s)^{q-1} + \sigma(s)\kappa^{q-1}(1-s)^{q-1}\Big]g(s)f(s,u(s))ds, \\ &\quad t \in [0,\kappa].\end{aligned}$$

So,

$$(Au)(t) \geq \int_0^\kappa \frac{t^{q-1}}{\kappa^{q-1}}G(\kappa,s)g(s)f(s,u(s))ds, \tag{3.5}$$

which (3.4) and (3.5) we know that

$$(Au)(t) \geq \frac{t^{q-1}}{\kappa^{q-1}}\|Au\|,\ t \in [0,\kappa],$$

which mean that $A : \overline{\Omega(d_2)} \setminus \Omega(d_1) \to P$. Noticing the continuity of $G(t,s)$ and $(H1)-(H2)$, it is easy to see that A is continuous in $\overline{\Omega(d_2)} \setminus \Omega(d_1)$. Next, we show A is completely continuous. For any $u \in \overline{\Omega(d_2)} \setminus \Omega(d_1)$, we have

$$\begin{aligned}|(Au)(t)| &= \Big|\int_0^\kappa G(t,s)g(s)f(s,u(s))ds\Big| \\ &\leq \int_0^\kappa G(\kappa,s)g(s)P_{d_1,d_2}(s)ds := M,\end{aligned}$$

for $t \in [0,\kappa]$, which implies that $A(\overline{\Omega(d_2)} \setminus \Omega(d_1))$ is uniformly bounded. Because $G(t,s)$ is uniformly continuous on $[0,\kappa]\times[0,1]$. Thus, for any $\epsilon > 0$, there exists $\delta > 0$

such that $|\tau_2-\tau_1|< \delta$ implies $|G(\tau_2, s)-G(\tau_1, s)|< \epsilon$ for all $(\tau_1, s), (\tau_2, s) \in [0, \kappa]\times[0, 1]$. Then, for any $u \in \overline{\Omega(d_2)} \setminus \Omega(d_1)$ and $\tau_1, \tau_2 \in [0, \kappa]$ such that $|\tau_2 - \tau_1|< \delta$, we have

$$\begin{aligned}|(Au)(\tau_2) - (Au)(\tau_1)| &= \Big|\int_0^\kappa \Big(G(\tau_2, s) - G(\tau_1, s)\Big)g(s)f(s, u(s))ds\Big| \\ &\le \epsilon\int_0^\kappa g(s)f(s, u(s))ds.\end{aligned}$$

We can see that the function in $A(\overline{\Omega(d_2)} \setminus \Omega(d_1))$ and equicontinuous.
So, $A(\overline{\Omega(d_2)} \setminus \Omega(d_1))$ is relative compact in $C[0, \kappa]$.
Thus $A : \overline{\Omega(d_2)} \setminus \Omega(d_1) \to P$ is completely continuous. □

Existence Result Via Guo-Krasnoselskii Fixed Point Theorem.

We introduce the following height functions to control the growth of the non-linear term $f(t, u)$. Let

$$\phi(t, d) = \max\{f(t, u) : \frac{t^{q-1}}{\kappa^{q-1}}d \le u \le d\},\ 0 < t < \kappa,\ d > 0$$

$$\varphi(t, d) = \min\{f(t, u) : \frac{t^{q-1}}{\kappa^{q-1}}d \le u \le d\},\ 0 < t < \kappa,\ d > 0.$$

Theorem 3.6. *Suppose the (H1)-(H3) hold and there exists two positive number $a < b$ such that one of the following condition is satisfied:*
$(H4)$ $a \le \int_0^\kappa G(\kappa, s)g(s)\varphi(s, a)ds < +\infty$ and $\int_0^\kappa G(\kappa, s)g(s)\phi(s, b)ds \le b$,
$(H5)$ $\int_0^\kappa G(\kappa, s)g(s)\phi(s, a)ds \le a$ and $b \le \int_0^\kappa G(\kappa, s)g(s)\varphi(s, b)ds < +\infty$.
Then the boundary value problem (1.3) has at least one non-decreasing positive solution $u^* \in P$ such that $a \le \|u^*\|\le b$.

