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A HYBRID PROJECTION ALGORITHM FOR FINDING FIXED POINTS OF BREGMAN QUASI-STRICT PSEUDO-CONTRACTIONS

ZI-MING WANG* AND SUN YOUNG CHO**

*Department of Foundation, Shandong Yingcai University, Jinan, China

**Research Center for Interneural Computing, China Medical University Hospital,
China Medical University, Taichung, Taiwan
E-mail: ooly61@hotmail.com
Corresponding author

Abstract. In this paper, a hybrid projection algorithm is investigated for finding fixed points of Bregman quasi-strict pseudo-contractions. Strong convergence theorems are established in the framework of reflexive Banach spaces.
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1. Introduction

Fixed point theory has emerged as a powerful and effective tool for studying many problems arising in various branches of physical, engineering, pure and applied sciences in a unified and general framework, see, for example, [1, 9, 11, 12, 15, 23]. The computation of fixed points is important in the study of many real world problems, including inverse problems; for instance, the split feasibility problem and the convex feasibility problem in signal processing and image reconstruction can both be formulated as a problem of finding fixed points of certain nonlinear operators, respectively; see [8] for more details and the references therein.

Recently, Mann's iterative algorithms for finding fixed points of nonexpansive mappings and strict pseudo-contractions has extensively been investigated. However, in an infinite-dimensional Hilbert space, the Mann's iterative algorithm has only weak convergence, in general, even for nonexpanisve mappings. In order to get strong convergence of the Mann's iterative algorithm, hybrid projection methods have been recently investigated by many authors; see [4, 3, 10, 14, 16, 17] and the references therein.

In this paper, we propose a hybrid projection algorithm for common fixed points of a finite family of Bregman quasi-strict pseudo-contractions. Strong convergence theorems are established in the framework of reflexive Banach spaces.

2. Preliminaries

Throughout this paper, we always that E is a real reflexive Banach space norm $\|\cdot\|$ and E^* is the dual space of E. The normalized duality mapping from E to 2^{E^*} denoted by J is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall\, x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. Let C be a nonempty closed and convex subset of E and let $T : C \to C$ be a mapping. We use $F(T) = \{x \in C : Tx = x\}$ to denote the set of fixed points of T. T is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y \in C$ as $n \to \infty$, then $Tx = y$. In this paper, we use $\mathbb{R}$ and $\mathbb{N}$ to stand for the sets of real numbers and positive integers, respectively.

Let $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function. We denote by dom$f$ the domain of f, that is, dom$f := \{x \in E : f(x) < +\infty\}$. For any $x \in$int dom$f$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^\circ(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be Gâteaux differentiable at x if

$$\lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}$$

exists for any y. In this case, $f^\circ(x, y)$ coincides with $\nabla f(x)$, the value of the gradient $\nabla f(x)$ of f at x. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$int dom$f$. The function f is said to be Fréchet differentiable at x if limit (2.1) is attained uniformly in $\|y\| = 1$. The function f is said to be Fréchet differentiable if it is Fréchet differentiable for any $x \in$int dom$f$. Finally, f is called be uniformly Fréchet differentiable on a subset C of E if limit (2.1) is attained uniformly for $x \in C$ and $\|y\| = 1$. It is well known that if a continuous convex function $f : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇f is norm-to-weak* continuous; see [6] and the references therein. Also, it is known that if f is said to be Fréchet differentiable, then ∇f is norm-to-norm continuous; see [13] and the references therein. The function f is said to be strongly coercive if

$$\lim_{\|x_n\|\to\infty} \frac{f(x_n)}{\|x_n\|} = \infty.$$

The following lemma play an important role in this paper.

