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SOME COMMON BEST PROXIMITY POINT THEOREMS VIA A FIXED POINT THEOREM IN METRIC SPACES

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Abstract. We prove that two common best proximity point theorems proved by Sadiq Basha [4] and by Mongkolkeha and Kumam [10] can be regarded as a direct consequence of Browder's fixed point theorem [5]. Moreover, the assumptions imposed in their results can be relaxed. We also present some supplementary results and some examples.
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1. Introduction

Suppose that X is a nonempty set and $S : X \to X$ a mapping. We say that an element $x \in X$ is a *fixed point* of S if $x = Sx$. If S has no fixed point, then it is interesting to find a point x in X such that x and Sx are very closed in some sense. In 2011, Basha [2] proposed the concept of a best proximity point of $S : A \to B$ where A, B are two nonempty subsets of a metric space (X, d), that is, $x \in A$ is a *best proximity point* of S if $d(x, Sx) = \inf\{d(a, b) : a \in A \text{ and } b \in B\} =: d(A, B)$. Motivated by the common fixed point theorem for two mappings proved by Jungck [7], Sadiq Basha [4] and Mongkolkeha and Kumam [10] presented the analogous results for best proximity points. In this paper, we show that both results of Sadiq Basha [4] and of Mongkolkeha and Kumam [10] are a consequence of a classical fixed point theorem of Browder [5]. Using our approach, we also show that some assumptions in their results can be weaken. Moreover, we provide another result for the existence of a common best proximity point under other assumptions. Some examples for our supplement result are illustrated.

2. Preliminaries and known results

The following is known as Banach contraction theorem.

Theorem B ([1]). *Let (X, d) be a complete metric space and $\alpha \in [0, 1)$. Let $S : X \to X$ be a mapping such that*

$$d(Sx, Sy) \leq \alpha d(x, y) \text{ for all } x, y \in X.$$

Then S has a unique fixed point $\widehat{x}$ in X and $\lim_{n\to\infty} S^n x = \widehat{x}$ for all $x \in X$.

In 1968, Browder [5] generalized Theorem B in the following way.

Theorem Br ([5]). *Let (X, d) be a complete metric space and let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing and right continuous function with $\psi(t) < t$ for all $t > 0$. Let $S : X \to X$ be a mapping satisfying that*

$$d(Sx, Sy) \leq \psi(d(x, y)) \text{ for all } x, y \in X.$$

Then S has a unique fixed point $\widehat{x}$ in X and $\lim_{n\to\infty} S^n x = \widehat{x}$ for all $x \in X$.

Remark 2.1. Theorem Br contains Theorem B as a special case. In fact, we set $\psi(t) \equiv \alpha t$.

In 1976, Jungck [7] proved the following theorem.

Theorem J. *Let (X, d) be a complete metric space and $\alpha \in [0, 1)$. Let $S, T : X \to X$ be mappings satisfying the following conditions:*

- *$d(Sx, Sy) \leq \alpha d(Tx, Ty)$ for all $x, y \in X$;*
- *$S(X) \subset T(X)$;*
- *S and T commute, that is, $STx = TSx$ for all $x \in X$;*
- *T is continuous (and hence S is continuous).*

Then S and T have a unique common fixed point, that is, there exists a unique element $\widehat{x} \in X$ such that $\widehat{x} = S\widehat{x} = T\widehat{x}$.

Remark 2.2. Theorem J contains Theorem B as a special case. In fact, we set $Tx \equiv x$.

Let (X, d) be a metric space and let $S : A \to B$ be a mapping where A and B are two nonempty subsets of X. Instead of finding a fixed point of S, we now find an element $x \in A$ such that

$$d(x, Sx) = \inf\{d(x, y) : x \in A \text{ and } y \in B\} =: d(A, B).$$

Such a point x is called a *best proximity point* of S. Suppose that $T : A \to B$ is another mapping. An element $x \in A$ such that

$$d(x, Sx) = d(x, Tx) = d(A, B)$$

is called a *common best proximity point* of S and T. Note that a (common) best proximity point of S (and T) becomes a (common) fixed point of S (and T) if $A = B$.

The following concept is an analogue of commutativity in the context of nonself-mappings.

Definition 2.3 ([3])**.** Let (X, d) be a metric space and let A and B be two nonempty subsets of X. We say that two mappings $S, T : A \to B$ *proximally commute* if

$$d(u, Sx) = d(v, Tx) = d(A, B) \Rightarrow Sv = Tu$$

for all $x, u, v \in A$.

Note that, if $A = B$ in the preceding definition, then the concept of proximal commutativity reduces to that of commutativity.

In 2013, Sadiq Basha [4] extended Theorem J for non-self mappings as follows.

