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VISCOSITY APPROXIMATION METHODS FOR GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND MONOTONE INCLUSIONS IN HADAMARD SPACES

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Abstract. The main purpose of this paper is to introduce and study some viscosity-type proximal point algorithms for approximating a common solution of monotone inclusion problem and fixed point problem. We obtained strong convergence of the proposed algorithms to a common solution of minimization problem and fixed point problem for a generalized asymptotically nonexpansive mapping which is also a unique solution of some variational inequality problems in Hadamard spaces. Our results extend and complement some recent results in this direction.
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1. Introduction

A metric space (X, d) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map c from $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be a uniquely geodesic if there is exactly

one geodesic joining x to y for each $x, y \in X$.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$$

for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [6]). A geodesic metric space is said to be a CAT(0) space [6] if all geodesic triangles of appropriate size satisfy the following comparison property. Let $\triangle$ be a geodesic triangle in X, and let $\bar{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $x, y \in \bar{\triangle}$, $d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})$.
We observe that if x, x_1, x_2 are points of a CAT(0) space and if x_0 is the midpoint of the segment $[x_1, x_2]$, then the CAT(0) inequality implies that

$$d^2(x, x_0) \leq \frac{1}{2}d^2(x, x_1) + \frac{1}{2}d^2(x, x_2) - \frac{1}{2}d^2(x_1, x_2). \tag{1.1}$$

The equality holds for the Euclidean metric. In fact (see [6, p.163]), a geodesic metric space is a CAT(0) space if and only if it satisfies inequality (1.1) (which is called the CN inequality of Bruhat and Tits [7]). For other equivalent definitions and basic properties of a CAT(0) space, see[4]. Complete CAT(0) spaces are often called Hadamard spaces. Let $x, y \in X$ and $\lambda \in [0, 1]$. We write $\lambda x \oplus (1 - \lambda)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = (1 - \lambda)d(x, y) \quad \text{and} \quad d(z, y) = \lambda d(x, y). \tag{1.2}$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is,

$$[x, y] = \{\lambda x \oplus (1 - \lambda)y : \lambda \in [0, 1]\}.$$

A subset C of a CAT(0) space is called convex if $[x, y] \subseteq C$ for all $x, y \in C$.
Berg and Nikolaev [5] introduced the concept of *quasilinearization* in a metric space X. Let denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. The quasilinearization is a map $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\Big(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\Big), \quad \forall a, b, c, d \in X. \tag{1.3}$$

It is easily seen that

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle, \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$$

and

$$\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$$

for all $a, b, c, d \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d),$$

for all $a, b, c, d, x \in X$. It is known that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality (see [5]).

In 2010, Kakavandi and Amini [19] introduced the concept of dual space for CAT(0) space, as follows. Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X)$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \tag{1.4}$$

where $C(X)$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t,a,b)$ is a Lipschitz semi-norm

$$L(\Theta(t,a,b)) = |t| d(a,b)$$

for all $a, b \in X$, where

$$L(f) = \sup \Big\{ \frac{f(x) - f(y)}{d(x,y)} : x, y \in X, x \neq y \Big\},$$

is the Lipschiz semi-norm of the function $f : X \to \mathbb{R}$. Now, define the pseudometric D on $\mathbb{R} \times X \times X$ by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)).$$

$D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in X$, see [19, Lemma 2.1]. For a complete CAT(0) space (X,d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space $(\mathrm{Lip}(X,\mathbb{R}), L)$ of all real-valued Lipschitz functions. Also, the metric D defines an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of $\overrightarrow{tab} : (t,a,b)$ is

$$[\overrightarrow{tab}] = \{ \overrightarrow{scd} : t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle \ \forall x, y \in X \}.$$

The set $X^* := \{[\overrightarrow{tab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with D, which is called the dual metric space of (X,d).