Lemma 2.1. [18] *If a function $f : X \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Then the Bregman distance with respect to f is the function D_f :dom$f\times$int dom$f \to [0, +\infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle.$$

With the function f we associate the bifunction $V_f : E \times E^* \to [0, +\infty)$ defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*.$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Recall that the Bregman projection [18] of $x \in$int dom$f$ onto the nonempty closed and convex set $C \subset dom f$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

It should be observed that if E is a smooth, and strictly convex Banach space, setting $f(x) = \|x\|^2$ for all $x \in E$, we have $\nabla f(x) = 2Jx$ for all $x \in E$. Hence $D_f(x, y)$ reduces to the Lyapunov function $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$, $\forall x,\ y \in E$ and the Bregman projection $P_C^f(x)$ reduces to the generalized projection $\Pi_C(x)$ which is defined by $\Pi_C(x) = \arg\min_{y \in C} \phi(y, x)$. If E is a Hilbert space H, then $D_f(x, y)$ becomes $\phi(x, y) = \|x - y\|^2$, $\forall x,\ y \in H$ and the Bregman projection $P_C^f(x)$ becomes the metric projection $P_C(x)$. Similarly to the metric projection in Hilbert space, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

Lemma 2.2. [7] *Suppose that f is Gâteaux differentiable and totally convex on int dom$f$. Let $x \in$ int dom$f$ and let $C \subset$ int dom$f$ be a nonempty, closed and convex set. If $\widehat{x} \in C$, then the following conditions are equivalent:*

(a) The vector $\widehat{x}$ is the Bregman projection of x onto C with respect to f, i.e., $\widehat{x} = P_C^f(x)$.

(b) The vector $\widehat{x}$ is the unique solution of the variational inequality

$$\langle \nabla f(x) - \nabla f(\widehat{x}), \widehat{x} - y \rangle \geq 0, \quad \forall\ y \in C.$$

(c) The vector $\widehat{x}$ is the unique solution of the inequality

$$D_f(y, \widehat{x}) + D_f(\widehat{x}, x) \leq D_f(y, x), \quad \forall\ y \in C.$$

Let E be a Banach space and let $B_r := \{z \in E : \|z\| \leq r\}$, for all $r > 0$ and $S_E = \{x \in E : \|x\| = 1\}$. Then a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E [24] if $\rho_r(t) > 0$ for all $r,\ t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is defined by

$$\rho_r(t) := \inf_{x, y \in B_r, \|x - y\| = t, \alpha \in (0,1)} \frac{\alpha f(x) + (1 - \alpha) f(y) - f(\alpha x + (1 - \alpha) y)}{\alpha(1 - \alpha)}.$$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Recall that the functon f is called totally convex at a point $x \in$ int dom$f$ if its modulus of total convexity at x, that is, the function ν_f : int dom $f \times [0, +\infty) \to [0, +\infty)$, defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in int\ dom f, \|y - x\| = t\},$$

is positive whenever $t > 0$. The function f is called totally convex when it is totally convex at every point $x \in$int dom f. Moreover, the function f is called totally convex on bounded subset of E if $\nu_f(C, t) > 0$ for any bounded subset C of E and for any

$t > 0$, where the modulus of total convexity of the function f on the set C is the function ν_f : int dom$f \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(C, t) := \inf\{\nu_f(x, t) : x \in C \cap int\ domf\}.$$

We remark in passing that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets.

Recall that the function f is said to be sequentially consistent [7] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

We have the following conclusions about totally convex functions which also play an important role in this paper.

Lemma 2.3. [6] *The function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 2.4. [19] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 2.5. [18] *Let $f : X \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:*

(a) f is strongly coercive and uniformly convex on bounded subsets of E;

(b) f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of $domf^* = E^*$.

Let $x \in int\ domf$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y), \quad \forall y \in E\}.$$

The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\},\ x^* \in E^*.$$

The function f is said to be essentially smooth if ∂f is both locally bounded and single-valued on its domain. It is called essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of dom ∂f. f is said to be a Legendre, if it is both essentially smooth and essentially strictly convex. When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \triangledown f$. From [2, 5], we also find that (i) f is essentially smooth if and only if f^* is essentially strictly convex; (ii) $(\partial f)^{-1} = \partial f^*$; (iii) f is Legendre if and only if f^* is Legendre and (iv) If f is Legendre, then $\triangledown f$ is bijection satisfying $\triangledown f = (\triangledown f^*)^{-1}$, $ran\triangledown f = dom\triangledown f^* = int\ domf^*$ and $ran\triangledown f^* = dom\triangledown f = int\ domf$.
Recall that a mapping T is said to be Bregman quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \le D_f(p, x), \quad \forall\ x \in C,\ p \in F(T).$$