Proof. Now, we prove $(H4)$,

- If $u \in \partial\Omega(a)$, then $\|u\|= a$ and $\frac{t^{q-1}}{\kappa^{q-1}}a \le u(t) \le a,\ 0 \le t \le \kappa$. By then definition of $\varphi(t, a)$, we know that

$$f(t, u(t)) \ge \varphi(t, a),\ 0 < t < \kappa, \tag{3.6}$$

by (3.6) and Lemma 3.4, we have

$$\begin{aligned}\|(Au)\| &= \max_{t\in[0,\kappa]}\int_0^\kappa G(t, s)g(s)f(s, u(s))ds \\ &\ge \max_{t\in[0,\kappa]}\int_0^\kappa \frac{t^{q-1}G(\kappa, s)}{\kappa^{q-1}}g(s)f(s, u(s))ds \\ &\ge \int_0^\kappa G(\kappa, s)g(s)\varphi(s, a)ds \ge a = \|u\|.\end{aligned}$$

So, $\|Au\|\ge \|u\|$ for $u \in P \cap \partial\Omega(a)$.

- If $u \in \partial\Omega(b)$, then $\|u\|= b$ and $\frac{t^{q-1}}{\kappa^{q-1}}b \le u(t) \le b,\ 0 \le t \le \kappa$. By then definition of $\phi(t, b)$, we get that

$$f(t, u(t)) \le \phi(t, b),\ 0 < t < \kappa, \tag{3.7}$$

by (3.7) and Lemma 3.4, we have

$$\begin{aligned}\|(Au)\| &= \max_{t\in[0,\kappa]}\int_0^\kappa G(t,s)g(s)f(s,u(s))ds\\ &\leq \int_0^\kappa G(\kappa,s)g(s)f(s,u(s))ds\\ &\leq \int_0^\kappa G(\kappa,s)g(s)\phi(s,b)ds \leq b = \|u\|.\end{aligned}$$

So, $\|Au\|\leq \|u\|$ for $u \in P\cap\partial\Omega(b)$. Next, we prove $(H5)$:

• If $u \in \partial\Omega(a)$, then $\|u\|= a$ and $\frac{t^{q-1}}{\kappa^{q-1}}a \leq u(t) \leq a,\ 0 \leq t \leq \kappa$. By then definition of $\phi(t,a)$, we get

$$f(t,u(t)) \leq \phi(t,a),\ 0 < t < \kappa, \tag{3.8}$$

by (3.8) and Lemma 3.4, we have

$$\begin{aligned}\|(Au)\| &= \max_{t\in[0,\kappa]}\int_0^\kappa G(t,s)g(s)f(s,u(s))ds\\ &\leq \int_0^\kappa G(\kappa,s)g(s)f(s,u(s))ds\\ &\leq \int_0^\kappa G(\kappa,s)g(s)\phi(s,a)ds \leq a = \|u\|.\end{aligned}$$

So, $\|Au\|\leq \|u\|$ for $u \in P\cap\partial\Omega(a)$.

• If $u \in \partial\Omega(b)$, then $\|u\|= b$ and $\frac{t^{q-1}}{\kappa^{q-1}}b \leq u(t) \leq b,\ 0 \leq t \leq \kappa$. By then definition of $\varphi(t,b)$, we get

$$f(t,u(t)) \geq \varphi(t,b),\ 0 < t < \kappa, \tag{3.9}$$

and in view of (3.9) and Lemma 3.4, we have

$$\begin{aligned}\|(Au)\| &= \max_{t\in[0,\kappa]}\int_0^\kappa G(t,s)g(s)f(s,u(s))ds\\ &\geq \max_{t\in[0,\kappa]}\int_0^\kappa \frac{t^{q-1}G(\kappa,s)}{\kappa^{q-1}}g(s)f(s,u(s))ds\\ &\geq \int_0^\kappa G(\kappa,s)g(s)\varphi(s,b)ds \geq b = \|u\|.\end{aligned}$$

