Fixed Point Theory, 21(2020), No. 1, 363-374
DOI: 10.24193/fpt-ro.2020.1.25
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

STABILITY OF MAXIMUM PRESERVING FUNCTIONAL EQUATION ON MULTI-BANACH LATTICE BY FIXED POINT METHOD

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Abstract. We prove the stability of maximum preserving functional equation by substituting the addition with the maximum operation in Cauchy functional equation in multi-Banach lattice by fixed point method.

Key Words and Phrases: Multi-Banach lattice, maximum preserving functional equation, Cauchy functional equation, multi-normed spaces.
2010 Mathematics Subject Classification: 46B42, 34K20.

1. Introduction

The stability problems for functional equations was raised by Ulam [24] in 1940 concerning the stability of group homomorphisms. This question has been affirmatively answered for Banach spaces by Hyers [14]. Later, Aoki [5] and Bourgin [6] considered the stability problem for the case of an additive mapping between Banach spaces subject to an unbounded Cauchy difference. In 1978, Th. M. Rassias [18]was the first to prove the stability of the linear mapping between Banach spaces subject to a continuity assumption on the mapping. In 1994, Găvruta [13]provided a further generalization of Th. M. Rassias result in which he replaced the bound $\varepsilon(\|x\|^r + \|y\|^r)$ by a general function $\phi(x, y)$ for the existence of unique linear mapping.

In 1996, Isac and Rassias [16] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By fixed point methods of several functional equations have been extensively investigated by a number of authors (see [8, 17]). The notion of multi-normed space was introduced by Dales and Polyakov (see [11, 10]). This concept is some what similar two operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples were given in [11].

Agbeko has studied the stability of maximum preserving functional equations motivated by the optimal average (see [1, 2, 4, 3]). He has replaced addition operation with the maximum operation on a given Banach lattice.

The most famous functional equation by Cauchy and known as linear functional equation reads: $f(x+y) = f(x) + f(y)$.

In 2003, Radu proved the Hyers-Ulam-Rassias stability of the additive Cauchy equation by using the fixed point method (see [7]).

In [20, 21], we have used the technique of [3] and obtained following results about quadratic functional equations. In this paper, we generalized Agbeko's theorem for Cauchy functional equation in multi-Banach lattice by fixed point method.

Definition 1.1. Let X be a set. A function $d : X^2 \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies
(M1) $d(x, y) = 0$ if and only if $x = y$;
(M2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We remark that the only difference between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We now introduce one of the fundamental results of the fixed point theory.

Theorem 1.2. *Let (X,d) be a generalized complete metric space. Assume that* $\Lambda : X \to X$ *is a strictly contractive operator with the Lipschitz constant* $L < 1$. *If there exists a nonnegative integer* n_0 *such that* $d(\Lambda^{n_0+1}x, \Lambda^{n_0}x) < \infty$ *for some* $x \in X$, *then the following statements are true:*
(i) The sequence $\{\Lambda^n x\}$ *converges to a fixed point* x^* *of* Λ;
(ii) x^* *is the unique fixed point of* Λ *in* $X^* = \{y \in X | d(\Lambda^{n_0}x, y) < \infty\}$;
(iii) If $y \in X^*$, *then*

$$d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$$

Now, recalling the notion of a multi-normed space from [11, 10]. In this paper, $(E, \|\cdot\|)$ denotes a complex normed space and let $k \in \mathbb{N}$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$ the linear operations E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \ldots, k\}$ and by $\mathfrak{S}_k$ the group of permutations on k symbols.

Definition 1.3. A multi-norm on $\{E^k : k \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_k) = (\|\cdot\|) : k \in \mathbb{N})$ such that $\|\cdot\|_k$ is a norm on E^k for each $k \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

N1: $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|x_1, \ldots, x_k)\|_k \quad (\sigma \in \mathfrak{S}_k; x_1, \ldots, x_k \in E)$;
N2: $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq (\max_{i \in \mathbb{N}_k} |\alpha_i|)\|(x_1, \ldots, x_k)\|_k$
$(\alpha_1, \ldots, \alpha_k \in \mathbb{C}; x_1, \ldots x_k \in E)$;
N3: $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \quad (x_1, \ldots, x_{k-1} \in E)$;
N4: $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1} \quad (x_1, \ldots, x_{k-1} \in E)$.