T is said to be Bregman quasi-strictly pseudo-contractive [22] if there exists a constant $k \in [0, 1)$ and $F(T) \neq \emptyset$ such that

$$D_f(p, Tx) \le D_f(p, x) + kD_f(x, Tx), \quad \forall\ x \in C,\ p \in F(T).$$

In addition, we also need the following lemmas.

Lemma 2.6. [22] *Let $f : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T : C \to C$ be a Bregman quasi-strictly pseudo-contractive mapping with respect to f. Then $F(T)$ is closed and convex.*

Lemma 2.7. [22] *Let $f : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty, closed, and convex subset of E and let $T : C \to C$ be a Bregman quasi-strictly pseudo-contractive mapping with respect to f. Then, for any $x \in C$, $p \in F(T)$, and $k \in [0, 1)$ the following hold:*

$$D_f(x, Tx) \leq \frac{1}{1-k}\langle \nabla f(x) - \nabla f(Tx), x - p\rangle.$$

3. Main results

Theorem 3.1. *Let E be a real reflexive Banach space and let C be a nonempty closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E and let $T_i : C \to C$, where $i = 1, 2, ..., N$, be a closed and Bregman quasi-k_i-strict pseudo-contraction. Assume $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_0^i = C,\ i = 1, 2, \cdots, N, \quad C_0 = \bigcap_{i=1}^N C_0^i, \\ y_n^i = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_i x_n)], \\ C_{n+1}^i = \{z \in C_n : D_f(z, y_n^i) \leq D_f(z, x_n) + \frac{k_i}{1-k_i}\langle x_n - z, \nabla f(x_n) - \nabla f(T_i x_n)\rangle\}, \\ C_{n+1} = \displaystyle\bigcap_{i=1}^N C_{n+1}^i, \\ x_{n+1} = P^f_{C_{n+1}}(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.1}$$

where $k_i \in [0, 1)$, $\{\alpha_n\}$ is a sequence in $[0, 1]$ with the control condition:

$$\liminf_{n\to\infty}(1-\alpha_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\widehat{p} = P_F^f(x_0)$, where P_F^f is the Bregman projection of E onto F.

Proof. The proof is split into seven steps.
Step 1. Show that $P_F^f(x_0)$ is well defined for every $x_0 \in C$.
From Lemma 2.6, one see that $F(T_i)$ is closed and convex for any $1 \leq i \leq N$. This shows that $F = \bigcap_{i=1}^N F(T_i)$ is also closed and convex. Therefore $P_F^f(x_0)$ is well defined for every $x_0 \in C$.
Step 2. Show that C_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$.

Indeed, it is obvious that $C_0 = C$ is closed and convex. Let C_m is closed and convex for some $m \in \mathbb{N}$. For $z \in C_m$, we see that

$$D_f(z, y_m^i) \leq D_f(z, x_m) + \frac{k_i}{1-k_i}\langle x_m - z, \nabla f(x_m) - \nabla f(T_i x_m)\rangle$$

is equivalent to

$$\langle z, \frac{1}{1-k_i}[\nabla f(x_m) - k_i \nabla f(T_i x_m)] - \nabla f(y_m^i)\rangle \leq f(y_m^i) - f(x_m) - \langle y_m^i, \nabla f(y_m^i)\rangle$$

$$+\langle x_m, \frac{1}{1-k_i}[\nabla f(x_m) - k_i \nabla f(T_i x_m)]\rangle.$$

From the above inequality, we find that C_{m+1} is closed and convex. Therefore C_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$.