Theorem S-B ([4])**.** *Let (X, d) be a complete metric space and let A and B be two nonempty closed subsets of X. Set*

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Let $S, T : A \to B$ be two mappings satisfying the following conditions:

- *there is a constant $\alpha \in (0, 1)$ such that*

$$\left.\begin{array}{l} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) \\ d(v_1, Tx_1) = d(v_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow d(u_1, u_2) \leq \alpha d(v_1, v_2)$$

  for all $u_1, u_2, v_1, v_2, x_1, x_2 \in A$;
- *S and T proximally commute;*
- *S and T are continuous;*
- *A_0 is nonempty and closed;*
- *$S(A_0) \subset T(A_0) \cap B_0$.*

Then there exists a unique common best proximity point of S and T.

Remark 2.4. The closedness of A_0 is missing from the statement of the original version of Theorem S-B in [4]. Moreover, in his proof, this assumption is needed. Hence the result stated above is a corrected one.

In 2013, Mongkolkeha and Kumam [10] proved the following theorem which is another extension of Theorem J. To state their result, we recall the following two definitions.

Definition 2.5 ([3])**.** Let (X, d) be a metric space and let A and B be two nonempty subsets of X. We say that A is *approximatively compact with respect to B* if every sequence $\{x_n\}$ in A satisfying the condition that $\lim_{n\to\infty} d(y, x_n) = d(y, A)$ for some $y \in B$ has a convergent subsequence.

Definition 2.6 ([3])**.** Let (X, d) be a metric space and let A and B be two nonempty subsets of X. We say that the mappings $S, T : A \to B$ can be *swapped proximally* if

$$\left.\begin{array}{c} d(u, y) = d(v, y) = d(A, B) \\ Su = Tv \end{array}\right\} \Rightarrow Sv = Tu$$

for all $u, v \in A$ and $y \in B$.

Theorem MK ([10])**.** *Let (X,d) be a complete metric space and let A and B be two nonempty closed subsets of X. Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. Let $S, T : A \to B$ be two mappings satisfying the following conditions:*

- *$d(Sx, Sy) \leq d(Tx, Ty) - \varphi(d(Tx, Ty))$ for all $x, y \in A$;*
- *T is continuous;*
- *S and T proximally commute;*
- *S and T can be swapped proximally;*
- *A is approximatively compact with respect to B*
- *A_0 is nonempty;*
- *$S(A_0) \subset T(A_0) \cap B_0$.*

Then there is a common best proximity point x of S and T. Moreover, if x^* is another common best proximity point of S and T, then $d(x, x^*) \leq 2d(A,B)$.

3. Main results

Inspired by the work of Jungck and Rhoades [9], we define the following concepts for nonself mappings.

Definition 3.1. Let (X,d) be a metric space and let A and B be two nonempty subsets of X. We say that two mappings $S, T : A \to B$ are

- *proximally compatible* if whenever $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ are sequences in A satisfying $d(u_n, Sx_n) = d(v_n, Tx_n) = d(A,B)$ for all $n \geq 1$ and one of the following conditions holds
  (i) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = w$ for some $w \in A$;
  (ii) $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \widehat{w}$ for some $\widehat{w} \in B$,
  it follows that
  $$\lim_{n\to\infty} d(Sv_n, Tu_n) = 0;$$
- *weakly proximally compatible* if
  $$\left.\begin{array}{c} d(u, Sx) = d(u, Tx) = d(A,B) \\ Sx = Tx \end{array}\right\} \Rightarrow Su = Tu$$
  for all $x, u \in A$.

Remark 3.2.

- Proximal commutativity implies proximal compatibility.
- Proximal compatibility implies weakly proximal compatibility.
- For self-mappings, proximal compatibility (weakly proximal compatibility, resp.) reduces to compatibility (weak compatibility, resp.). Recall that the two mappings $S, T : X \to X$ are
  (i) *compatible* [8] if
  $$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x \Rightarrow \lim_{n\to\infty} d(STx_n, TSx_n) = 0;$$
  (ii) *weakly compatible* [9] if $Sx = Tx \Rightarrow TSx = STx$.

3.1. **Auxiliary result.** We start with the following three lemmas which play a crucial role in this paper.

Lemma 3.3. *Let (X, d) be a metric space and let A and B be two nonempty subsets of X. Let $S, T : A \to B$ be two mappings. Suppose that*

- *S and T are weakly proximally compatible;*
- *$S(A_0) \subset B_0$;*
- *there exists $x \in A_0$ such that $Sx = Tx$.*

Assume that one of the following conditions holds:

(a) *for all $u_1, u_2, v_1, v_2, x_1, x_2 \in A$ with $v_1 \neq v_2$,*

$$\left.\begin{array}{l} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) \\ d(v_1, Tx_1) = d(v_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow d(u_1, u_2) < d(v_1, v_2);$$

(b) *for all $x_1, x_2 \in A$ with $Tx_1 \neq Tx_2$*

$$d(Sx_1, Sx_2) < d(Tx_1, Tx_2).$$

Then S and T have a common best proximity point.

