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STRONG CONVERGENCE OF AN INERTIAL FORWARD-BACKWARD SPLITTING METHOD FOR ACCRETIVE OPERATORS IN REAL BANACH SPACE

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Abstract. The main purpose of this paper is to introduce a modified inertial forward-backward splitting method and prove its strong convergence to a zero of the sum of two accretive operators in real uniformly convex Banach space which is also uniformly smooth. We then apply our results to solve variational inequality problem and convex minimization problem. We also give a numerical example of our algorithm to show that it converges faster than the un-accelerated modified forward-backward algorithm.

Key Words and Phrases: Monotone inclusion problem, inertial iterative algorithm, Banach space, forward-backward splitting method, inertial extrapolation.
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1. Introduction

Let E be a real uniformly convex Banach space which is also uniformly smooth. Let $B : E \to 2^E$ be an m-accretive operator and $A : E \to E$ be an α-inverse strongly accretive operator. We shall study in this paper, the following Monotone Variational Inclusion Problem (MVIP): Find $x \in E$ such that

$$0 \in Ax + Bx. \tag{1.1}$$

Many mathematical problems emanating from machine learning, image processing, linear inverse problems, optimization problems, among others can be posed as problem

(1.1). The traditional method for solving problem (1.1) is the forward-backward splitting method, which has many applications in diverse fields (see [11, 14]).

One of the best ways to speed up the convergence rate of iterative algorithms is to combine the iterative scheme with the inertial term. This term which is represented by $\theta_n(x_n - x_{n-1})$, is a remarkable tool for improving the performance of algorithms and it is known to have some nice convergence characteristics. Thus, there are growing interests by authors working in this direction (see [3, 5, 8, 10, 12, 16]). Having reviewed the literature where inertial type algorithms were studied, we observed that the authors obtained mostly weak convergence results. In the very few cases where strong convergence results were obtained, the authors employed the inertial type algorithm which involves the construction of the sets C_n or Q_n (or both) (see [3, 5, 4]).

Remark 1.1. We remark here that, in general, algorithms that does not involve the construction of C_n or both C_n and Q_n are more desirable and interesting since they are easy to compute than those that involves these computations. Thus, it is of practical computational importance to study the strong convergence of inertial type algorithms which does not involve any of the above mentioned computations in each of the process. We also observe that, in the proof of the strong convergence theorems in [13, 14] and other related results in literature, the authors considered the *two cases approach*: that is, when $\{||x_n - Q_{(A+B)^{-1}}(0)f(z)||\}$ is monotonically decreasing and when it is not monotonically decreasing. These *two cases approach* often result to a very long proof.

In view of Remark 1.1, our contribution in this paper for solving problem (1.1) is in two-fold: First, we obtain strong convergence of a modified inertial forward-backward splitting iteration to a solution of (1.1) in real Banach space, which does not involve the construction of any of the subsets used in [3, 5, 4]. Second, our method of proof does not involve the *two cases approach*. Thus, our iteration is easier to compute and the method used in this paper is shorter (with respect to the forward-backward splitting iteration). We also apply our results to solve variational inequality problem and convex minimization problem. Furthermore, we give a numerical example of our result to illustrate the performance of our algorithm. Our results improve and generalize many recent results previously obtained in this direction.

2. Preliminaries

Let $D(A)$ and $R(A)$ be the domain and range of a set-valued operator $A : E \to 2^E$, then A is said to be accretive (see [6]) if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0, \ \forall u \in Ax, v \in Ay, \tag{2.1}$$

where J is the normalized duality setvalued mapping and j is the corresponding single-valued mapping.

More so, A is said to be m-accretive if it is an accretive operator and the range $R(I + \lambda A) = E$ for all $\lambda > 0$. Let $\alpha > 0$ and $q \in (1, \infty)$, then A is said to be α-inverse strongly accretive (α-isa) of order q (see [14]), if for each $x, y \in D(A)$, there exists

$j_q(x-y) \in J_q(x-y)$ such that

$$\langle u-v, j_q(x-y)\rangle \geq \alpha||u-v||^q,\ \forall u \in Ax, v \in Ay. \tag{2.2}$$

If $q = 2$, then we simply write α-inverse strongly accretive. Clearly, every α- inverse strongly accretive operator A is accretive (see [14, 13] for details). Let A be an m-accretive operator, then the resolvent of A, $J_\mu : R(I + \mu A) \to D(A)$ with parameter $\mu > 0$, is a nonexpansive single-valued mapping defined by $J_\mu := (I + \mu A)^{-1}$ (see [14]). It is well known that $F(J_\mu) = A^{-1}(0)$, where $A^{-1}(0)$ denotes the set of zeroes of A.