So, $\|Au\|\geq \|u\|$ for $u \in P\cap\partial\Omega(b)$. By Guo-Krasnoselskii fixed point theorem A has a fixed point $u^* \in \overline{\Omega(b)}\setminus\Omega(a)$. Since u^* is a solution of (1.3) and $a \leq \|u^*\|\leq b$. It is not difficult to see that, $u^* \geq \frac{t^{q-1}}{\kappa^{q-1}}\|u^*\|\geq \frac{t^{q-1}}{\kappa^{q-1}}a > 0$, $0 \leq t \leq \kappa$, which implies that u^* is a positive solution for (1.3). From Lemma 3.4 we have

$$\begin{aligned}(u^*)' &= (Au^*)'(t)\\ &\geq \int_0^\kappa \frac{\partial}{\partial t}G(t,s)g(s)f(s,u^*(s))ds \geq 0,\ t\in[0,\kappa],\end{aligned}$$

which shows that u^* is a non-decreasing positive solution on $[0,\kappa]$. □

Existence and Uniqueness Result Via Boyd and Wong Fixed Point Theorem.

Theorem 3.7. *Let* $f : [0,\kappa] \times \mathbb{R}^+ \to \mathbb{R}^+$ *be a continuous function satisfying the assumption:*
$(H6)$ $|f(t,u) - f(t,v)| \leq \frac{\theta(t)|u-v|}{\Theta+|u-v|}$, *for* $t \in [0,\kappa],\ u,v \geq 0$,
where $\theta(t) : [0,\kappa] \to \mathbb{R}^+$ *is continuous and* Θ *the constant defined by*

$$\Theta := \int_0^\kappa G(t,s)g(s)\theta(s)ds \neq 0.$$

Then the problem (1.3) *has a unique solution on* $[0,\kappa]$.

Proof. Consider a non-decreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ define by

$$\Psi(\epsilon) = \frac{\Theta\epsilon}{\Theta+\epsilon},\ \forall\epsilon > 0,$$

such that $\Psi(0) = 0$ and $\Psi(\epsilon) < \epsilon,\ \forall\epsilon > 0$. For any $u, v \in E$ and for each $t \in [0,\kappa]$, yields:

$$\begin{aligned} |Au(t) - Av(t)| &\leq \left| \int_0^\kappa G(t,s)g(s)\Big(f(s,u(s)) - f(s,v(s))\Big)ds \right| \\ &\leq \int_0^\kappa G(t,s)g(s)\frac{\theta(s)|u-v|}{\Theta+|u-v|}ds \\ &\leq \frac{\Psi(\|u-v\|)}{\Theta}\int_0^\kappa G(t,s)g(s)\theta(s)ds \\ &\leq \Psi(\|u-v\|). \end{aligned}$$

This implies that $\|Au - Av\| \leq \Psi(\|u-v\|)$. There for A is a non-linear contraction. Hence, by theorem (Boyd and Wong). The operator A has a unique fixed point, which is the unique solution of the problem (1.3). ∎

Example 3.8. Consider the following fractional boundary value problems

$$\begin{cases} {}_{RL}D_{0^+}^{\frac{8}{3}}u(t) + \frac{1}{1-t}\Big(2 - \frac{2}{u+1} + \sqrt{t}\Big) = 0,\ t \in (0,1), \\ u(0) = u'(0) = 0, \\ u(1) = -\frac{1}{2}\,{}_{RL}D_{0^+}^{\frac{5}{2}}u(\frac{1}{5}) - \frac{1}{3}\,{}_{RL}D_{0^+}^{\frac{7}{3}}u(\frac{3}{10}) - \frac{1}{4}\,{}_{RL}D_{0^+}^{\frac{9}{4}}u(\frac{2}{5}). \end{cases} \tag{3.10}$$