Proof. Since $Sx \in S(A_0) \subset B_0$, there is an element $u \in A_0$ such that

$$d(u, Sx) = d(u, Tx) = d(A, B).$$

Then $Tu = Su$ because S and T are weakly proximally compatible. We prove that u is a common best proximity point of S and T.
Case 1: Assume that the condition (a) holds. Since $Su \in S(A_0) \subset B_0$, there is an element $z \in A_0$ such that

$$d(z, Su) = d(z, Tu) = d(A, B).$$

If $u \neq z$, then we have that $d(u, z) < d(u, z)$ by (a) which is a contradiction. Thus $u = z$ and hence $d(u, Su) = d(u, Tu) = d(A, B)$.
Case 2: Assume that the condition (b) holds. If $Tx \neq Tu$, then

$$d(Tx, Tu) = d(Sx, Su) < d(Tx, Tu)$$

which is a contradiction. This implies that $Tx = Tu$ and hence $Sx = Su$. Then

$$d(u, Su) = d(u, Tu) = d(A, B).$$

This completes the proof. □

Lemma 3.4. *Let (X, d) be a complete metric space and let Y be a subset of X. Suppose that $U : Y \to Y$ is a mapping such that it preserves Cauchy sequences, that is, if a sequence $\{x_n\} \subset Y$ is Cauchy, then so is $\{Ux_n\}$. Then there exists an extension $\boldsymbol{U} : \overline{Y} \to \overline{Y}$ of U such that for each $\overline{x} \in \overline{Y}$,*

$$\boldsymbol{U}\overline{x} = \lim_{n\to\infty} Ux_n$$

where $\{x_n\}$ is a sequence in Y such that $\lim_{n\to\infty} x_n = \overline{x}$.

Proof. We assume that $\{x_n\}$ and $\{x'_n\}$ are two sequences in Y such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x'_n = x$ for some $x \in \overline{Y}$. Set $y_n := x_n$ if n is odd and $y_n := x'_n$ if n is even. Note that $\{x_n\}$, $\{x'_n\}$, and $\{y_n\}$ are Cauchy sequences. It follows that $\{Ux_n\}$, $\{Ux'_n\}$, and $\{Uy_n\}$ are all Cauchy sequences. Note that $\{Uy_n\}$ is a subsequence of $\{Ux_n\}$ and of $\{Ux'_n\}$. It follows from the completeness of $\overline{Y}$ that $\lim_{n\to\infty} Ux_n = \lim_{n\to\infty} Ux'_n$. So we can define a mapping $\boldsymbol{U} : \overline{Y} \to \overline{Y}$ by for each $\overline{x} \in \overline{Y}$, $\boldsymbol{U}\overline{x} = \lim_{n\to\infty} Ux_n$ where $\{x_n\}$ is a sequence in Y such that $\lim_{n\to\infty} x_n = \overline{x}$. The proof is finished. □

Lemma 3.5. *Let (X, d) be a complete metric space and let Y be a subset of X. Suppose that $U : Y \to Y$ is a mapping satisfying*

$$d(Ux, Uy) \le \psi(d(x, y)) \text{ for all } x, y \in Y$$

where $\psi : [0,\infty) \to [0,\infty)$ is upper semicontinuous with $\lim_{t\to 0^+} \psi(t) = 0$. Then there exists a unique extension $\boldsymbol{U} : \overline{Y} \to \overline{Y}$ of U such that

$$d(\boldsymbol{U}\overline{x}, \boldsymbol{U}\overline{y}) \le \psi(d(\overline{x}, \overline{y}))$$

for all $\overline{x}, \overline{y} \in \overline{Y}$.

Proof. It is clear that the mapping U preserves Cauchy sequences. By Lemma 3.4, there exists an extension $\boldsymbol{U} : \overline{Y} \to \overline{Y}$ of U. In fact, for each $\overline{x} \in \overline{Y}$, $\boldsymbol{U}\overline{x} = \lim_{n\to\infty} Ux_n$ where $\{x_n\}$ is a sequence in Y such that $\lim_{n\to\infty} x_n = \overline{x}$. Moreover, let $\overline{x}, \overline{y} \in \overline{Y}$ together with two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n\to\infty} x_n = \overline{x} \text{ and } \lim_{n\to\infty} y_n = \overline{y}.$$