Let C be a nonempty, closed and convex subset of E and P be a mapping of E onto C. Then P is said to be sunny if $P(P(x) + tVP(x))) = P(x)$ for all $x \in E$ and $t \geq 0$. A mapping P of E into E is said to be a retraction if $P^2 = P$. If a mapping P is a retraction, then $P(x) = (x)$ for every $x \in R(P)$. Let D be a fixed point set of a nonexpansive mapping from C into itself, then a retraction $P : C \to D$ is sunny and nonexpansive iff (see [17])

$$\langle x - P(x), j(z - P(x))\rangle \leq 0,\ \forall x \in C, z \in D. \tag{2.3}$$

Lemma 2.1. [15] *Let E be a real Banach space with Fréchet differentiable norm and $\beta^*(t)$ be defined by*

$$\beta^*(t) = \sup\left\{\left|\frac{||x+ty||^2 - ||x||^2}{t} - 2\langle y, j(x)\rangle\right| : ||y|| = 1\right\},\ \forall x \in E \text{ and } 0 < t < \infty. \tag{2.4}$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0$, and for all $h \in E$ such that $h \neq 0$, we have

$$||x+h||^2 \leq ||x||^2 + 2\langle h, j(x)\rangle + ||h||\beta^*(||h||). \tag{2.5}$$

Remark 2.2. In Lemma 2.1, if $\beta^*(t) \leq ct$, for $t > 0$ and for some $c > 1$, then we obtain from (2.5) that

$$2\langle h, j(x)\rangle \leq ||x||^2 + c||h||^2 - ||x-h||^2. \tag{2.6}$$

Lemma 2.3. [1] *Let E be a real uniformly convex Banach space,*

$$B_r(0) := \{x \in E :\ ||x|| \leq r\}$$

be a closed ball with center 0 and radius $r > 0$. Then there exists a continuous strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that

$$||\sum_{i=1}^{\infty}\alpha_i x_i||^2 \leq \sum_{i=1}^{\infty}\alpha_i||x_i||^2 - \alpha_i\alpha_j g(||x_i - x_j||),$$

for any $i, j \in \mathbb{N},\ i < j$, where $\{x_i\}_{i\geq 1}$ is a sequence in $B_r(0)$ and $\alpha_i \in (0,1)$ such that

$$\sum_{i=1}^{\infty}\alpha_i = 1.$$

Remark 2.4. Throughout this paper, we may assume in Lemma 2.3 that $g(t) \geq kt$, for all $t \geq 0,\ k > 0$.

Lemma 2.5. [9] *Let* $\{a_n\}$ *be a sequence of non-negative number such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n r_n,$$

where $\{r_n\}$ *is a sequence of real numbers bounded from above and* $\{\alpha_n\} \subset [0,1]$ *satisfies* $\sum \alpha_n = \infty$. *Then* $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} r_n$.

Lemma 2.6. [14] *Let* E *be a real uniformly convex Banach space with Frèchet differentiable norm. Let* $B : E \to 2^E$ *be an m-accretive operator and* $A : E \to E$ *be an* α*-inverse strongly accretive mapping on* E. *Then, given* $s > 0$, *there exists a continuous, strictly increasing and convex function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(0) = 0$ *such that for all* $x, y \in E$ *and* $\mu > 0$, *we have*

$$\begin{aligned} ||K_n x - K_n y||^2 &\leq ||x-y||^2 - \mu(2\alpha - \mu c)||Bx - By||^2 \\ &- \phi(||(I - J_\mu^B)(I-\mu A)x - (I-J_\mu^B)(I-\mu A)y||), \end{aligned} \quad (2.7)$$

where $K_n = J_\mu^B(I - \mu A)$.

Lemma 2.7. [7] *Let* E *be a real Banach space. Let* $B : E \to 2^E$ *be an m-accretive operator and* $A : E \to E$ *be an* α*-inverse strongly accretive mapping on* E. *Then we have*

(i) *for* $\mu > 0$, $F(J_\mu^B(I-\mu A)) = (A+B)^{-1}(0)$,
(ii) *for* $0 < \lambda \leq \mu$ *and* $x \in E$, $||x - J_\lambda^B(I-\lambda A)x|| \leq 2||x - J_\mu^B(I-\mu A)x||$.

3. Main results

Lemma 3.1. *Let* E *be a real Banach space with Frèchet differentiable norm. Let* $A : E \to E$ *be an* ρ*-inverse strongly accretive mapping and* $B : E \to 2^E$ *be an m-accretive operator. Let* $f : E \to E$ *be a contraction with constant* $\tau \in (0,1)$ *and* $\{v_n\}$ *be a sequence in* E *such that* $\{v_n\}$ *converges to* $v \in E$. *Assume that* $\Theta := (A+B)^{-1}(0) \neq \emptyset$ *and for arbitrary* $x_0, x_1 \in E$ *the sequence* $\{x_n\}$ *is generated iteratively by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = (1-\alpha_n)u_n + \alpha_n f(v_n); \\ z_n = (1-\lambda_n)y_n + \lambda_n J_{\mu_n}^B(I-\mu_n A)y_n; \\ x_{n+1} = (1-\beta_n)y_n + \beta_n z_n; \end{cases} \quad (3.1)$$

where $\{\alpha_n\}, \{\lambda_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$, $\{\theta_n\} \subset [0,\theta)$, $\theta \in [0,1)$ *and* $\{\mu_n\}$ *is a sequence of positive real numbers such that the following conditions are satisfied:*
(i) $0 < \mu_n \leq \frac{2\rho}{c}$ $\forall n \geq 1$;

(ii) $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$. *Then, the sequence* $\{x_n\}$ *is bounded.*