Let X be a Hadamard space and X^* be its dual space. A multivalued operator $A : X \to 2^{X^*}$ with domain $\mathbb{D}(A) := \{x \in X : Ax \neq \emptyset\}$ is monotone if and only if for all $x, y \in \mathbb{D}(A),\ x^* \in Ax,\ y^* \in Ay$,

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0 \quad \text{(see [20])}.$$

A monotone operator A is called a maximal monotone operator if the graph $G(A)$ of A defined by

$$G(A) := \{(x,x^*) \in X \times X^* :\ x^* \in A(x)\},$$

is not properly contained in the graph of any other monotone operator. The resolvent of a monotone operator A of order $\lambda > 0$ is the multivalued mapping $J_\lambda^A : X \to 2^X$ defined by (see [20])

$$J_\lambda^A(x) := \left\{ z \in X \ |\ \left[\frac{1}{\lambda} \overrightarrow{zx} \right] \in Az \right\}.$$

We say that the operator A satisfies the range condition if for every $\lambda > 0$, $\mathbb{D}(J_\lambda^A) = X$ (see [20]). For simplicity, we shall write J_λ for the resolvent of a monotone operator A.

The theory of monotone operators known as one of the most important theory in nonlinear and convex analysis is a vital tool in optimization theory, variational inequalities, semi group theory, evolution equations, among others. One of the most important problems in the theory of monotone operators is the problem of finding the solution of the following Monotone Inclusion Problem (MIP).

$$\text{Find } x \in \mathbb{D}(A) \text{ such that } 0 \in Ax, \tag{1.5}$$

where $A : X \to 2^{X^*}$ is a monotone operator. Throughout this paper, we shall denote the solution set of problem (1.5) by $A^{-1}(0)$, which is known to be closed and convex (see [32], Remark 3.1). Many mathematical problems such as optimization problems, equilibrium problems, variational inequality problems, saddle point problems, among others, can be modeled as a MIP (1.5). Thus, MIP is of central importance in nonlinear and convex analysis. The most popular and successful method for finding solution of MIP, is the Proximal Point Algorithm (PPA) introduced in Hilbert space by Martinet [27] and further developed by Rockafellar [33], as follows:

$$x_{n-1} - x_n \in \lambda_n A(x_n),\ x_0 \in H, \tag{1.6}$$

where $\{\lambda_n)$ is a sequence of positive real numbers. Rockafellar [33] proved that the sequence $\{x_n\}$ generated by Algorithm (1.6) is weakly convergent to a solution of MIP (1.5), provided $\lambda_n \geq \lambda > 0$ for each $n \geq 1$. The PPA was later introduced in CAT(0) spaces by Bačák [3], who proved the Δ-convergence of it when the operator A is a subdifferential of a convex, proper and lower semicontinuous function. In 2016, Khatibzadeh and Ranjbar [20] introduced and studied the following PPA in CAT(0) spaces for the case when the operator A is a monotone operator:

$$\begin{cases} x_0 \in X, \\ \left[\frac{1}{\lambda_n}\overrightarrow{x_n x_{n-1}}\right] \in Ax_n. \end{cases} \tag{1.7}$$

They obtained a strong and Δ-convergence results of (1.7) to a solution of (1.5). Very recently, Ranjbar and Khatibzadeh [32] proposed the following Mann-type PPA in CAT(0) spaces for finding the solution of (1.5) and obtained a Δ-convergence result.

$$\begin{cases} x_0 \in X, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) J_{\lambda_n} x_n. \end{cases} \tag{1.8}$$

In the same paper, Ranjbar and Khatibzadeh [32] proposed the following Halpern-type PPA in order to obtain a strong convergence result:

$$\begin{cases} u, x_0 \in X, \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J_{\lambda_n} x_n,\ n \geq 1, \end{cases} \tag{1.9}$$

where $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\} \subset [0,1]$.
Let C be a nonempty closed convex subset of a CAT(0) space X. Then, for any $x \in X$ there exists a unique point $u \in C$ such that

$$d(x,u) = \min_{y \in C} d(x,y).$$

The mapping $P_C : X \to C$ defined by $P_C x = u$ is called the metric projection from X onto C (see [12]). The metric projection is characterized by the following (see [12, Theorem 3.1]):

$$u = P_C x \text{ if and only if } \langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geq 0, \text{ for all } y \in C.$$

Let C be a nonempty subset of a CAT(0) space X, then a mapping T from C into itself is called

(i) *firmly nonexpansive* if (see [20])
$$d^2(Tx,Ty) \le \langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle \ \forall x,y \in C,$$
(ii) *nonexpansive* if
$$d(Tx,Ty) \le d(x,y) \ \forall x,y \in C.$$