This implies that $\lim_{n\to\infty} d(x_n, y_n) = d(\overline{x}, \overline{y})$ and $\lim_{n\to\infty} d(Ux_n, Uy_n) = d(\boldsymbol{U}\overline{x}, \boldsymbol{U}\overline{y})$. Therefore,

$$d(\boldsymbol{U}\overline{x}, \boldsymbol{U}\overline{y}) = \lim_{n\to\infty} d(Ux_n, Uy_n) \le \limsup_{n\to\infty} \psi(d(x_n, y_n)) \le \psi(d(\overline{x}, \overline{y})).$$

To prove the uniqueness, let $V : \overline{Y} \to \overline{Y}$ be an extension of U such that $d(V\overline{x}, V\overline{y}) \le \psi(d(\overline{x}, \overline{y}))$ for all $\overline{x}, \overline{y} \in \overline{Y}$. We show that $V\overline{x} = \boldsymbol{U}\overline{x}$ for all $\overline{x} \in \overline{Y}$. To see this, let $\overline{x} \in \overline{Y}$ and $\{x_n\} \subset Y$ such that $\lim_{n\to\infty} x_n = \overline{x}$. It follows that

$$d(V\overline{x}, Ux_n) = d(V\overline{x}, Vx_n) \le \psi(d(\overline{x}, x_n)).$$

This implies that $V\overline{x} = \lim_{n\to\infty} Ux_n = \boldsymbol{U}\overline{x}$. This completes the proof. □

3.2. **Theorem S-B is a consequence of Theorem Br.**

Theorem 3.6. *Let (X, d) be a complete metric space and let A and B be two nonempty subsets of X. Let $\psi : [0,\infty) \to [0,\infty)$ be a nondecreasing and right continuous function such that $\psi(t) < t$ for all $t > 0$. Let $S, T : A \to B$ be two mappings satisfying the following conditions:*

- *for all* $u_1, u_2, v_1, v_2, x_1, x_2 \in A$,
$$\left.\begin{array}{l} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A,B) \\ d(v_1, Tx_1) = d(v_2, Tx_2) = d(A,B) \end{array}\right\} \Rightarrow d(u_1, u_2) \le \psi(d(v_1, v_2));$$
- S *and* T *are proximally compatible;*
- S *and* T *are continuous;*
- A_0 *is nonempty and closed.*
- $S(A_0) \subset T(A_0) \cap B_0$.

Then S *and* T *have a unique common best proximity point.*

Proof. We define
$$Y := \{x \in A_0 : d(x, Tw) = d(A,B) \text{ for some } w \in A_0\}.$$
Note that Y is nonempty because $\varnothing \ne S(A_0) \subset T(A_0) \cap B_0$. For $x \in Y$, we suppose that there are two elements $w_1, w_2 \in A_0$ such that
$$d(x, Tw_1) = d(x, Tw_2) = d(A,B).$$
Since $Sw_1, Sw_2 \in S(A_0) \subset B_0$, there are two elements $y_1, y_2 \in A_0$ such that
$$d(y_1, Sw_1) = d(y_2, Sw_2) = d(A,B).$$
It follows that
$$d(y_1, y_2) \le \psi(d(x,x)) = 0.$$
That is, $y_1 = y_2$.

Using this observation, we define a self-mapping $U : Y \to Y$ as follows: for each $x \in Y$,
$$Ux := y$$
where y is the element in A_0 such that
$$d(y, Sw) = d(x, Tw) = d(A,B)$$
for some $w \in A_0$. Obviously, every common best proximity point of S and T is a fixed point of U.

We claim that $d(Ux, Ux') \le \psi(d(x,x'))$ for all $x, x' \in Y$. To see this, let $x, x' \in Y$. We assume that there are two elements $w, w' \in A_0$ such that
$$d(Ux, Sw) = d(x, Tw) = d(Ux', Sw') = d(x', Tw') = d(A,B).$$
It follows that
$$d(Ux, Ux') \le \psi(d(x,x')).$$
Note that ψ is upper semicontinuous and $\lim_{t\to 0^+} \psi(t) = 0$. Now, we apply Lemma 3.5 to obtain the extension $\boldsymbol{U} : \overline{Y} \to \overline{Y}$ of U. Note that
$$d(\boldsymbol{U}\overline{x}, \boldsymbol{U}\overline{y}) \le \psi(d(\overline{x}, \overline{y}))$$
for all $\overline{x}, \overline{y} \in \overline{Y}$. Note that $\overline{Y}$ is complete. As a consequence of Theorem Br, there exists a unique fixed point z of $\boldsymbol{U}$. Then there exists a sequence $\{z_n\}$ in Y such that $\lim_{n\to\infty} z_n = z$. Note that
$$\lim_{n\to\infty} Uz_n = \boldsymbol{U}z = z = \lim_{n\to\infty} z_n.$$