In this case, we say that $((E^k, \|\cdot\|) : k \in \mathbb{N})$ is a *multi-normed space*.

The motivation for the study of multi-normed spaces (and multi-normed algebras) and many examples are detailed in the earlier investigation [11].

Suppose that $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-normed space, and take $k \in \mathbb{N}$. The following properties are almost immediate consequences of the axioms.

(a) $\|(x, \ldots, x)\|_k = \|x\| \qquad (x \in E);$

(b) $\max_{i\in\mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i\in\mathbb{N}_k \|x_i\|} \quad (x_1, \ldots, x_k \in E).$

It follows from the item (b) above that, if $(E, \|\cdot\|)$ is a Banach space, then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbb{N}$; in this case, $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space.

Example 1.4. Let $(E, \|\cdot\|)$ be Banach lattice and define

$$\|(x_1, \ldots, x_k)\|_k := \||x_1| \vee \cdots \vee |x_k|\| \qquad (x_1, \ldots, x_k) \in E$$

Then $((E^k, \|\cdot\|_k) : k \in \mathbb{N})$ is a multi-Banach space(see [11]). We say it *multi-Banach lattice.*

2. Main results

Throughout this section, let $(E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $p : [0, \infty) \to [0, \infty)$ be continuous function and $\tau, \eta \in \mathbb{R}^+$. For convenience we use the following abbreviation for a given mapping $f : E_1 \to E_2$

$$Df(x, y) = f(\tau|x| \vee \eta|y|) - \frac{(\tau p(\tau) f(|x|) \vee \eta p(\eta) f(|y|))}{p(\tau) \vee p(\eta)}$$

Let us recall some necessary definitions.

If B is a Banach lattice, then B^+ stands for its positive cone, i.e.

$$B^+ = \{x \in B : x \geq 0\} = \{|x| : x \in B\}.$$

Given two Banach lattices X and Y we say that a functional $f : X \to Y$ is cone-related if $f(X^+) = \{f(|x|) : x \in B\} \subset Y^+$(see [3]).

Let X and Y be two Banach lattices and $f : X \to Y$ be a cone-related functional, with following properties:

I) Maximum Preserving Functional Equation: $f(|x| \vee |y|) = f(|x|) \vee f(|y|)$ for all members $x, y \in X$(see [3]).

II) Semi-homogeneity: $f(\alpha|x|) = \alpha f(|x|)$ for all $x \in X$ and every number $\alpha \in [0, \infty)$.

We shall use the technics in [3] to prove the following two theorems.

Theorem 2.1. *Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice. Suppose $\phi : E_1^{2k} \to [0, \infty)$ is a given function and there exists a constant L, $0 < L < 1$, such that*

$$\phi(tx_1, ty_1, \ldots, tx_k, ty_k) \leq tL\phi(x_1, y_1, \ldots, x_k, y_k) \tag{2.1}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E_1$, $t \in [0, \infty)$. Furthermore, let $f : E_1 \to E_2$ be a cone-related function with $f(0) = 0$ which satisfies

$$\|Df(x_1, y_1), \ldots, Df(x_k, y_k)\|_k \leq \phi(\tau x_1, \eta y_1, \ldots, \tau x_k, \eta y_k) \tag{2.2}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E_1$. *If* ϕ *satisfies*

$$\lim_{n\to\infty} t^{-n}\phi(t^n x_1, t^n y_1, \ldots, t^n x_k, t^n y_k) = 0 \tag{2.3}$$

for any $x_1, \ldots, x_k, y_1, \ldots, y_k \in E_1$, *then there is a unique cone-related mapping* $T : E_1 \to E_2$ *which satisfies properties I, II and the inequality.*

$$\|T(|x_1|) - f(|x_1|), \ldots, T(|x_k|) - f(|x_k|)\|_k \leq \frac{L}{1-L}\phi(x_1, x_1, \ldots, x_k, x_k) \tag{2.4}$$