Step 3. Show that $F = \bigcap_{i=1}^{N} F(T_i) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

It is obvious that $F(T) \subset C = C_0$. Suppose that $F(T) \subset C_m$ for some $m \in \mathbb{N}$. For any $p \in F(T) \subset C_m$, we obtain

$$\begin{aligned}
D_f(p, y_m^i) &= D_f(p, \nabla f^*[\alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(T_i x_m)]) \\
&= V(p, \alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(T_i x_m)) \\
&= f(p) - \langle p, \alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(T_i x_m)\rangle \\
&+ f^*(\alpha_m \nabla f(x_m) + (1-\alpha_m)\nabla f(T_i x_m)) \\
&\leq \alpha_m[f(p) - \langle p, \nabla f(x_m)\rangle + f^*(\nabla f(x_m))] \\
&+ (1-\alpha_m)[f(p) - \langle p, \nabla f(T_i x_m)\rangle + f^*(\nabla f(T_i x_m))] \\
&= \alpha_m V(p, \nabla f(x_m)) + (1-\alpha_m)V(p, \nabla f(T_i x_m)) \\
&= \alpha_m D_f(p, x_m) + (1-\alpha_m)D_f(p, T_i x_m) \\
&\leq \alpha_m D_f(p, x_m) + (1-\alpha_m)[D_f(p, x_m) + k_i D_f(x_m, T_i x_m)] \\
&\leq D_f(p, x_m) + \frac{(1-\alpha_m)k_i}{1-k_i}\langle x_m - p, \nabla f(x_m) - \nabla f(T_i x_m)\rangle, \\
&\leq D_f(p, x_m) + \frac{k_i}{1-k_i}\langle x_m - p, \nabla f(x_m) - \nabla f(T_i x_m)\rangle.
\end{aligned}$$

This implies that $p \in C_{m+1}$. Thus, we have $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Step 4. Show that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

In fact, since $x_n = P^f_{C_n}(x_0)$, from Lemma 2.2 (c), one has

$$D_f(x_n, x_0) = D_f(P^f_{C_n}(x_0), x_0) \leq D_f(p, x_0) - D_f(p, P^f_{C_n}(x_0)) \leq D_f(p, x_0), \qquad (3.2)$$

for each $p \in F(T)$ and for each $n \geq 1$. Therefore, $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is bounded. In view of Lemma 2.4, one has $\{x_n\}$ is also bounded. On the other hand, noticing that $x_n = P^f_{C_n}(x_0)$ and $x_{n+1} = P^f_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, one has

$$D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0)$$

for all $n \geq 1$. This implies that $\{D_f(x_n, x_0)\}_{n\in\mathbb{N}}$ is a nondecreasing sequence. Therefore, $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

Step 5. Show that $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $\widehat{p} \in C$.

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup \widehat{p} \in C = C_0$. Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $\widehat{p} \in C_n$ for all $n \geq 0$. In view of $x_{n_i} = P^f_{C_{n_i}}(x_0)$, one has $D_f(x_{n_i}, x_0) \leq D_f(\widehat{p}, x_0)$, $\forall n_i \geq 1$. Since f is a lower semi-continuous function on convex set C, it is weakly lower semi-continuous on C. Hence we have

$$\begin{aligned}\liminf_{i\to\infty} D_f(x_{n_i}, x_0) &= \liminf_{i\to\infty}\{f(x_{n_i}) - f(x_0) - \langle \nabla f(x_0), x_{n_i} - x_0\rangle\}\\ &\geq f(\widehat{p}) - f(x_0) - \langle \nabla f(x_0), \widehat{p} - x_0\rangle\\ &= D_f(\widehat{p}, x_0).\end{aligned}$$