Since $\{z_n\}$ is a sequence in Y, for each $n \geq 1$, there exists an element $w_n \in A_0$ such that

$$d(z_n, Tw_n) = d(Uz_n, Sw_n) = d(A, B).$$

Since S and T are continuous,

$$\lim_{n\to\infty} Sz_n = Sz \text{ and } \lim_{n\to\infty} TUz_n = Tz.$$

Since S and T are proximally compatible,

$$Sz = \lim_{n\to\infty} Sz_n = \lim_{n\to\infty} TUz_n = Tz.$$

Note that $z \in A_0$ because A_0 is closed. By using Lemma 3.3, there is an element $\widehat{z} \in A$ such that

$$d(\widehat{z}, S\widehat{z}) = d(\widehat{z}, T\widehat{z}) = d(A, B).$$

In particular, $\widehat{z}$ is a fixed point of $\boldsymbol{U}$. Since $\boldsymbol{U}$ has a unique fixed point, $z = \widehat{z}$. Hence the uniqueness of a common best proximity point of S and T follows. This completes the proof. □

Remark 3.7. Our Theorem 3.6 extends Theorem S-B in the following ways.

- The term $\alpha d(v_1, v_2)$ where $\alpha \in (0, 1)$ is relaxed to $\psi(d(v_1, v_2))$ where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing and right continuous function with $\psi(t) < t$ for all $t > 0$.
- The proximal commutativity is relaxed to the proximal compatibility.

Remark 3.8. Using the same method of the proof of Theorem 3.6, we can show that Theorem J even with a weaker assumption of commutativity is a consequence of Theorem B. Haghi, et al. [6, Theorem 2.4] proved that Theorem J where the continuity of T is replaced by the closedness of $T(X)$ is a consequence of Theorem B. It is worth mentioning that the technique used here is totally different from the one used in [6].

We discuss the following variants of Theorem 3.6 where the continuities of S and T are dropped. These result can be regarded as supplementary results of Theorem 3.6 (and hence Theorem S-B).

Theorem 3.9. *Let (X, d) be a complete metric space and let A and B be two nonempty subsets of X. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing and right continuous function such that $\psi(t) < t$ for all $t > 0$. Let $S, T : A \to B$ be two mappings satisfying the following conditions:*

- *for all $u_1, u_2, v_1, v_2, x_1, x_2 \in A$,*
$$\left.\begin{array}{l} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) \\ d(v_1, Tx_1) = d(v_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow d(u_1, u_2) \leq \psi(d(v_1, v_2));$$
- *S and T are proximally compatible;*
- *A is closed;*
- *$A_0 \neq \varnothing$ and $T(A_0)$ is compact;*
- *$S(A_0) \subset T(A_0) \cap B_0$.*

Then S and T have a unique common best proximity point.

Theorem 3.10. *Let (X, d) be a complete metric space and let A and B be two nonempty subsets of X. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing and right continuous function such that $\psi(t) < t$ for all $t > 0$. Let $S, T : A \to B$ be two mappings satisfying the following conditions:*

- *for all $u_1, u_2, v_1, v_2, x_1, x_2 \in A$,*
$$\left.\begin{array}{l} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) \\ d(v_1, Tx_1) = d(v_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow d(u_1, u_2) \le \psi(d(v_1, v_2));$$
- *S and T are proximally compatible;*
- *A is closed;*
- *B is approximatively compact with respect to A;*
- *$A_0 \neq \varnothing$ and $T(A_0)$ is closed;*
- *$S(A_0) \subset T(A_0) \cap B_0$.*

Then S and T have a unique common best proximity point.

The proofs of the preceding two theorems are based on the following lemma.

Lemma 3.11. *Let (X, d) be a metric space and let A and B be two nonempty subsets of X such that A is closed. Let $T : A \to B$ be a nonself mapping. Suppose that $T(A_0) \cap B_0 \neq \varnothing$. Assume that one of the followings is satisfied:*

- *$T(A_0)$ is compact;*
- *$T(A_0)$ is closed and B is approximatively compact with respect to A.*

Then $Y := \{x \in A : d(x, Tw) = d(A, B)$ for some $w \in A_0\}$ is closed.