Proof. Let $K_n = J_{\mu_n}^B(I - \mu_n A)$ for all $n \geq 1$, where $J_{\mu_n}^B = (I + \mu_n B)$. Then it follows from Lemma 2.6 that K_n is nonexpansive for all $n \geq 1$. Now, let $p \in \Theta$, then from Lemma 2.7 (i), we have that $p = K_n p$ for all $n \geq 1$. Thus, we obtain from (3.1) and the convexity of norm that

$$\begin{aligned}||x_{n+1}-p|| &= ||(1-\beta_n)(y_n-p)+\beta_n(z_n-p)|| \\ &\le (1-\beta_n)||y_n-p||+\beta_n||z_n-p|| \\ &\le (1-\beta_n)||y_n-p||+\beta_n\left[(1-\lambda_n)||y_n-p||+\lambda_n||K_ny_n-p||\right] \\ &\le (1-\beta_n)||y_n-p||+\beta_n\left[(1-\lambda_n)||y_n-p||+\lambda_n||y_n-p||\right] \\ &= ||y_n-p|| \\ &= ||(1-\alpha_n)(u_n-p)+\alpha_n(f(v_n)-p)|| \\ &\le (1-\alpha_n)||u_n-p||+\alpha_n||f(v_n)-p|| \\ &\le (1-\alpha_n)||u_n-p||+\alpha_n\tau||v_n-p||+\alpha_n||f(p)-p||. \end{aligned} \tag{3.2}$$

Also, we obtain from (3.1) that

$$\begin{aligned}||u_n-p|| &\le ||x_n+\theta_n(x_n-x_{n_1})-p|| \\ &\le ||x_n-p||+\theta_n||x_n-x_{n-1}||,\end{aligned}$$

which implies from (3.2) that

$$\begin{aligned}||x_{n+1}-p|| &\le (1-\alpha_n)||x_n-p||+(1-\alpha_n)\theta_n||x_n-x_{n-1}|| \\ &\quad +\alpha_n\tau||v_n-p||+\alpha_n||f(p)-p|| \\ &\le \max\left\{||x_n-p||,||f(p)-p||\right\}+\max\{\theta_n||x_n-x_{n-1}||,||v_n,p||\}.\end{aligned}$$

From condition (ii), we obtain that $\lim_{n\to\infty}\theta_n||x_n-x_{n-1}||=0$. Hence, $\{\theta_n||x_n-x_{n-1}||\}$ is bounded. So, there exists $M_1>0$ such that $\theta_n||x_n-x_{n-1}||\le M_1$ $\forall n\ge 1$. Similarly, since $\{v_n\}$ converges, there exists $M_2>0$ such that $||v_n-p||\le M_2$ $\forall n\ge 1$. Thus by induction, we obtain that

$$||x_n-p||\le \max\{||x_1-p||,||f(p)-p||\}+\max\{M_1,M_2\}.$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{u_n\},\{y_n\},\{z_n\}$ and $\{K_nu_n\}$ are all bounded.

Theorem 3.2. *Let E be a real uniformly convex Banach space which is also uniformly smooth. Let $A:E\to E$ be an ρ-inverse strongly accretive mapping and $B:E\to 2^E$ be an m-accretive operator. Let $f:E\to E$ be a contraction with constant $k\in(0,1)$ and $\{v_n\}$ be a sequence in E such that $\{v_n\}$ converges to v.*
Assume that $\Theta:=(A+B)^{-1}(0)\ne\emptyset$ and for arbitrary $x_1,u\in E$, the sequence $\{x_n\}$ is generated iteratively by (3.1), where $\{\alpha_n\},\{\lambda_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, $\{\theta_n\}\subset[0,\theta)$, $\theta\in[0,1)$ and $\{\mu_n\}$ is a sequence of positive real numbers such that the following conditions are satisfied:
(i) $0<\mu\le\mu_n<\frac{2\rho}{c}$ $\forall n\ge 1$;
(ii) $0<\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<1$;
(iii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;
(iv) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(v) $\sum_{n=1}^{\infty}\theta_n||x_n-x_{n-1}||<\infty$;

(vi) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0.$

Then the sequence $\{x_n\}$ *generated by* (3.1) *converges strongly to* $p = P_\Theta f(v)$, *where* P_Θ *is the unique sunny nonexpansive retraction of* E *onto* Θ.

Proof. Let $p = P_\Theta f(v)$, then from (3.1), Lemma 2.3 and Remark 2.2, we obtain that

$$\begin{aligned} ||z_n - p||^2 &= ||(1-\lambda_n)y_n + \lambda_n K_n y_n - p||^2 \\ &\le (1-\lambda_n)||y_n - p||^2 + \lambda_n ||K_n y_n - p||^2 - \lambda_n(1-\lambda_n)g(||y_n - K_n y_n||) \\ &\le (1-\lambda_n)||y_n - p||^2 + \lambda_n ||y_n - p||^2 - \lambda_n(1-\lambda_n)k||y_n - K_n y_n||^2 \\ &= ||y_n - p||^2 - \lambda_n(1-\lambda_n)k||y_n - K_n y_n||^2 \\ &\le ||y_n - p||^2. \end{aligned} \tag{3.3}$$

Also, we obtain from (3.1) that

$$\begin{aligned} ||z_n - y_n||^2 &= \Big|\Big|\frac{1}{\beta_n}(x_{n+1} - y_n)\Big|\Big|^2 \\ &= \frac{1}{\beta_n^2}||x_{n+1} - y_n||^2 \\ &= \frac{\alpha_n}{\beta_n}\Big(\frac{||x_{n+1} - y_n||^2}{\alpha_n \beta_n}\Big). \end{aligned} \tag{3.4}$$