Therefore, one has

$$D_f(\widehat{p}, x_0) \leq \liminf_{i\to\infty} D_f(x_{n_i}, x_0) \leq \limsup_{i\to\infty} D_f(x_{n_i}, x_0) \leq D_f(\widehat{p}, x_0),$$

which implies that

$$\lim_{i\to\infty} D_f(x_{n_i}, x_0) = D_f(\widehat{p}, x_0), \tag{3.3}$$

In view of Lemma 2.2 (c), we have that

$$D_f(\widehat{p}, x_{n_i}) \leq D_f(\widehat{p}, x_0) - D_f(x_{n_i}, x_0).$$

By taking $i \to \infty$ in the above inequality and using (3.3), we can obtain that

$$\lim_{n_i\to\infty} D_f(\widehat{p}, x_{n_i}) = 0,$$

which implies from Lemma 2.3 and (2.3) that

$$\lim_{n_i\to\infty} x_{n_i} = \widehat{p}.$$

On the other hand, noticing that $\{D_f(x_n, x_0)\}$ is convergent, this together with (3.3) implies that

$$\lim_{n\to\infty} D_f(x_n, x_0) = D_f(\widehat{p}, x_0). \tag{3.4}$$

From Lemma 2.2 (c), we also have that

$$D_f(\widehat{p}, x_n) \leq D_f(\widehat{p}, x_0) - D_f(x_n, x_0),$$

by taking $n \to \infty$ in the above inequality and (3.4), we obtain that

$$\lim_{n\to\infty} D_f(\widehat{p}, x_n) = 0,$$

which implies from Lemma 2.3 and (2.3) that

$$\lim_{n\to\infty} x_n = \widehat{p}. \tag{3.5}$$

Step 6. Show that the limit of $\{x_n\}_{n\in\mathbb{N}}$ belongs to $F = \bigcap_{i=1}^{N} F(T_i)$.

Since $x_n = P^f_{C_n} x_0$, one has from Lemma 2.2 (c) that

$$D_f(x_{n+1}, x_n) \leq D_f(x_{n+1}, x_0) - D_f(x_n, x_0),$$

From Step 4, one has

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.6}$$

Since f is totally convex on bounded subsets of E, f is sequentially consistent. It follows from (3.6) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

From the uniform continuity of ∇f and (3.7), one also has

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0. \tag{3.8}$$

On the other hand, since $x_{n+1} = P^f_{C_{n+1}} x_0 \in C_{n+1}$, one has that

$$D_f(x_{n+1}, y_n^i) \le D_f(x_{n+1}, x_n) + \frac{k_i}{1-k_i}\langle x_n - x_{n+1}, \nabla f(x_n) - \nabla f(T_i x_n)\rangle,$$

which implies that

$$\lim_{n\to\infty} D_f(x_{n+1}, y_n^i) = 0, \quad \forall\ i = 1, 2, \cdots, N. \tag{3.9}$$

Since f is totally convex on bounded subsets of E, f is sequentially consistent. It follows from (3.9) that

$$\lim_{n\to\infty} \|x_{n+1} - y_n^i\| = 0, \quad \forall\ i = 1, 2, \cdots, N. \tag{3.10}$$

From the uniform continuity of ∇f and (3.10), one has

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(y_n^i)\| = 0, \quad \forall\ i = 1, 2, \cdots, N. \tag{3.11}$$

From $y_n^i = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_i x_n)]$, We find that

$$\begin{aligned}
&\|\nabla f(x_{n+1}) - \nabla f(y_n^i)\| \\
&= \|\nabla f(x_{n+1}) - [\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_i x_n)]\| \\
&= \|\alpha_n[\nabla f(x_{n+1}) - \nabla f(x_n)] + (1-\alpha_n)[\nabla f(x_{n+1}) - \nabla f(T_i x_n)]\| \\
&= \|(1-\alpha_n)[\nabla f(x_{n+1}) - \nabla f(T_i x_n)] - \alpha_n[\nabla f(x_n) - \nabla f(x_{n+1})]\| \\
&\ge (1-\alpha_n)\|\nabla f(x_{n+1}) - \nabla f(T_i x_n)\| - \alpha_n\|\nabla f(x_n) - \nabla f(x_{n+1})\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|\nabla f(x_{n+1}) - \nabla f(T_i x_n)\| \le \frac{1}{(1-\alpha_n)}[&\|\nabla f(x_{n+1}) - \nabla f(y_n^i)\| \\
&+ \alpha_n\|\nabla f(x_n) - \nabla f(x_{n+1})\|].
\end{aligned}$$