Proof. If we define

$$X = \{g : E_1 \to E_2 | \;\; g(0) = 0\}$$

and introduce a generalized metric on X as follows:

$$\begin{aligned} d(g,h) = \inf\{c \in [0,\infty] : &\|g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\|_k \\ &\leq c\phi(x_1, x_1, \ldots, x_k, x_k), \quad \text{for all } x_1, \ldots, x_k \in E_1\} \end{aligned}$$

then (X, d) is complete. We define an operator $\Lambda : X \to X$ by

$$(\Lambda g)(x) = \frac{g(tx)}{t}$$

for all $x \in E_1$. First, we assert that Λ is strictly contractive on X. Given $g, h \in X$, let $c \in [0,\infty)$ be an arbitrary constant with $d(g,h) \leq c$, i.e.,

$$\|g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k)\|_k \leq c\phi(x_1, x_1, \ldots, x_k, x_k),$$

for all $x_1, \ldots, x_k \in E_1$. If we replace x in the last inequality with tx and make use of (2.1), then we have

$$\begin{aligned} &\|\Lambda g(x_1) - \Lambda h(x_1), \ldots, \Lambda g(x_k) - \Lambda h(x_k)\|_k \\ &= \frac{1}{t}\ \|g(tx_1) - h(tx_1), \ldots, g(tx_k) - h(tx_k)\|_k \\ &\leq \frac{1}{t}\ c\phi(tx_1, tx_1, \ldots, tx_k, tx_k) \leq Lc\phi(x_1, x_1, \ldots, x_k, x_k) \end{aligned}$$

for every $x_1, \ldots, x_k \in E_1$, i.e., $d(\Lambda g, \Lambda h) \leq Lc$.
Hence, we conclude that $d(\Lambda g, \Lambda h) \leq Ld(g,h)$ for any $g, h \in X$. Next, we assert that $d(\Lambda f, f) < \infty$. If we substitute x for y in (2.2) and $\tau = \eta = t$, then (2.1) establishes

$$\begin{aligned} &\|f(t|x_1|) - tf(|x_1|), \ldots, f(t|x_k|) - tf(|x_k|)\|_k \leq \phi(tx_1, tx_1, \ldots, tx_k, tx_k) \\ \Rightarrow &\left\|\frac{f(t|x_1|)}{t} - f(|x_1|), \ldots, \frac{f(t|x_k|)}{t} - f(|x_k|)\right\|_k \leq \frac{1}{t}\ \phi(tx_1, tx_1, \ldots, tx_k, tx_k) \\ &\leq L\phi(x_1, x_1, \ldots, x_k, x_k) \\ \Rightarrow &\|\Lambda f(|x_1|) - f(|x_1|), \ldots, \Lambda f(|x_k|) - f(|x_k|)\| \leq L\phi(x_1, x_1, \ldots, x_k, x_k) \end{aligned}$$

for any $x_1, \ldots, x_k \in E_1$, i.e.,

$$d(\Lambda f, f) \leq L \leq \infty \tag{2.5}$$

Now, it follows from Theorem 1.2 (i) that there exists a function $T : E_1 \to E_2$ with $T(0) = 0$, which is a fixed point of Λ, such that $\Lambda^n f \to T$, i.e.,

$$T(x) = \lim_{n\to\infty} \frac{f(t^n x)}{t^n}$$

for all $x \in E_1$. Since the integer n_0 of Theorem 1.2 is 0 then $f \in X^*$ which

$$X^* = \{y \in X : \quad d(\Lambda^{n_0} f, y) < \infty\}.$$

By Theorem 1.2 (iii) and (2.5) we obtain

$$d(f,T) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{L}{1-L}$$

i.e., the inequality (2.4) is true for all $x \in E_1$.
Clearly, T is a cone-related operator. Let us show that T is maximum preserving. Let $\tau = \eta = t^n$ in (2.3) we have

$$\begin{aligned}\|f(t^n(|x_1| \vee |y_1|)) - t^n(f(|x_1|) \vee f(|y_1|)), \dots, f(t^n(|x_k| \vee |y_k|))\\ - t^n(f(|x_k|) \vee f(|y_k|))\|_k \leq \phi(t^n x_1, t^n y_1, \dots, t^n x_k, t^n y_k).\end{aligned}$$

Substituting x with $t^n x$ and y with $t^n y$ in the last inequality:

$$\begin{aligned}\|f(t^{2n}(|x_1| \vee |y_1|)) - t^n(f(t^n|x_1|) \vee f(t^n|y_1|)), \dots, f(t^{2n}(|x_k| \vee |y_k|))\\ - t^n(f(t^n|x_k|) \vee f(t^n|y_k|))\|_k \leq \phi(t^{2n} x_1, t^{2n} y_1, \dots, t^{2n} x_k, t^{2n} y_k).\end{aligned}$$