Proof. Note that $Y \neq \varnothing$. Let $\{x_n\}$ be a sequence in Y such that $\lim_{n\to\infty} x_n = x$ for some $x \in A$. Since $\{x_n\}$ is in Y, there is a sequence $\{w_n\}$ in A_0 such that

$$d(x_n, Tw_n) = d(A, B) \text{ for all } n \ge 1.$$

Case 1: $T(A_0)$ is compact. Then there exists a subsequence $\{Tw_{n_k}\}$ of $\{Tw_n\}$ such that $\lim_{k\to\infty} Tw_{n_k} = Tw$ for some $w \in A_0$. Then

$$d(x, Tw) = \lim_{k\to\infty} d(x_{n_k}, Tw_{n_k}) = d(A, B).$$

Case 2: $T(A_0)$ is closed and B is approximatively compact with respect to A. Since $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} d(x, Tw_n) = d(A, B)$. Since B is approximatively compact with respect to A, there is a subsequence $\{Tw_{n_k}\}$ of $\{Tw_n\}$ such that $\lim_{n\to\infty} Tw_{n_k} = Tw$ for some $w \in A_0$ because $T(A_0)$ is closed. Thus

$$d(x, Tw) = \lim_{k\to\infty} d(x_{n_k}, Tw_{n_k}) = d(A, B).$$

It follows from both cases that $x \in Y$ and hence Y is closed. □

Proofs of Theorem 3.9 and Theorem 3.10. We will follow the proof of Theorem 3.6. Set $Y := \{x \in A_0 : d(x, Tw) = d(A, B)$ for some $w \in A_0\}$. We define $U : Y \to Y$ as in Theorem 3.6, that is, for each $x \in Y$,

$$Ux := y,$$

where $d(y, Sw) = d(x, Tw) = d(A, B)$ for some $w, y \in A_0$. Note that for each $x, x' \in Y$

$$d(Ux, Ux') \leq \psi(d(x, x')).$$

By Lemma 3.11, Y is closed and hence complete. It follows from Theorem Br that there exists a unique fixed point $z \in Y$. The rest of the proof is exactly the same as the proof of Theorem 3.6. □

We now illustrate our supplementary results with the following two examples.

Example 1. Let $X := \mathbb{R}^2$ be equipped with the usual Euclidean metric. Let $A := \{(\alpha, 1) \in X : \alpha \geq 0\}$ and $B := \{(\alpha, 0) \in X : \alpha \geq 0\}$. Define $T : A \to B$ by

$$T(\alpha, 1) := (\lfloor \alpha \rfloor, 0) \text{ for all } \alpha \geq 0$$

where $\lfloor \cdot \rfloor$ is the floor function, that is, $\lfloor \beta \rfloor$ is the greatest integer which is less than or equal to β.

We also define $S : A \to B$ by

$$S(\alpha, 1) := (0, 0) \text{ for all } \alpha \geq 0.$$

It follows that

- for all $u_1, u_2, v_1, v_2, x_1, x_2 \in A$,
$$\left.\begin{array}{l} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) \\ d(v_1, Tx_1) = d(v_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow d(u_1, u_2) \leq \frac{1}{2} d(v_1, v_2);$$
- $d(Sx, Sy) \leq \frac{1}{2} d(Tx, Ty)$ for all $x, y \in A$;
- S and T proximally commute;
- $A_0 = A$ and $B_0 = B$;
- $S(A_0) \subset T(A_0) \cap B_0$;
- A and B are closed;
- T is not continuous;
- B is approximatively compact with respect to A;
- $T(A_0) = \{(\alpha, 0) : \alpha = 0, 1, 2, \dots\}$ is closed but not compact.

Example 2. Let $X := \mathbb{R}^2$ be equipped with the usual Euclidean metric. Let $A := \{(\alpha, 1) \in X : 0 \leq \alpha \leq 1\}$ and $B := \{(\alpha, 0) \in X : 0 \leq \alpha < 1\}$. Define $T : A \to B$ by

$$T(\alpha, 1) := \begin{cases} (0, 0) & \text{if } \alpha \in [0, 1] \cap \mathbb{Q}; \\ (1/2, 0) & \text{if } \alpha \in [0, 1] \cap \mathbb{Q}^c. \end{cases}$$

We also define $S : A \to B$ by

$$S(\alpha, 1) := (0, 0) \text{ for all } \alpha \in [0, 1].$$

It follows that

- for all $u_1, u_2, v_1, v_2, x_1, x_2 \in A$,
$$\left.\begin{array}{l} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) \\ d(v_1, Tx_1) = d(v_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow d(u_1, u_2) \leq \frac{1}{2} d(v_1, v_2);$$
- S and T proximally commute;
- $A_0 = \{(\alpha, 1) \in X : 0 \leq \alpha < 1\}$ and $B_0 = B$;

- $S(A_0) \subset T(A_0) \cap B_0$;
- A is closed;
- T is not continuous;
- B is not approximatively compact with respect to A;
- $T(A_0) = \{(0,0), (1/2, 0)\}$ is compact.