Again, from (3.1), (3.3), (3.4), Lemma 2.3 and Remark 2.4, we obtain that

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||(1-\beta_n)y_n + \beta_n z_n - p||^2 \\ &\le (1-\beta_n)||y_n - p||^2 + \beta_n ||z_n - p||^2 - \beta_n(1-\beta_n)g(||y_n - z_n||) \\ &\le (1-\beta_n)||y_n - p||^2 + \beta_n ||y_n - p||^2 - \beta_n(1-\beta_n)k||y_n - z_n||^2 \\ &\le ||y_n - p||^2 - \frac{1}{\beta_n}(1-\beta_n)k||x_{n+1} - y_n||^2 \\ &\le c\alpha_n^2||f(v_n) - p||^2 + (1-\alpha_n)^2||u_n - p||^2 \\ &- 2\alpha_n(1-\alpha_n)\langle p - f(v_n), j(u_n - p)\rangle \\ &- \frac{1}{\beta_n}(1-\beta_n)k||x_{n+1} - y_n||^2. \end{aligned} \tag{3.5}$$

Again, from (3.1), Lemma 2.1 and Remark 2.4, we obtain that

$$\begin{aligned} ||u_n - p||^2 &= ||x_n + \theta_n(x_n - x_{n-1}) - p||^2 \\ &\le ||x_n - p||^2 + 2\theta_n\langle x_n - x_{n-1}, j(x_n - p)\rangle \\ &+ c\theta_n^2||x_n - x_{n-1}||^2 \\ &\le ||x_n - p||^2 + \theta_n\big[||x_n - p||^2 + c||x_n - x_{n-1}||^2 - ||x_{n-1} - p||^2\big] \\ &+ c\theta_n^2||x_n - x_{n-1}||^2 \\ &\le ||x_n - p||^2 + \theta_n\big[||x_n - p||^2 - ||x_{n-1} - p||^2\big] + 2c\theta_n||x_n - x_{n-1}||^2. \end{aligned} \tag{3.6}$$

On substituting (3.6) into (3.5), we have that

$$\begin{aligned}
||x_{n+1}-p||^2 &\leq c\alpha_n^2||f(v_n)-p||^2+(1-\alpha_n)\big[||x_n-p||^2\\
&+\theta_n(||x_n-p||^2-||x_{n-1}-p||^2)+2c\theta_n||x_n-x_{n-1}||^2\big]\\
&-2\alpha_n(1-\alpha_n)\langle p-f(v_n),j(u_n-p)\rangle-\frac{1}{\beta_n}(1-\beta_n)k||x_{n+1}-y_n||^2\\
&\leq(1-\alpha_n)||x_n-p||^2+\theta_n(||x_n-p||^2-||x_{n-1}-p||^2)\\
&+2c\theta_n||x_n-x_{n-1}||^2-2\alpha_n(1-\alpha_n)\langle p-f(v_n),j(u_n-p)\rangle\\
&-\frac{1}{\beta_n}(1-\beta_n)k||x_{n+1}-y_n||^2+c\alpha_n^2||f(v_n)-p||^2\\
&=(1-\alpha_n)||x_n-p||^2\\
&-\alpha_n\bigg(-\alpha_n c||f(v_n)-p||^2-\frac{\theta_n}{\alpha_n}\big[||x_n-p||^2-||x_{n-1}-p||^2\big]\\
&-2\frac{c}{\alpha_n}\theta_n||x_n-x_{n-1}||^2+2(1-\alpha_n)\langle p-f(v_n),j(u_n-p)\rangle\\
&+\frac{1}{\alpha_n\beta_n}(1-\beta_n)k||x_{n+1}-y_n||^2\bigg).
\end{aligned}\tag{3.7}$$

Let

$$\begin{aligned}
\Omega_n:&=-\alpha_n c||f(v_n)-p||^2-\frac{\theta_n}{\alpha_n}\big[||x_n-p||^2-||x_{n-1}-p||^2\big]-2c\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||^2\\
&+2(1-\alpha_n)\langle p-f(v_n),j(u_n-p)\rangle+\frac{1}{\alpha_n\beta_n}(1-\beta_n)k||x_{n+1}-y_n||^2.
\end{aligned}\tag{3.8}$$

Thus, (3.7) becomes

$$||x_{n+1}-p||^2\leq(1-\alpha_n)||x_n-p||^2-\alpha_n\Omega_n.$$

From Lemma 3.1, we have that $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are bounded, thus they are bounded below. It then follows from condition (vi) that $\{\Omega_n\}$ is bounded below. Hence, by condition (iv) and applying Lemma 2.5 in (3.1), we obtain that

$$\begin{aligned}
\limsup_{n\to\infty}||x_n-p||^2&\leq\limsup_{n\to\infty}(-\Omega_n)\\
&=-\liminf_{n\to\infty}\Omega_n.
\end{aligned}\tag{3.9}$$

Therefore, $\liminf\limits_{n\to\infty}\Omega_n$ exists. Thus, we obtain from (3.8), condition (iv), (v) and (vi) that

$$\liminf_{n\to\infty}\Omega_n=\liminf_{n\to\infty}\bigg(2\langle p-f(v_n),j(u_n-p)\rangle+\frac{1}{\alpha_n\beta_n}(1-\beta_n)k||x_{n+1}-y_n||^2\bigg).\tag{3.10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup q$ for some $q \in E$, and

$$\liminf_{n\to\infty} \Omega_n = \lim_{j\to\infty} \big(2\langle p - f(v_{n_j}), j(u_{n_j} - p)\rangle + \frac{1}{\alpha_{n_j}\beta_{n_j}}(1-\beta_{n_j})k||x_{n_j+1} - y_{n_j}||^2\big). \tag{3.11}$$