Thus

$$\begin{aligned}\|t^{-2n} f(t^{2n}(|x_1| \vee |y_1|)) - t^{-n}(f(t^n|x_1|) \vee f(t^n|y_1|)), \dots\\ , t^{-2n} f(t^{2n}(|x_k| \vee |y_k|)) - t^{-n}(f(t^n|x_k|) \vee f(t^n|y_k|))\|_k\\ \leq t^{-2n}\phi(t^{2n} x_1, t^{2n} y_1, \dots, t^{2n} x_k, t^{2n} y_k).\end{aligned}$$

with use of (2.1)

$$\begin{aligned}\|t^{-2n} f(t^{2n}(|x_1| \vee |y_1|)) - t^{-n}(f(t^n|x_1|) \vee f(t^n|y_1|)), \dots\\ , t^{-2n} f(t^{2n}(|x_k| \vee |y_k|)) - t^{-n}(f(t^n|x_k|) \vee f(t^n|y_k|))\|_k\\ \leq t^{-n} L^n \phi(t^n x_1, t^n y_1, \dots, t^n x_k, t^n y_k).\end{aligned}$$

By letting $n \to \infty$ and considering (2.2), replace $x_1, \dots, x_k$ with x and $y_1, \dots, y_k$ with y in the last inequality conclude

$$\lim_{n\to\infty} \|t^{-2n} f(t^{2n}(|x| \vee |y|)) - t^{-n}(f(t^n|x|) \vee f(t^n|y|))\| = 0$$

we get for all $x, y \in X$ the equality

$$\|T(|x| \vee |y|) - T(|x|) \vee T(|y|)\| = 0$$

or equivalently

$$T(|x| \vee |y|) = T(|x|) \vee T(|y|),$$

because,

$$\lim_{n\to\infty} t^{-2n} f(t^{2n}|z|) = \lim_{m\to\infty} t^{-m} f(t^m|z|) = T(|z|), \quad z \in X$$

Now, we must show $T(r|x|) = rT(|x|)$ for all $x \in X$ and $r \in [0, \infty)$. Using the inequality (2.2) with $\eta = \tau$, $y_1, \dots, y_k = 0$ and substituting τ with $t^n\tau$:

$$\begin{aligned}\|f(t^n\tau(|x_1|)) - t^n\tau(f(|x_1|)), \dots, f(t^n\tau(|x_k|) - t^n\tau(f(|x_k|))\|_k\\ \leq \phi(t^n\tau x_1, 0, \dots, t^n\tau x_k, 0).\end{aligned}$$

If we replace $x_1, \dots, x_k$ with $t^n x_1, \dots, t^n x_k$ respectively, then:

$$\|f(t^{2n}\tau(|x_1|)) - t^n\tau(f(t^n|x_1|)), \dots, f(t^{2n}\tau(|x_k|) - t^n\tau(f(t^n|x_k|))\|_k \\ \leq \phi(t^{2n}\tau x_1, 0, \dots, t^{2n}\tau x_k, 0).$$

Divide by t^{2n} both side of above inequality and use the inequality (2.1) :

$$\|t^{-2n}f(t^{2n}\tau(|x_1|)) - t^{-n}\tau(f(t^n|x_1|)), \dots, t^{-2n}f(t^{2n}\tau(|x_k|) \\ -t^{-n}\tau(f(t^n|x_k|))\|_k \leq t^{-2n}\phi(t^{2n}\tau x_1, 0, \dots, t^{2n}\tau x_k, 0) \\ \leq t^{-n}L^n\phi(t^n\tau x_1, 0, \dots, t^n\tau x_k, 0).$$

By letting $n \to \infty$ and considering (2.3), replace $x_1, \dots, x_k$ with x in the last inequality conclude

$$\lim_{n\to\infty} \|t^{-2n}f(t^{2n}(\tau|x|)) - t^{-n}\tau(f(t^n|x|))\| = 0$$

we get for all $x \in X$ the equality

$$\lim_{n\to\infty} t^{-2n}f(t^{2n}\tau|x|) = \tau \lim_{n\to\infty} t^{-n}f(t^n|x|) = \tau T(|x|),$$

by taking $z = \tau|x|$, we have

$$\tau T(|x|) = \lim_{n\to\infty} t^{-2n}f(t^{2n}\tau|x|) = \lim_{n\to\infty} t^{-2n}f(t^{2n}|z|) = T(|z|) = T(\tau|x|).$$