An example of a nonexpansive mapping is the metric projection mapping (see [13, Proposition 2.4]). Also recall that, a mapping T is said to be asymptotically nonexpansive [17], if there is a sequence $\{u_n\} \subseteq [0,\infty)$ with $u_n \to 0$ as $n \to \infty$ such that
$$d(T^nx,T^ny) \le (1+u_n)d(x,y) \ \forall n \ge 1, \ x,y \in C.$$
T is said to be uniformly L-Lipschitzian, if there exists a constant $L > 0$ such that
$$d(T^nx,T^ny) \le Ld(x,y); \ \forall n \ge 1; \ x,y \in C$$
and T is said to be asymptotically regular, if
$$\lim_{n\to\infty} d(T^nx,T^{n+1}x) = 0 \ \forall x \in C.$$
Furthermore, a mapping $T : C \to C$ is called generalized asymptotically nonexpansive if there exist nonnegative sequences $\{\mu_n\},\{\nu_n\}$ with $\mu_n \to 0, \nu_n \to 0$ as $n \to \infty$ such that
$$d(T^nx,T^ny) \le (1+\nu_n)d(x,y) + \mu_n, \ \forall n \ge 1, \ x,y \in C.$$
Clearly, every asymptotically nonexpansive mapping is a generalized asymptotically nonexpansive mapping. However, we shall see in Example 1 that there exists an asymptotically nonexpansive mapping which is not a generalized asymptotically nonexpansive mapping.

Example 1.1. [1]

(1) Let $X = \mathbb{R}, C = [0,\infty)$ and $T : C \to C$ be defined by $Tx = \sin x$. Then T is asymptotically nonexpansive, hence, a generalized asymptotically nonexpansive mapping.
(2) Let $X = \mathbb{R}, C = [-\frac{1}{\pi}, \frac{1}{\pi}]$ be defined by $Tx = kx\sin\frac{1}{x}$, where $k \in (0,1)$. Then T is generalized asymptotically nonexpansive.

Example 1.2. [46] Let $X = \mathbb{R}, C = (-\infty,1]$ and for $k \in (0,1)$ define $T : C \to C$ by
$$Tx = \begin{cases} x, & x \in (-\infty,0) \\ kx, & x \in [0,\frac{1}{2}] \\ 0, & x \in (\frac{1}{2},1]. \end{cases}$$
Then, T is generalized asymptotically nonexpansive with $u_n = 2k^n$ and $v_n = k^n$. But T is not asymptotically nonexpansive.

A mapping T of C into itself is called a contraction with coefficient $\alpha \in [0,1)$ if and only if $d(T(x),T(y)) \le \alpha d(x,y)$ for all $x,y \in C$. Recall that a point $x \in C$ is called a fixed point of T if $Tx = x$. We denote by $F(T)$ the set of all the fixed points of T. Banach contraction principle [6] guarantees that T has a unique fixed point when C is a nonempty, closed, and convex subset of a complete metric space. Iterative methods for finding fixed points of nonexpansive mappings have received vast investigations due to their extensive applications in a variety of applied areas of inverse problems, partial differential equations, image recovery, and signal processing; see [11, 26, 30, 37, 43, 45] and the references therein. One of the difficulties in carrying out results from Banach spaces to Hadamard spaces lies in the heavy use of the linear structure of the Banach spaces. Recently, fixed points results were studied by many authors in the setting of CAT(0) metric spaces for example see [8, 9, 2, 41, 16, 18, 21, 22, 24, 23, 25, 29] and the references therein.