Using the fact that $\{x_n\}$ is bounded and $\liminf_{n\to\infty}\Omega_n$ exists, we have that $\left\{\frac{1}{\alpha_{n_j}\beta_{n_j}}(1-\beta_{n_j})||x_{n_j+1} - y_{n_j}||^2\right\}$ is bounded. Also, by condition (iii), there exists $b \in (0,1)$ such that $\beta_n \leq b < 1$ which implies that $\frac{1}{\alpha_{n_j}\beta_{n_j}}(1-\beta_{n_j}) \geq \frac{1}{\alpha_{n_j}\beta_{n_j}}(1-b) > 0$. Hence, we have that $\left\{\frac{1}{\alpha_{n_j}\beta_{n_j}}||x_{n_j+1} - y_{n_j}||^2\right\}$ is bounded. Observe from condition (iii) and (iv) that there exits $a \in (0,1)$ such that

$$0 < \frac{\alpha_{n_j}}{\beta_{n_j}} \leq \frac{\alpha_{n_j}}{a} \to 0, k \to \infty.$$

Therefore, we obtain from (3.4) that

$$\lim_{j\to\infty} ||z_{n_j} - y_{n_j}|| = 0. \tag{3.12}$$

From (3.1) and (3.12), we obtain

$$||x_{n_j+1} - y_{n_j}|| = \beta_{n_j}||z_{n_j} - y_{n_j}|| \to 0, \text{ as } j \to \infty. \tag{3.13}$$

From (3.1) and condition (v), we obtain

$$||u_{n_j} - x_{n_j}|| = \theta_{n_j}||x_{n_j} - x_{n_j-1}|| \to 0, \text{ as } j \to \infty. \tag{3.14}$$

Using (3.1) and condition (iv), we obtain

$$||y_{n_j} - u_{n_j}|| = \alpha_{n_j}||u - u_{n_j}|| \to 0, \text{ as } j \to \infty. \tag{3.15}$$

From (3.14) and (3.15), we have that

$$||y_{n_j} - x_{n_j}|| \leq ||y_{n_j} - u_{n_j}|| + ||u_{n_j} - x_{n_j}|| \to 0, \text{ as } j \to \infty. \tag{3.16}$$

From (3.1), (3.12) and condition (ii), we obtain that

$$||K_{n_j}y_{n_j} - y_{n_j}|| = \frac{1}{\lambda_{n_j}}||z_{n_j} - y_{n_j}|| \to 0, \text{ as } j \to \infty. \tag{3.17}$$

From condition (i), Lemma 2.7 (ii) and (3.17), we obtain that

$$||Ky_{n_j} - y_{n_j}|| \leq 2||K_{n_j}y_{n_j} - y_{n_j}|| \to 0, \text{ as } j \to \infty, \tag{3.18}$$

where $K = J_\mu^B(I - \mu A)$.
Since $x_{n_j} \rightharpoonup q$, we obtain from (3.14) and (3.16) that $u_{n_j} \rightharpoonup q$ and $y_{n_j} \rightharpoonup q$ respectively. It then follows from the demicloseness of K, (3.18) and Lemma 2.7 (i) that $q \in F(K) = \Theta$.
Now, since $v_n \to v$ and f is a contraction (which implies that f is continuous), then $f(v_n) \to f(v)$. Furthermore, since $u_{n_j} \rightharpoonup q$ and the duality map is norm-to-norm

uniformly continuous on bounded sets, we obtain from (3.10), (3.13) and (2.3) that

$$\begin{aligned}\liminf_{n\to\infty}\Omega_n &= \lim_{j\to\infty}\big(2\langle p-f(v_{n_j}), j(u_{n_j}-p)\rangle + \frac{1}{\alpha_{n_j}\beta_{n_j}}(1-\beta_{n_j})k\|x_{n_j+1}-y_{n_j}\|^2\big)\\ &\geq 2\lim_{j\to\infty}\langle p-f(v), j(u_{n_j}-p\rangle\\ &= 2\langle p-f(v), j(q-p)\rangle \geq 0. \qquad (3.19)\end{aligned}$$

Hence, from (3.9), we have that

$$\limsup_{n\to\infty}\|x_n-p\|^2 \leq -\liminf_{n\to\infty}\Omega_n \leq 0.$$

Therefore, $\lim_{n\to\infty}\|x_n-p\|=0$ and this implies that $\{x_n\}$ converges strongly to $p=P_\Theta f(v)$.

Theorem 3.3. *Let E be a real uniformly convex Banach space which is also uniformly smooth. Let $T:E\to E$ be a nonexpansive mapping. Let $f:E\to E$ be a contraction with constant $k\in(0,1)$ and $\{v_n\}$ be a sequence in E such that $\{v_n\}$ converges to v. Assume that $\Theta := F(T)\neq\emptyset$ and for arbitrary $x_0,x_1\in E$, the sequence $\{x_n\}$ is generated iteratively by*

$$\begin{cases} u_n = x_n+\theta_n(x_n-x_{n-1});\\ y_n=(1-\alpha_n)u_n+\alpha_n f(v_n);\\ z_n=(1-\lambda_n)y_n+\lambda_n Ty_n;\\ x_{n+1}=(1-\beta_n)y_n+\beta_n z_n; \end{cases} \qquad (3.20)$$

where $\{\alpha_n\},\{\lambda_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$, $\{\theta_n\}\subset[0,\theta)$ and $\theta\in[0,1)$ satisfies the conditions in Theorem 3.2. Then the sequence $\{x_n\}$ converges strongly to $p=P_\Theta f(v)$, where P_Θ is the unique sunny nonexpansive retraction of E onto Θ.