For uniqueness of T: Assume that the inequality (2.4) is also satisfied with another homogenous function of degree two $S : E_1 \to E_2$ besides T. (As S is a homogeneous function of degree two, S satisfies that

$$S(x) = \frac{S(tx)}{t} = \Lambda S(x)$$

for all $x \in E_1$. That is, S is a fixed point of Λ.) In view of (2.4) and the definition of d, we know that

$$d(f, S) \leq \frac{L}{1-L} < \infty$$

i.e.,$S \in X^*$. (In view of (2.5), the integer n_0 of Theorem 1.2 is 0.) Thus, Theorem 1.2 (ii) implies that $S = T$. This proves the uniqueness of T. $\square$

Theorem 2.2. *Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $f : E_1 \to E_2$ be a cone-related functional for which there are numbers $\theta > 0$ and $0 \leq r < 1$ such that*

$$\|Df(x_1, y_1), \dots, Df(x_k, y_k)\|_k \leq \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r) \tag{2.6}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E_1$; then there is a unique cone-related mapping $T : E_1 \to E_2$ such that

$$\|T(|x_1|) - f(|x_1|), \dots, T(|x_k|) - f(|x_k|)\|_k \leq \frac{2\theta}{2-2^r} \sum_{i=1}^{k} \|x_i\|^r \tag{2.7}$$

and satisfies properties I, II.

Proof. If we define

$$X = \{g : E_1 \to E_2| \quad g(0) = 0\}$$

and introduce a generalized metric on X as follows:

$$d(g,h) = \inf\{c \in [0,\infty] : \|g(x_1)-h(x_1),\dots,g(x_k)-h(x_k)\|_k \le c\sum_{i=1}^{k}\|x_i\|^r, \quad \text{for all } x_1,\dots,x_k \in E_1\}.$$

then (X, d) is complete. We define an operator $\Lambda : X \to X$ by

$$(\Lambda g)(x) = \frac{g(2|x|)}{2},$$

for all $x \in E_1$. First, we assert that Λ is strictly contractive on X. Given $g,h \in X$, let $c \in [0,\infty)$ be an arbitrary constant with $d(g,h) \le c$, i.e.,

$$\|g(x_1)-h(x_1),\dots,g(x_k)-h(x_k)\|_k \le c\sum_{i=1}^{k}\|x_i\|^r,$$

for all $x \in E_1$. If we replace x in the last inequality with $2x$, then we have

$$\begin{aligned}
&\|\Lambda g(x_1)-\Lambda h(x_1),\dots,\Lambda g(x_k)-\Lambda h(x_k)\|_k\\
&=\frac{1}{2}\|g(2x_1)-h(2x_1),\dots,g(2x_k)-h(2x_k)\|_k\\
&\le \frac{c}{2}\sum_{i=1}^{k}\|2x_i\|^r \le 2^{r-1}c\sum_{i=1}^{k}\|x_i\|^r,
\end{aligned}$$

for every $x_1,\dots,x_k \in E_1$, i.e., $d(\Lambda g,\Lambda h) \le 2^{r-1}c$. Hence, we conclude that

$$d(\Lambda g,\Lambda h) \le 2^{r-1}d(g,h),$$

for any $g,h \in X$, and so Λ is strictly contortive with constant $L = 2^{r-1} < 1$ on X.

Next, we assert that $d(\Lambda f, f) < \infty$. If we substitute x for y in (2.6) and $\tau = \eta = 2$ divide both sides by 2, then (2.6) establishes

$$\begin{aligned}
&\|f(2|x_1|)-2f(|x_1|),\dots,f(2|x_k|)-2f(|x_k|)\|_k \le 2\theta\sum_{i=1}^{k}\|x_i\|^r\\
&\Rightarrow \|\frac{1}{2}f(2|x_1|)-f(|x_1|),\dots,\frac{1}{2}f(2|x_k|)-f(|x_k|)\|_k \le \theta\sum_{i=1}^{k}\|x_i\|^r\\
&\Rightarrow \|\Lambda f(|x_1|)-f(|x_1|),\dots,\Lambda f(|x_k|)-f(|x_k|)\|_k \le \theta\sum_{i=1}^{k}\|x_i\|^r
\end{aligned}$$