In 2012, Shi and Chen [35], studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive mapping T and a contraction f on C and $t \in (0,1)$. Let $x_t \in C$ be a unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$; i.e.,

$$x_t = tf(x_t) \oplus (1-t)Tx_t. \tag{1.10}$$

Consider that $x_0 \in C$ is arbitrarily chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \tag{1.11}$$

where $\{\alpha_n\} \subset (0,1)$. They proved that $\{x_t\}$ defined by (1.10) converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ in the framework of CAT(0) space and satisfy property $\mathcal{P}$, i.e., if for $x, u, y_1, y_2 \in X$,

$$d(x, P_{[x,y_1]}u)d(x,y_1) \leq d(x, P_{[x,y_2]}u)d(x,y_2) + d(x,u)d(y_1,y_2).$$

Furthermore, they obtained that $\{x_n\}$ defined by (1.11) converges strongly as $n \to \infty$ to $\tilde{x} \in F(T)$ under appropriate conditions on $\{\alpha_n\}$. By using the concept of quasilinearization, Wangkeeree and Preechasilp [42] improved Shi and Chen's results. In fact, they proved the strong convergence theorems for two given iterative schemes (1.10) and (1.11) in a complete CAT(0) space without the property $\mathcal{P}$. They proved that the iterative schemes (1.10) and (1.11) converges strongly to $\tilde{x} := P_{F(T)}f(\tilde{x})$ which is a unique solution of the variational inequality (VIP):

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{1.12}$$

In 2013, Shi et al. [36] studied the Δ-convergence of the iterative sequence (1.10) and (1.11) for asymptotically nonexpansive mappings in CAT(0) spaces. Wangkeeree et al. [40] studied the strong convergence theorems of the Moudafi's viscosity approximation methods for an asymptotically nonexpansive mapping in CAT(0) spaces: Let C be a closed convex subset of a complete CAT(0) space X and $T : C \to C$ be an asymptotically nonexpansive mapping. For given a contraction f on C and $\alpha_n \in (0,1)$, let $x_n \in C$ be a unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1-\alpha_n)T^n x$; i.e.

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n x_n, \quad n \geq 1, \tag{1.13}$$

and $x_1 \in C$ is arbitrary chosen and

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n x_n, \quad n \geq 1. \tag{1.14}$$

They proved that the iterative schemes (1.13) and (1.14) converge strongly to the same point $\tilde{x} := P_{F(T)}f(\tilde{x})$ which is the unique solution of the variational inequality:

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{1.15}$$

Motivated and inspired by the above results, we study some strong convergence of the viscosity-type proximal point algorithms for approximating a common solution of monotone inclusion problem and fixed point problem for a generalized asymptotically nonexpansive mapping which is also a unique solution of some variational inequality problems in Hadamard space. Our results extend and complement the results of Bačák [3], Khatibzadeh and Ranjbar [20], Ranjbar and Khatibzadeh [32], Shi and Chen [35], Wangkeeree and Preechasilp [42], and Wangkeeree et al. [40].

The paper is organized as follows. The next section presents some preliminary results. In section 3, strong convergence of both implicit and explicit of the modified viscosity-type proximal point algorithms to a common solution of monotone inclusion problem and fixed point problem for a uniformly asymptotically regular and uniformly L-Lipschitzian generalized asymptotically nonexpansive mapping in Hadamard space are presented in Theorem 3.1 and Theorem 3.3 respectively.

2. Preliminaries

We denote by $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ the set of natural numbers, the set of nonnegative real numbers and the set of real numbers, respectively. We also denote by $\to$ and $\rightharpoonup$ strong convergence and Δ-convergence respectively. In the sequel, we shall use the following results:

Lemma 2.1. [6] *Let X be a CAT(0) space, $w, x, y, z \in X$ and $t \in [0,1]$. Then*

$$d(tw \oplus (1-t)x, ty \oplus (1-t)z) \leq td(w,y) + (1-t)d(x,z).$$

Lemma 2.2. *Let X be a CAT(0) space, $x, y, z \in X$ and $t, s \in [0,1]$. Then*

(1) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$.(see[16])
(2) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$. (see [16])
(3) $d^2(tx \oplus (1-t)y, z) \leq t^2d^2(x,z) + (1-t)^2d^2(y,z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$. (see [12])
(4) $z = tx \oplus (1-t)y$ implies $\langle \overrightarrow{zy}, \overrightarrow{zw} \rangle \leq t\langle \overrightarrow{xy}, \overrightarrow{zw} \rangle$, $\forall\ w \in X$. (see [12])
(5) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) \leq |t-s|d(x,y)$. (see [10])

Let $\{x_n\}$ be a bounded sequence in a Hadamard space X. For $x \in X$, define

$$r(x, \{x_n\}) := \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is defined by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is well known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see [15, Proposition 7]). A sequence $\{x_n\}$ in X is said to be $\triangle$-convergent to a point w, if

w is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. This is written as $\triangle - \lim_{n\to\infty} x_n = w$.