Proof. The proof follows from the proof of Theorem 3.2 by setting $J^B_{\mu_n}(I-\mu_n A)=T$ in Algorithm (3.1).
By setting $v_n=v\ \forall\ n\geq 1$ and $\Theta_n=0\ \forall\ n\geq 1$ in Theorem 3.2, we obtain the following result.

Corollary 3.4. *Let E be a real uniformly convex Banach space which is also uniformly smooth. Let $A:E\to E$ be an ρ-inverse strongly accretive mapping and $B:E\to 2^E$ be an m-accretive operator. Assume that $\Theta:=(A+B)^{-1}(0)\neq\emptyset$ and for arbitrary $x_1,x_0,v\in E$, the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n=(1-\alpha_n)x_n+\alpha_n f(v);\\ z_n=(1-\lambda_n)y_n+\lambda_n J^B_{\mu_n}(I-\mu_n A)y_n;\\ x_{n+1}=(1-\beta_n)y_n+\beta_n z_n; \end{cases} \qquad (3.21)$$

where $\{\alpha_n\},\{\lambda_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$, $\{\theta_n\}\subset[0,\theta)$ and $\theta\in[0,1)$ satisfies the conditions in Theorem 3.2. Then the sequence $\{x_n\}$ converges strongly to $p=P_\Theta f(v)$, where P_Θ is the unique sunny nonexpansive retraction of E onto Θ.

Recall that for a real Hilbert space H, a mapping $B : H \to H$ is said to be
(i) monotone, if

$$\langle Bx - By, x - y\rangle \geq 0, \ \forall \ x, y \in H.$$

(ii) ρ-inverse strongly monotone if there exists a constant $\rho > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \rho||Bx - By||^2, \forall \ x, y \in H.$$

Since the normalized duality mapping is simply the identity mapping in H, then it is clear that the class of accretive and inverse strongly accretive operators coincides with the class of monotone and inverse strongly monotone operators respectively in real Hilbert space. Thus, we obtain the following result which improves and complements the results of [5, 4].

Corollary 3.5. *Let H be a real Hilbert space. Let $A : H \to H$ be a ρ-inverse strongly monotone mapping and $B : H \to 2^H$ be a maximal monotone operator. Let $f : H \to H$ be a contraction with constant $k \in (0, 1)$ and $\{v_n\}$ be a sequence in H such that $\{v_n\}$ converges to v. Assume that $\Theta := (A + B)^{-1}(0) \neq \emptyset$ and for arbitrary $x_1, x_0 \in H$, the sequence $\{x_n\}$ is generated iteratively by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = (1 - \alpha_n)u_n + \alpha_n f(v_n); \\ z_n = (1 - \lambda_n)y_n + \lambda_n J^B_{\mu_n}(I - \mu_n A)y_n; \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n z_n; \end{cases} \tag{3.22}$$

where $\{\alpha_n\}, \{\lambda_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$, $\{\theta_n\} \subset [0, \theta)$ and $\theta \in [0, 1)$ satisfies the conditions in Theorem 3.2. Then the sequence $\{x_n\}$ converges strongly to $p = P_\Theta f(v)$, where P_Θ is the projection of H onto Θ.

We also note that the proofs of the above two corollaries also follows from Theorem 3.3 just by setting $T = J^B_{\mu_n}(I - \mu_n A)$ in Theorem 3.3.

4. Applications

4.1. **Variational inequality problem.** Let C be a nonempty closed and convex subset of a real Hilbert space H. The classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall \ y \in C. \tag{4.1}$$

It is known that $VI(C, A) = P_C(I - \lambda A)$, where $VI(C, A)$ denotes the solution set of (4.1) and P_C is the metric projection from H onto C.

The subdifferential of the function $g : H \to (-\infty, +\infty]$ is a set-valued function $\partial g : H \to 2^H$ defined by

$$\partial g(x) = \{z \in H : g(x) + \langle y - x, z\rangle \leq g(y), \forall \ y \in H\}.$$

While the indicator function $i_C : H \to (-\infty, +\infty]$ is defined by

$$i_C x = \begin{cases} 0, \ x \in C, \\ \infty, \ x \notin C. \end{cases}$$

We know that the subdifferential ∂i_C of i_C is a maximal monotone operator,

$$J_\lambda^{\partial i_C} = P_C \text{ and } (A + \partial i_C)^{-1}(0) = VI(A, C).$$

Thus, by setting $B = \delta i_C$ in Corollary 3.5, we apply Corollary 3.5 to obtain the following result:

Theorem 4.1. *Let C be a nonempty, closed and convex subset of real Hilbert space H. Let $A : H \to H$ be a ρ-inverse strongly monotone mapping and $g : H \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Let $f : C \to C$ be a contraction with constant $k \in (0, 1)$ and $\{v_n\}$ be a sequence in H such that $\{v_n\}$ converges to v. Assume that $\Theta := VI(C, A) \neq \emptyset$ and for arbitrary $x_1, x_0 \in H$, the sequence $\{x_n\}$ is generated iteratively by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}); \\ y_n = (1 - \alpha_n)u_n + \alpha_n f(v_n); \\ z_n = (1 - \lambda_n)y_n + \lambda_n P_C(I - \mu_n A)y_n; \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n z_n; \end{cases} \tag{4.2}$$

where $\{\alpha_n\}, \{\lambda_n\}, \{\beta_n\}$ are sequences in $(0, 1)$, $\{\theta_n\} \subset [0, \theta)$ and $\theta \in [0, 1)$ satisfies the conditions in Theorem 3.2. Then the sequence $\{x_n\}$ converges strongly to $p = P_\Theta f(v)$, where P_Θ is the projection of H onto Θ.