for any $x_1,\dots,x_k \in E_1$, i.e.,

$$d(\Lambda f, f) \le \theta \le \infty \tag{2.8}$$

Now, it follows from Theorem 1.2 (i) that there exists a function $T : E_1 \to E_2$ with $T(0) = 0$, which is a fixed point of Λ, such that $\Lambda^n f \to T$, i.e.,

$$T(|x|) = \lim_{n\to\infty} 2^{-n} f(2^n|x|) \tag{2.9}$$

for all $x \in E_1$. Since the integer n_0 of Theorem 1.2 is 0 then $f \in X^*$ which

$$X^* = \{y \in X : \quad d(\Lambda^{n_0} f, y) < \infty\}.$$

By Theorem 1.2 (iii) and (2.8) we obtain

$$d(f,T) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{2\theta}{2-2^r}$$

i.e., the inequality (2.7) is true for all $x_1, \dots, x_k \in E_1$.
Clearly, T is a cone-related operator. Let us show that T is maximum preserving. Let $\tau = \eta = 2^n$ in (2.6). We have

$$\|f(2^n(|x_1| \vee |y_1|)) - 2^n(f(|x_1|) \vee f(|y_1|)), \dots, f(2^n(|x_k| \vee |y_k|)) - 2^n(f(|x_k|) \vee f(|y_k|))\|_k \leq \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r)$$

Substituting $x_1, \dots, x_k$ with $2^n x_1, \dots, 2^n x_k$ and $y_1, \dots, y_k$ with $2^n y_1, \dots, 2^n y_k$, respectively in the last inequality:

$$\|f(4^n(|x_1| \vee |y_1|)) - 2^n(f(2^n|x_1|) \vee f(|2^n y_1|)), \dots, f(4^n(|x_k| \vee |y_k|)) - 2^n(f(2^n|x_k|) \vee f(2^n|y_k|))\|_k \leq 2^{nr}\theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r)$$

Thus

$$\|4^{-n} f(4^n(|x_1| \vee |y_1|)) - 2^{-n}(f(2^n|x_1|) \vee f(2^n|y_1|)), \dots, 4^{-n} f(4^n(|x_k| \vee |y_k|)) - 2^{-n}(f(2^n|x_k|) \vee f(2^n|y_k|))\|_k \leq 2^{n(r-2)}\theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r)$$

By letting $n \to \infty$, replace $x_1, \dots, x_k$ with x and $y_1, \dots, y_k$ with y in the last inequality conclude

$$\lim_{n\to\infty} \|4^{-n} f(4^n(|x| \vee |y|)) - 2^{-n}(f(2^n|x|) \vee f(2^n|y|))\| = 0,$$

because $r < 1$, by considering (2.9) we get for all $x, y \in X$ the equality

$$\|T(|x| \vee |y|) - T(|x|) \vee T(|y|)\| = 0$$

or equivalently

$$T(|x| \vee |y|) = T(|x|) \vee T(|y|),$$

because,

$$\lim_{n\to\infty} 4^{-n} f(4^n|z|) = \lim_{m\to\infty} 2^{-m} f(2^m|z|), \quad z \in X$$

Now, we must show $T(r|x|) = rT(|x|)$ for all $x \in X$ and $r \in [0, \infty)$. Using the inequality (2.3) with $\eta = \tau$, $y_1, \dots, y_k = 0$ and substituting τ with $2^n\tau$:

$$\|f(2^n\tau|x_1|) - 2^n\tau f(|x_1|), \dots, f(2^n\tau|x_k|) - 2^n\tau f(|x_k|)\|_k \leq \theta \sum_{i=1}^{k} \|x_i\|^r.$$

If we replace $x_1, \dots, x_k$ with $2^n x_1, \dots, 2^n x_k$ respectively then:

$$\|f(4^n\tau|x_1|) - 2^n\tau f(2^n|x_1|), \dots, f(4^n\tau|x_k|) - 2^n\tau f(|2^n x_k|)\|_k \leq 2^{nr}\theta\sum_{i=1}^{k}\|x_i\|^r.$$

Divide by 4^n both side of above inequality and use inequality (2.1):

$$\|4^{-n}f(4^n\tau|x_1|) - 2^{-n}\tau f(2^n|x_1|), \dots$$
$$, 4^{-n}f(4^n\tau|x_k|) - 2^{-n}\tau f(|2^n x_k|)\|_k \leq 2^{n(r-2)}\theta\sum_{i=1}^{k}\|x_i\|^r.$$