Let $\{x_n\}$ be a bounded sequence in a Hadamard space X, and C be a closed and convex subset of X which contains $\{x_n\}$. We note that $\{x_n\} \rightharpoonup w$ if and only if $A(\{x_n\}) = \{w\}$ (see [28]).

Lemma 2.3. [24] *Every bounded sequence in a Hadamard space always has a Δ-convergent subsequence.*

Recall that a mapping T is called total asymptotically nonexpansive, if there exist sequences of nonnegative numbers $\{u_n\}$ and $\{v_n\}$, and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$d(T^n x, T^n y) \le d(x,y) + u_n \phi(d(x,y)) + v_n, \tag{2.1}$$

$\forall n \ge 1,\ x, y \in C, u_n \to 0, v_n \to 0$ as $n \to \infty$.

Lemma 2.4. [8] *Let C be a closed convex subset of a Hadamard space X and $T : C \to X$ be a uniformly L-Lipschitzian and $(\{u_n\},\{v_n\},\phi)$-total asymptotically nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $Tp = p$.*

Remark 2.5. If $\phi(\lambda) = \lambda$ in (2.1), then in Lemma 2, T is generalized asymptotically nonexpansive mapping.

Lemma 2.6. [18] *Let X be a Hadamard space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}\triangle-$converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0$ for all $y \in C$.*

Lemma 2.7. [38] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a metric space of hyperbolic type X and $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$\liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = \beta_n x_n \oplus (1-\beta_n) y_n$ for all $n \ge 0$ and

$$\limsup_{n\to\infty} (d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \le 0.$$

Then $\lim_{n\to\infty} d(y_n, x_n) = 0$.

Lemma 2.8. [42] *Let X be a CAT(0) space. For any $t \in [0,1]$ and $u, v \in X$, let*

$$u_t = tu \oplus (1-t)v.$$

Then, for all $x, y \in X$,

(1) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y}\rangle \le t\langle \overrightarrow{ux}, \overrightarrow{u_t y}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y}\rangle$;
(2) $\langle \overrightarrow{u_t x}, \overrightarrow{uy}\rangle \le t\langle \overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{ux}\rangle$ *and*
(3) $\langle \overrightarrow{u_t x}, \overrightarrow{vy}\rangle \le t\langle \overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy}\rangle$.

Lemma 2.9. [20] *Let X be a CAT(0) space and J_λ be the resolvent of the operator A of order λ. We have the following:*

(i) *For any $\lambda > 0$, $\mathbb{R}(J_\lambda) \subset \mathbb{D}(A)$ and $F(J_\lambda) = A^{-1}(0)$.*
(ii) *If A is monotone, then J_λ is a single-valued and firmly nonexpansive mapping.*
(iii) *If A is monotone and $0 < \lambda \le \mu$, then*

$$d^2(J_\lambda x, J_\mu x) \le \frac{\mu - \lambda}{\mu + \lambda} d^2(x, J_\mu x)\ \forall x \in X.$$

Remark 2.10. From Cauchy-Schwartz inequality, it is not difficult to see that every firmly nonexpansive mapping is a nonexpansive mapping. Therefore, J_λ is nonexpansive.

Lemma 2.11. (Xu, [44]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0,$$

where
(i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$;
(ii) $\limsup \sigma_n \leq 0$;
(iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$.
Then, $a_n \to 0$ as $n \to \infty$.