4.2. **Convex minimization problem.** Let $F : H \to \mathbb{R}$ be a convex and continuously differentiable function, and $G : H \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, the gradient ∇F of F is monotone and continuous, and the subdifferential ∂G of G is maximal monotone. Moreover,

$$F(x^*) + G(x^*) = \min_{x \in H} [F(x) + G(x)] \Leftrightarrow 0 \in \nabla F(x^*) + \partial G(x^*).$$

We now consider the following Minimization Problem (MP): Find

$$x^* \in H \text{ such that } F(x^*) + G(x^*) = \min_{x \in H} [F(x) + G(x)], \tag{4.3}$$

where F and G are as defined above. Suppose the solution set of problem (4.3) is Ω, then setting $B = \partial G$ and $A = \nabla F$ in Corollary 3, we apply Corollary 3.5 to approximate solutions of (4.3).

4.3. **Numerical example.** We now give a numerical example of Algorithm 3.1. Let $E = \mathbb{R}^4$ with the euclidean norm and $B : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by

$$B(x) = (x_1, -x_2, -x_3, x_4).$$

Then B is maximal monotone. Hence, we obtain that

$$J_{\mu_n}^B(x) = \left(\frac{x_1}{1 + 3\mu_n}, \frac{x_2}{1 + 3\mu_n}, \frac{x_3}{1 + 3\mu_n}, \frac{x_4}{1 + 3\mu_n} \right).$$

Let $A : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $A(x) = (2x_1, 2x_2, 2x_3, 2x_4)$. Then, A is ρ-inverse strongly monotone mapping with $\rho = \frac{1}{2}$. Let $f(x) = \frac{1}{3}x$ and $v_n = \frac{n}{2n+1}$, then

$v_n \to \frac{1}{2} = v$. Take $\mu_n = \frac{n}{3n+2}$ $\forall n \geq 1$, $\lambda_n = \frac{n+1}{3n}, \alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{n+1}{5n}$. Then, conditions (i)-(iv) of Theorem 3.2 are satisfied. Hence, Algorithm 3.1 becomes:

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \ n \geq 1; \\ y_n = \frac{n}{n+1}u_n + \frac{n}{(n+1)(6n+1)}; \\ z_n = \frac{2n-1}{3n}y_n + \frac{n+1}{3n}J^B_{\mu_n}(I - \mu_n A)y_n; \\ x_{n+1} = \frac{4n-1}{5n}y_n + \frac{n+1}{5n}z_n, \ n \geq 1 \end{cases} \tag{4.4}$$

and Algorithm (3.1) of Shehu and Cai [14] becomes:

$$x_{n+1} = \frac{1}{3(n+1)}x_n + \frac{n}{n+1}J^B_{\mu_n}(I - \mu_n A)x_n, \ n \geq 1. \tag{4.5}$$

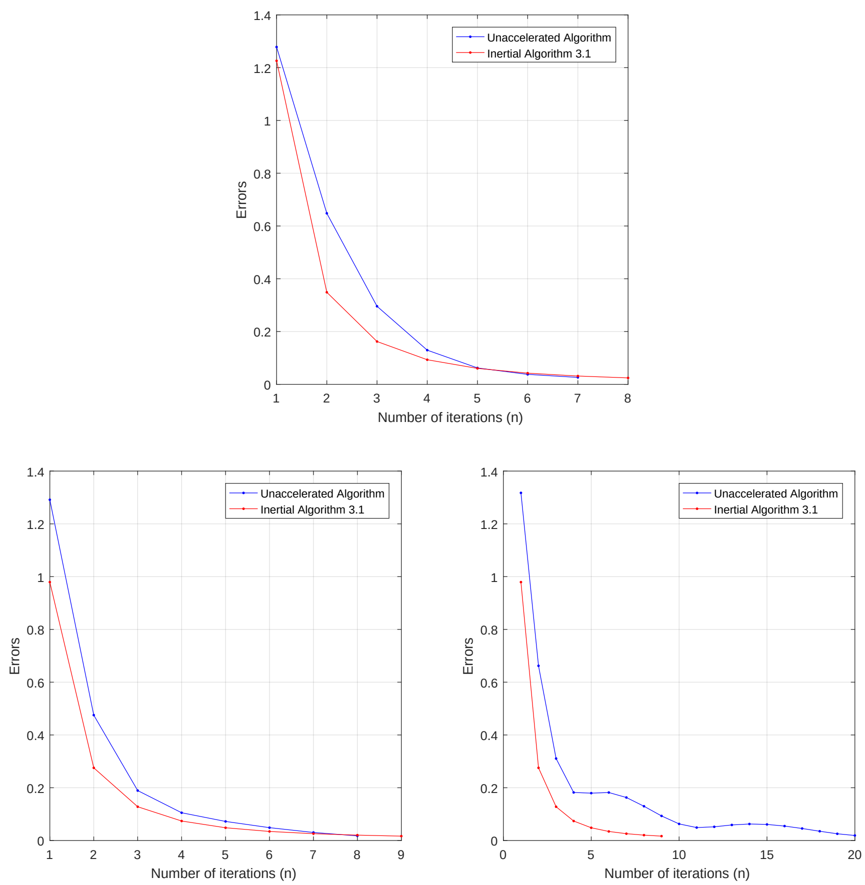

FIGURE 1. Errors vs Iteration numbers(n): **Case I(a)** (top); **Case I(b)** (bottom left); **Case I(c)** (bottom right).