From (3.8), (3.11) and the control condition $\liminf_{n\to\infty}(1-\alpha_n) > 0$, one obtains that

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(T_i x_n)\| = 0, \quad \forall\ i = 1, 2, \cdots, N.$$

Since f is strongly coercive and uniformly convex on bounded subset of E, f^* is uniformly Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, one arrives at

$$\lim_{n\to\infty} \|x_{n+1} - T_i x_n\| = 0, \quad \forall\ i = 1, 2, \cdots, N. \tag{3.12}$$

From triangle inequality principal, one has

$$\|x_n - T_i x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - T_i x_n\|. \tag{3.13}$$

From (3.7), (3.12) and (3.13), one obtains

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \forall\, i = 1, 2, \cdots, N. \tag{3.14}$$

For any $i = 1, 2, 3, \cdots, N$, in view of the closedness of T_i, $\lim_{n\to\infty} x_n = \widehat{p}$ and (3.14), one has $T_i p = p$, that is, $p \in \bigcap_{i=1}^{N} F(T_i) = F$.

Step 7. Show that $\widehat{p} = P^f_{F(T)}(x_0)$.

From $x_n = P^f_{C_n} x_0$, one has $\langle y - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \le 0$, $\forall y \in C_n$. Since $F \subset C_n$ for each $n \in \mathbb{N}$, one obtains

$$\langle y - x_n, \nabla f(x_0) - \nabla f(x_n)\rangle \le 0, \quad \forall y \in F. \tag{3.15}$$

Taking $n \to \infty$ in (3.15), one has

$$\langle y - \widehat{p}, \nabla f(x_0) - \nabla f(\widehat{p})\rangle \le 0, \quad \forall y \in F.$$

In view of Lemma 2.2 (a) and Lemma 2.2 (b), one has $\widehat{p} = P^f_{F(T)}(x_0)$. This completes the proof of Theorem 3.1. □

For a single closed and Bregman quasi-strict pseudo-contraction T, we find the following result immediately.

Corollary 3.2. *Let E be a real reflexive Banach space and let C be a nonempty closed and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E, and let $T : C \to C$ be a Bregman quasi-k-strict pseudo-contraction such that $F(T) \ne \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_0 = C, \\ y_n = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)], \\ C_{n+1} = \{z \in C_n : D_f(z, y_n) \le D_f(z, x_n) + \frac{k}{1-k}\langle x_n - z, \nabla f(x_n) - \nabla f(Tx_n)\rangle\}, \\ x_{n+1} = P^f_{C_{n+1}}(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $k \in [0, 1)$, $\{\alpha_n\}$ is sequence in $[0, 1]$ with the control condition:

$$\liminf_{n\to\infty} (1 - \alpha_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\widehat{p} = P^f_{F(T)}(x_0)$, where $P^f_{F(T)}$ is the Bregman projection of E onto $F(T)$.

It is clear that $F(P^f_{K_i}) = K_i$ for any $i = 1, 2, 3, \cdots, N$. If the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the Bregman projection $P^f_{K_i}$ is a closed Bregman relatively nonexpansive mapping, so is a closed Bregman quasi-strict pseudo-contraction. In the following, we employ Theorem 3.1 in solving the following convex feasibility problems.

Corollary 3.3 *Let E be a real reflexive Banach space and let C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E, and K_i, $i = 1, 2, ..., N$, be a finite family of closed and nonempty subset of C such that $F = \bigcap_{i=1}^{N} K_i \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_0^i = C, \ i = 1, 2, \cdots, N, \quad C_0 = \bigcap_{i=1}^N C_0^i, \\ y_n^i = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(P_{K_i} x_n)], \\ C_{n+1}^i = \{z \in C_n : D_f(z, y_n^i) \leq D_f(z, x_n)\rangle\}, \\ C_{n+1} = \bigcap_{i=1}^N C_{n+1}^i, \\ x_{n+1} = P_{C_{n+1}}^f(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with the control condition:

$$\liminf_{n\to\infty}(1-\alpha_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\widehat{p} = P_F^f(x_0)$, where P_F^f is the Bregman projection of E onto F.