Remark 3.12. The preceding two examples are supplements to Theorem 3.6. Moreover, Theorem 3.9 and Theorem 3.10 are independent.

3.3. **Theorem MK is a consequence of Theorem Br.** We note that both Theorem S-B and Theorem MK are generalizations of Theorem J. But the conclusion of Theorem MK cannot conclude the uniqueness of a common best proximity point. We now discuss first the following result.

Lemma 3.13. *Let (X, d) be a metric space and let A and B be two nonempty subsets of X. Let $S, T : A \to B$ be two nonself mappings such that*

$$d(Sx, Sy) < d(Tx, Ty)$$

for each $x, y \in A$ with $Tx \neq Ty$. Suppose that S and T proximally commute. If x and y are two common best proximity points of S and T, then $d(x, y) \leq 2d(A, B)$.

Proof. Suppose that x and y are two common best proximity points of S and T, that is,

$$d(x, Sx) = d(x, Tx) = d(y, Sy) = d(y, Ty) = d(A, B).$$

Since S and T commute proximally, we obtain

$$Sx = Tx \text{ and } Sy = Ty.$$

Note that $Tx = Ty$. Otherwise, $d(Sx, Sy) < d(Tx, Ty) = d(Sx, Sy)$ which is a contradiction. It follows then that

$$d(x, y) \leq d(x, Tx) + d(Tx, Ty) + d(Ty, y) = 2d(A, B).$$

This completes the proof. □

The following example shows that the constant 2 in Lemma 3.13 is best possible.

Example 3. We consider the set $X := \mathbb{R}$ equipped with the usual metric. Let $A := \{-1, 1\}$ and $B := \{0\}$. Define $S, T : A \to B$ by $Sx = Tx = 0$ for all $x \in A$. Note that $d(A, B) = 1$ and the set of all common best proximity points of S and T is $\{-1, 1\}$. It is clear that S and T proximally commute; and $d(Sx, Sy) = \frac{1}{2} d(Tx, Ty)$ for each $x, y \in A$. Moreover, $d(-1, 1) = 2$.

We are now ready to state the following improvement of Theorem MK.

Theorem 3.14. *Let (X, d) be a complete metric space and let A and B be two nonempty subsets of X. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing and right continuous function such that $\psi(t) < t$ for all $t > 0$. Let $S, T : A \to B$ be two mappings satisfying the following conditions:*

- *$d(Sx, Sy) \leq \psi(d(Tx, Ty))$ for all $x, y \in A$;*
- *T is continuous;*
- *B is closed;*

- *S and T are proximally compatible;*
- *S and T can be swapped proximally;*
- *A is approximatively compact with respect to B;*
- *A_0 is nonempty;*
- *$S(A_0) \subset T(A_0) \cap B_0$.*

Then S and T have a common best proximity point. If, in addition, S and T proximally commute, then $d(x, y) \leq 2d(A, B)$ whenever x and y are two common best proximity points of S and T.

Proof. We first observe the following statement. For $x \in T(A_0)$, we suppose that there are two elements $x_1, x_2 \in A_0$ such that $x = Tx_1 = Tx_2$. By the assumption,

$$d(Sx_1, Sx_2) \leq \psi(d(Tx_1, Tx_2)) = 0.$$

That is, $Sx_1 = Sx_2$. By using this observation, we define a mapping $V : T(A_0) \to T(A_0)$ in the following way: For each $x \in T(A_0)$,

$$Vx := S\widehat{x}$$

where $\widehat{x} \in A_0$ and $x = T\widehat{x}$.

We claim that for all $x, y \in T(A_0)$

$$d(Vx, Vy) \leq \psi(d(x, y)).$$

To see this, let $x, y \in T(A_0)$ and let $\widehat{x}, \widehat{y} \in A_0$ such that

$$x = T\widehat{x} \text{ and } y = T\widehat{y}.$$

So we have

$$d(Vx, Vy) = d(S\widehat{x}, S\widehat{y}) \leq \psi(d(T\widehat{x}, T\widehat{y})) = \psi(d(x, y)).$$

We set

$$Y := T(A_0) \cap B_0.$$

In particular,

$$Y = \{T\widehat{x} : \widehat{x} \in A_0 \text{ and } d(u, T\widehat{x}) = d(A, B) \text{ for some } u \in A_0\}.$$

Note that Y is nonempty and $S(A_0) \subset Y \subset T(A_0)$. Set $U := V|_Y$. It follows that $U : Y \to Y$ and $d(Ux, Uy) \leq \psi(d(x, y))$ for all $x, y \in Y$.