Case I:

(a) $x_0 = (0.5,\ 3,\ 1,\ 4)^T$, $x_1 = (0.5,\ 3,\ 1,\ 4)^T$ and $\theta_n = \frac{n}{4n^4+1}$.

(b) $x_0 = (0.5,\ 3,\ 1,\ 2)^T$, $x_1 = (0.1,\ 0.01,\ 1,\ 2)^T$ and $\theta_n = \frac{n}{9n^3+1}$.

(c) $x_0 = (0.1,\ 0.01,\ 1,\ 2)^T$, $x_1 = (0.5,\ 3,\ 1,\ 2)^T$ and $\theta_n = \frac{n}{9n^3+1}$.

Case II:

(a) $x_0 = (1,\ 2,\ 1,\ 3)^T$, $x_1 = (1,\ 0.1,\ 1,\ 1.2)^T$ and $\theta_n = \frac{n}{4n^4+1}$.

(b) $x_0 = (-0.1,\ 0.01,\ 1,\ -2)^T$, $x_1 = (-0.5,\ 3,\ -1,\ 2)^T$ and $\theta_n = \frac{n}{9n^3+1}$.

(c) $\bar{x}_0 = (-0.5,\ 3,\ -1,\ 2)^T$, $\bar{x}_1 = (-0.1,\ 0.01,\ 1,\ -2)^T$ and $\theta_n = \frac{n}{9n^3+1}$.

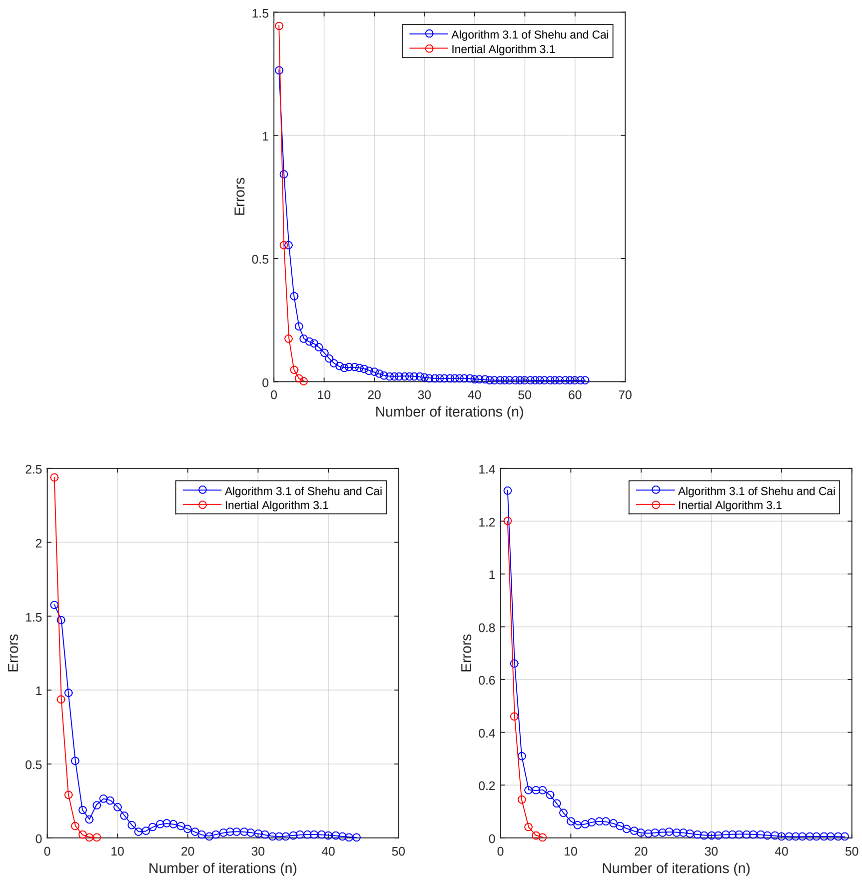

FIGURE 2. Errors vs Iteration numbers(n): **Case II(a)** (top); **Case II(b)** (bottom left); **Case II(c)** (bottom right).

Remark 4.2. By considering **Case I (a)-(c)**, we compared our inertial Algorithm 3.1 with its corresponding unaccelerated algorithm. Also, by considering **Case II (a)-(c)**, we compared our algorithm with Algorithm 3.1 of Shehu and Cai [14].
We can see from Figure 1 that our accelerated algorithm converges faster than its corresponding unaccelerated algorithm. In particular, Figure 2 shows that our inertial-type algorithm performs well and have competitive advantage over the unaccelerated algorithm of Shehu and Cai [14] (and other corresponding unaccelerated algorithms).

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