Next, we give some applications of the main results.

1. Applications to equilibrium problems

Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $G : C \times C \to \mathbb{R}$ be a bifunction that satisfies the following conditions:

(A1) $G(x, x) = 0$ for all $x \in C$;

(A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for all $x, \ y \in C$;

(A3) for all $x, \ y, \ z \in C$, $\limsup_{t\downarrow 0} G(tz + (1-t)x, y) \leq G(x, y)$;

(A4) for each $x \in C$, $G(x, \cdot)$ is convex and lower semicontinuous.

The "so-called" equilibrium problem corresponding to G is to find $\bar{x} \in C$ such that $G(\bar{x}, y) \geq 0, \quad \forall y \in C$. The set of its solutions is denoted by $EP(G)$. The resolvent of a bifunction $G : C \times C \to \mathbb{R}$ is the operator $Res_G^f : E \to 2^C$ defined by

$$Res_G^f(x) = \{z \in C : G(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \quad \forall \ y \in C\}. \tag{3.16}$$

It is well known that Res_G^f has the following properties:

(1) Res_G^f is single-valued;

(2) The set of fixed points of Res_G^f is the solution set of the corresponding equilibrium problem, i.e., $F(Res_G^f) = EP(G)$;

(3) Res_G^f is a closed Bregman quasi-nonexpansive mapping, so is a closed Bregman quasi-strict pseudo-contraction.

Theorem 3.4. *Let E be a real reflexive Banach space let C be a nonempty, closed, and convex subset of E. Let $G_i : C \times C \to \mathbb{R}$, $i = 1, 2, \cdots, N$, be a finite family of bifunctions* *that satisfy conditions (A1)-(A4) such that* $F = \bigcap_{i=1}^{N} EP(G_i) \neq \emptyset$. *Let* $f : E \to \mathbb{R}$ *be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* E, *and* $Res^f_{G_i} : E \to 2^C$ *be resolvent operator defined as (3.16). Let* $\{x_n\}$ *be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_0^i = C,\ i = 1, 2, \cdots, N, \quad C_0 = \bigcap_{i=1}^{N} C_0^i, \\ y_n^i = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res^f_{G_i} x_n)], \\ C_{n+1}^i = \{z \in C_n : D_f(z, y_n^i) \leq D_f(z, x_n)\rangle\}, \\ C_{n+1} = \bigcap_{i=1}^{N} C_{n+1}^i, \\ x_{n+1} = P^f_{C_{n+1}}(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $\kappa \in [0, 1)$.
Then $\{x_n\}$ *converges strongly to* $\widehat{p} = P^f_F(x_0)$, *where* P^f_F *is the Bregman projection of* E *onto* F.

Proof. Since $Res^f_{G_i}$ is a closed Bregman quasi-strict pseudo-contraction for each $i = 1, 2, \cdots, N$, by applying Theorem 3.1, we find that $\{x_n\}$ converges strongly to $\widehat{p} = P^f_F(x_0)$. □

2. Applications to zero point problem of maximal monotone operators

Let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by dom A, that is, dom $A = \{x \in E : Ax \neq \emptyset\}$. The range of A is denoted by ran A, that is, ran $A = \{Ax : x \in dom\ A\}$. A mapping $A : E \to 2^{E^*}$ is said to be monotone if for any $x, y \in$ dom A, we have

$$u \in Ax,\ v \in Ay \Rightarrow \langle u - v, x - y\rangle \geq 0.$$

A monotone mapping A is said to be maximal if graph A, the graph of A, is not a proper subset of the graph of any other monotone mapping.

Let E be a real reflexive Banach space, $A : E \to 2^{E^*}$ be a maximal monotone operator. The problem of finding an element $x \in E$ such that $0^* \in Ax$ is very important in optimization theory and related fields.