By comparing problem (1.3) and (3.10), we obtain the following parameters:

$$q = \frac{8}{3},\ p_1 = \frac{5}{2},\ p_2 = \frac{7}{3},\ p_3 = \frac{9}{4},\ f(t,u) = 2 - \frac{2}{u+1} + \sqrt{t},\ g(t) = \frac{1}{1-t},$$

$$\beta_1 = -\frac{1}{2},\ \beta_2 = -\frac{1}{3},\ \beta_3 = -\frac{1}{4},\ \eta_1 = \frac{1}{5},\ \eta_2 = \frac{3}{10},\ \eta_3 = \frac{2}{5},$$

we choose $\kappa = 1$, obvious $f \in C((0,1) \times \mathbb{R}^+, \mathbb{R}^+)$ and $g \in C((0,1), \mathbb{R}^+)$. For and positive number $d_1 < d_2$, we can see that $(H1) - (H3)$ hold for $P_{d_1,d_2}(t) = 3 - \frac{2}{d_2+1}$.

$$\phi(t,d) = \max\{2 - \frac{2}{u+1} + \sqrt{t} : t^{\frac{5}{3}}d < u < d\} \leq 3 - \frac{2}{d+1},$$

$$\varphi(t,d)=\min\{2-\frac{2}{u+1}+\sqrt{t}:t^{\frac{5}{3}}d<u<d\}\geq 3-\frac{2}{t^{\frac{5}{3}}d+1}.$$

It follow that $\int_0^1 G(1,s)g(s)\phi(s,b)ds\leq b$.
So,

$$\begin{aligned}\int_0^1 G(1,s)g(s)\phi(s,1)ds &\leq \frac{1}{\Gamma(\frac{8}{3})\sigma(0)}\int_0^1\Big[-\sigma(0)(1-s)^{\frac{5}{3}}+\sigma(s)(1-s)^{\frac{5}{3}}\Big]\frac{2}{1-s}ds\\ &\leq \frac{2}{\Gamma(\frac{8}{3})\sigma(0)}\int_0^1(1-s)^{\frac{2}{3}}ds\\ &\leq 0.3572<1,\end{aligned}$$

and it follows that $\int_0^1 G(1,s)g(s)\varphi(s,a)ds\geq a$. Thus,

$$\begin{aligned}\int_0^1 G(1,s)g(s)\varphi(s,0.1)ds &\geq \frac{1}{\Gamma(\frac{8}{3})\sigma(0)}\int_0^1\Big[-\sigma(0)(1-s)^{\frac{5}{3}}\\ &\quad+\sigma(s)(1-s)^{\frac{5}{3}}\Big]\frac{1}{1-s}(3-\frac{2}{\frac{s^{\frac{5}{3}}}{10}+1})ds\\ &\geq \frac{1}{\Gamma(\frac{8}{3})\sigma(0)}\int_0^1(-\sigma(0)+\sigma(s))(1-s)^{\frac{2}{3}}ds\\ &\geq \frac{1}{\Gamma(\frac{8}{3})\sigma(0)}\int_0^1(-\sigma(1)+\sigma(0))(1-s)^{\frac{2}{3}}ds\\ &\geq 0.2208>0.1.\end{aligned}$$

According to Theorem 3.6, we get that (3.10), has at least one non-decreasing positive solution u^* and $\frac{1}{10}\leq\|u^*\|\leq 1$. Hence, if we choose $\theta(t)=2$. We find

$$\Theta=\int_0^1 G(t,s)g(s)\theta(s)\approx 0.3572.$$

Clealy,

$$\begin{aligned}|f(t,u)-f(t,v)| &= \Big|-\frac{2}{u+1}-\frac{2}{v+1}\Big|\\ &\leq \frac{2|u-v|}{|1+u+v+uv|}\\ &\leq \frac{2|u-v|}{0.3572+|u-v|}.\end{aligned}$$

Hence, by Theorem 3.7, problem (3.10) has a unique solution on $(0,1)$.

4. Conclusions

In this paper, we proved the existences and uniqueness of positive solution of fractional differential equation with nonlocal multi-point condition by the fixed point theorems of Guo-Krasnoselskii, Boyd and Wong non-linear contraction in the interval such that the Green's function is non-decreasing. Finally some example are provided to illustrate our result.

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