Now, we apply Lemma 3.5 to obtain the extension $\boldsymbol{U} : \overline{Y} \to \overline{Y}$ of U. Note that

$$d(\boldsymbol{U}\overline{x}, \boldsymbol{U}\overline{y}) \leq \psi(d(\overline{x}, \overline{y}))$$

for all $\overline{x}, \overline{y} \in \overline{Y}$. As a consequence of Theorem Br, there is a unique fixed point $z \in \overline{Y}$ of $\boldsymbol{U}$. Then there is a sequence $\{z_n\}$ in Y such that $\lim_{n\to\infty} z_n = z$. Since $\{z_n\}$ is a sequence in Y, there are two sequences $\{u_n\}$ and $\{\widehat{z}_n\}$ in A_0 such that for each $n \geq 1$

$$z_n = T\widehat{z}_n \text{ and } d(u_n, T\widehat{z}_n) = d(A, B).$$

Since $S\widehat{z}_n \in S(A_0) \subset B_0$, there is a sequence $\{v_n\}$ in A_0 such that, for each $n \geq 1$,

$$d(v_n, S\widehat{z}_n) = d(A, B).$$

Note that $z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} T\widehat{z}_n$ and $z = \boldsymbol{U}z = \lim_{n\to\infty} Uz_n = \lim_{n\to\infty} S\widehat{z}_n$.
In particular,

$$\lim_{n\to\infty} d(u_n, z) = \lim_{n\to\infty} d(v_n, z) = d(A, B).$$

Since A is approximatively compact with respect to B, there is a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$\lim_{k\to\infty} u_{n_k} = u \text{ and } \lim_{k\to\infty} v_{n_k} = v$$

for some $u, v \in A$. In particular, $d(u, z) = d(v, z) = d(A, B)$ and hence $u, v \in A_0$. Because T is continuous, so is S. Hence $\lim_{k\to\infty} Su_{n_k} = Su$ and $\lim_{k\to\infty} Tv_{n_k} = Tv$. Since S and T are proximally compatible,

$$\lim_{k\to\infty} Tv_{n_k} = \lim_{k\to\infty} Su_{n_k}.$$

It follows that $Tv = Su$. Since S and T can be swapped proximally,

$$Sv = Tu.$$

Hence

$$d(Tv, Tu) = d(Sv, Su) \le \psi(d(Tv, Tu))$$

which implies that $Tv = Tu$ and hence $Sv = Tv$. By using Lemma 3.3, there is $\widehat{x} \in A$ such that

$$d(\widehat{x}, S\widehat{x}) = d(\widehat{x}, T\widehat{x}) = d(A, B).$$

The proof is complete. □

Remark 3.15. Our Theorem 3.14 extends Theorem MK in the following ways.

- The term $d(Tx, Ty) - \varphi(d(Tx, Ty))$, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that φ vanishes only at zero, is replaced by the more general term $\psi(d(Tx, Ty))$ where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing and right continuous function such that $\psi(t) < t$ for all $t > 0$.
- The proximal commutativity is relaxed to the proximal compatibility.

Theorem 3.16 below is analogous to Theorem 3.14. In the presence of the closedness of $T(A_0)$ in Theorem 3.16, the following conditions:

- T is continuous;
- B is closed;
- S and T can be swapped proximally;
- A is approximatively compact with respect to B;

are not required. Moreover, the proximal compatibility is relaxed to the weakly proximal compatibility.

Theorem 3.16. *Let (X, d) be a complete metric space and let A and B be two nonempty subsets of X. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing and right continuous function such that $\psi(t) < t$ for all $t > 0$. Let $S, T : A \to B$ be two mappings satisfying the following conditions:*

- *$d(Sx, Sy) \le \psi(d(Tx, Ty))$ for all $x, y \in A$;*
- *S and T are weakly proximally compatible;*
- *A_0 is nonempty;*

- *$T(A_0)$ is closed;*
- *$S(A_0) \subset T(A_0) \cap B_0$.*

Then S and T have a common best proximity point.

Proof. We follow the proof of Theorem 3.14. Set $Y := T(A_0)$ and define $U : Y \to Y$ by for each $x \in Y$

$$Ux := S\widehat{x} \text{ where } x = T\widehat{x} \text{ for some } \widehat{x} \in A_0.$$

Then

$$d(Ux, Uy) \leq \psi(d(x, y)) \text{ for all } x, y \in Y.$$

Using Theorem Br, there is $z \in Y$ such that $z = Uz$. That is, $T\widehat{z} = z = Uz = S\widehat{z}$ where $z = T\widehat{z}$ for some $\widehat{z} \in A_0$. Then the existence of a common best proximity point of S and T follows from Lemma 3.3. □

Remark 3.17. Theorem 3.16 is a supplement to Theorem 3.14. Moreover, Example 1 is applicable to Theorem 3.16 but not to Theorem 3.14.

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