Recall that the resolvent of A, denoted by $Res^f_A : E \to 2^E$, is defined as follows:

$$Res^f_A(x) = (\nabla f + A)^{-1} \circ \nabla f(x). \tag{3.17}$$

It is well known that the fixed point set of the resolvent Res^f_A is equal to the set of zeroes of the mapping A, that is, $F(Res^f_A) = A^{-1}(0^*)$. In fact,

$$\begin{aligned} u \in F(Res^f_A) &\Leftrightarrow u = Res^f_A(u) = (\nabla f + A)^{-1} \circ \nabla f(u) \Leftrightarrow \nabla f(u) \in \nabla f(u) + A(u) \\ &\Leftrightarrow 0^* \in A(u) \Leftrightarrow u \in (A)^{-1} 0^*. \end{aligned}$$

From [20], we know that Res_A^f is a closed Bregman quasi-strict pseudo-contraction. So the following result is obtained easily by applying Theorem 3.1.

Theorem 3.5. *Let E be a real reflexive Banach space with the dual E^*, $A_i : E \to 2^{E^*}$, $i = 1, 2, \cdots, N$, be a finite family of maximal monotone operators with*

$$F = \bigcap_{i=1}^{N} A_i^{-1}(0^*) \neq \emptyset.$$

Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $Res_{A_i}^f : E \to 2^E$ be the resolvent with respect to A_i. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_0^i = C, \quad i = 1, 2, \cdots, N, \\ C_0 = \bigcap_{i=1}^{N} C_0^i, \\ y_n^i = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res_{A_i}^f x_n)], \\ C_{n+1}^i = \{z \in C_n : D_f(z, y_n^i) \leq D_f(z, x_n)\rangle\}, \\ C_{n+1} = \bigcap_{i=1}^{N} C_{n+1}^i, \\ x_{n+1} = P_{C_{n+1}}^f(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ with the control condition:

$$\liminf_{n\to\infty}(1-\alpha_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\widehat{p} = P_F^f(x_0)$, where P_F^f is the Bregman projection of E onto F.

3. Application to minimizers of proper, lower semicontinuous, and convex functionals

For a proper lower semicontinuous convex function $g : E \to (-\infty, +\infty]$, the subdifferential mapping $\partial g \subset E \times E^*$ of g is defined as follows:

$$\partial g = \{x^* \in E^* : g(y) \geq g(x) + \langle y - x, x^*\rangle, \ \forall\, y \in \ E\}, \quad \forall\, x \in E.$$

From Rockafellar [21], we know that ∂g is maximal monotone. It is easy to verify that $0^* \in \partial g(v)$ if and only if $g(v) = \min_{x\in E} g(x)$. Emulating (3.17), the resolvent of ∂g, denoted by $Res_{\partial g}^f : E \to 2^E$, is defined as follows:

$$Res_{\partial g}^f(x) = (\nabla f + \partial g)^{-1} \circ \nabla f(x).$$

Theorem 3.6. *Let E be a real reflexive Banach space with the dual E^*, $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $g_i : E \to (-\infty, \infty]$, $i = 1, 2, \cdots, N$, be a finite family of proper, lower semicontinuous, and convex function, ∂g_i the subdifferential* *mapping of* g_i*,* $Res^f_{\partial g_i}$ *the resolvent of* ∂g_i*. Assume that* $F = \bigcap_{i=1}^{N} (\partial g_i)^{-1}(0^*) \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated by the following iterative algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_0^i = C, \quad i = 1, 2, \cdots, N, \\ C_0 = \bigcap_{i=1}^{N} C_0^i, \\ y_n^i = \nabla f^*[\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res^f_{\partial g_i} x_n)], \\ C_{n+1}^i = \{z \in C_n : D_f(z, y_n^i) \leq D_f(z, x_n)\rangle\}, \\ C_{n+1} = \bigcap_{i=1}^{N} C_{n+1}^i, \\ x_{n+1} = P^f_{C_{n+1}}(x_0), \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *with the control condition:*

$$\liminf_{n\to\infty}(1-\alpha_n) > 0.$$

Then $\{x_n\}$ *converges strongly to* $\widehat{p} = P^f_F(x_0)$*, where* P^f_F *is the Bregman projection of* E *onto* F*.*

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