By letting $n \to \infty$, replace $x_1, \dots, x_k$ with x and $y_1, \dots, y_k$ with y in the last inequality conclude

$$\lim_{n\to\infty}\|4^{-n}f(4^n\tau|x|) - 2^{-n}\tau f(2^n|x|)\| = 0,$$

by (2.9) we get for all $x, y \in X$ the equality

$$\lim_{n\to\infty} 4^{-n}f(4^n\tau|x|) = \tau\lim_{n\to\infty} 2^{-n}f(2^n|x|) = \tau T(|x|),$$

by taking $z = \tau|x|$, we have

$$\tau T(|x|) = \lim_{n\to\infty} 4^{-n}f(4^n\tau|x|) = \lim_{n\to\infty} 4^{-n}f(4^n|z|) = T(|z|) = T(\tau|x|).$$

For uniqueness of T: Assume that the inequality (2.7) is also satisfied with another homogeneous function of degree two $S : E_1 \to E_2$ besides T. (As S is a homogeneous function of degree two, S satisfies that

$$S(x) = \frac{S(2x)}{2} = \Lambda S(x),$$

for all $x \in E_1$. That is, S is a fixed point of Λ.) In view of (2.7) and the definition of d, we know that

$$d(f, S) \leq \frac{L}{1-L} < \infty$$

i.e., $S \in X^*$. (In view of (2.8), the integer n_0 of Theorem 1.2 is 0.) Thus, Theorem 1.2 (ii) implies that $S = T$. This proves the uniqueness of T. □

Corollary 2.3. *Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $p : [0, \infty) \to [0, \infty)$ be a continuous function $f : E_1 \to E_2$ be a cone-related functional for which there are numbers $\theta > 0$ and $0 \leq r < 1$ such that*

$$\|f(\tau|x_1| \vee \eta|y_1|) - \tau f(|x_1|) \vee \eta f(|y_1|), \dots$$
$$, f(\tau|x_k| \vee \eta|y_k|) - \tau f(|x_k|) \vee \eta f(|y_k|)\|_k \leq \theta\sum_{i=1}^{k}(\|x_i\|^r + \|y_i\|^r)$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E_1$ and $\tau, \eta \in \mathbb{R}^+$; then there is a unique cone-related mapping $T : E_1 \to E_2$ such that

$$\|T(|x_1|) - f(|x_1|), \dots, T(|x_k|) - f(|x_k|)\|_k \leq \frac{2\theta}{2-2^r}\sum_{i=1}^{k}\|x_i\|^r$$

and satisfies properties I, II.

Proof. Enough, we put $p(t) = 1$ in above theorem for $t \in [0, \infty)$. In this case, the sense of stability in multi-Banach lattice is similarity with stability of additive functional equation in Banach space. $\square$

Corollary 2.4. *Let E_1 and E_2 be two Banach lattices and $((E_1^k, \|\cdot\|_k) : k \in \mathbb{N})$ be a multi-Banach lattice and $p : [0, \infty) \to [0, \infty)$ be a continuous function $f : E_1 \to E_2$ be a cone-related functional for which there are numbers $\theta > 0$ and $0 \le r < 1$ such that*

$$\Bigg\| f(\tau|x_1| \vee \eta|y_1|) - \frac{\tau^2 f(|x_1|) \vee \eta^2 f(|y_1|)}{\tau \vee \eta}., \dots$$

$$, f(\tau|x_k| \vee \eta|y_k|) - \frac{\tau^2 f(|x_k|) \vee \eta^2 f(|y_k|)}{\tau \vee \eta} \Bigg\|_k \le \theta \sum_{i=1}^{k} (\|x_i\|^r + \|y_i\|^r).$$

for all $x_1, \dots, x_k, y_1, \dots, y_k \in E_1$ and $\tau, \eta \in \mathbb{R}^+$; then there is a unique cone-related mapping $T : E_1 \to E_2$ such that

$$\|T(|x_1|) - f(|x_1|), \dots, T(|x_k|) - f(|x_k|)\|_k \le \frac{2\theta}{2 - 2^r} \sum_{i=1}^{k} \|x_i\|^r$$

and satisfies properties I, II.

Proof. Enough, we put $P(t) = t$ in above theorem. $\square$

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Received: September 30, 2016; Accepted: February 23, 2017.