3. Main results

Theorem 3.1. *Let X be a Hadamard space and X^* be its dual space. Let $T : X \to X$ be uniformly asymptotically regular and uniformly L-Lipschitzian generalized asymptotically nonexpansive mapping with sequences $\{u_n\}, \{v_n\} \subset [0,\infty)$ and $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} v_n = 0$. Let $A : X \to 2^{X^*}$ be a multivalued monotone mapping which satisfies the range condition and f be a contraction mapping on X with coefficient $\gamma \in (0,1)$. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$ and for arbitrary $x_1 = x \in C$, the sequence $\{x_n\}_{n=1}^\infty$ is generated by*

$$\begin{cases} y_n = J_{\lambda_n}(x_n), \\ x_n = \alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n \ n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and

$$\lim_{n\to\infty} \frac{u_n}{\alpha_n} = 0, \ \lim_{n\to\infty} \frac{v_n}{\alpha_n} = 0,$$

assuming that $L < (1-\alpha_n\gamma)/(1-\alpha_n)$ and $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$. Then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in \Gamma$ which solves the variational inequality

$$\langle \overrightarrow{pf(p)}, \overrightarrow{qp} \rangle \geq 0, \quad q \in \Gamma. \tag{3.2}$$

Proof. First, we show that $\{x_n\}$ defined by (3.1) is well defined. For each $n \geq 1$, define the mapping $T_n^f : X \to X$ as follows:

$$T_n^f x = \alpha_n f(J_{\lambda_n} x) \oplus (1-\alpha_n)T^n J_{\lambda_n} x.$$

Now, using the method in Qin et al. [31] and Saluja [34], and by the uniformly L-Lipschitzian of T, we obtain from Lemma 2.1 that

$$\begin{aligned} d(T_n^f x, T_n^f y) &= d(\alpha_n f(J_{\lambda_n} x) \oplus (1-\alpha_n)T^n J_{\lambda_n} x, \alpha_n f(J_{\lambda_n} y) \oplus (1-\alpha_n)T^n J_{\lambda_n} y) \\ &\leq \alpha_n d(f(J_{\lambda_n} x), f(J_{\lambda_n} y)) + (1-\alpha_n) d(T^n J_{\lambda_n} x, T^n J_{\lambda_n} y) \\ &\leq \gamma\alpha_n d(J_{\lambda_n} x, J_{\lambda_n} y) + (1-\alpha_n) L d(J_{\lambda_n} x, J_{\lambda_n} y) \\ &\leq (\gamma\alpha_n + (1-\alpha_n)L)\, d(x,y). \end{aligned}$$

Since $L < (1-\alpha_n\gamma)/(1-\alpha_n)$, we obtain that T_n^f is a contraction for each $n \geq 1$. Therefore, by Banach contraction principle, there exists a unique fixed point x_n of T_n^f for each $n \geq 1$. Hence, (3.1) is well defined.
Let $p \in F(T)$, then from (3.1) and Lemma 2.2 (1), we obtain

$$\begin{aligned}
d(x_n,p) &= d(\alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n, p) \\
&\leq \alpha_n d(f(y_n),p) + (1-\alpha_n) d(T^n y_n, p) \\
&\leq \alpha_n \gamma d(y_n,p) + \alpha_n d(f(p),p) \\
&+ (1-\alpha_n)[(1+u_n)d(y_n,p) + v_n] \\
&= \Big(1-\alpha_n[(1-\gamma)-(1-\alpha_n)u_n/\alpha_n]\Big) d(J_{\lambda_n}x_n,p) \\
&+ \alpha_n[d(f(p),p) + (1-\alpha_n)v_n/\alpha_n] \\
&\leq \Big(1-\alpha_n[(1-\gamma)-(1-\alpha_n)u_n/\alpha_n]\Big) d(x_n,p) \\
&+ \alpha_n[d(f(p),p) + (1-\alpha_n)v_n/\alpha_n]
\end{aligned} \tag{3.3}$$

Therefore

$$d(x_n,p) \leq \frac{d(f(p),p) + (1-\alpha_n)v_n/\alpha_n}{(1-\gamma) - (1-\alpha_n)u_n/\alpha_n}.$$

Since $\lim_{n\to\infty}(1-\alpha_n)v_n/\alpha_n = 0$ and $\lim_{n\to\infty}(1-\alpha_n)u_n/\alpha_n = 0$, then there exist $n_0 \in \mathbb{N}$ such that $(1-\alpha_n)v_n/\alpha_n < (1-\gamma)/4$ and $(1-\alpha_n)u_n/\alpha_n < (1-\gamma)/4$ respectively for all $n \geq n_0$. Hence

$$d(x_n,p) \leq \left[\frac{4d(f(p),p)}{1-\gamma} + \frac{1}{3}\right],$$

for all $n \geq n_0$. Thus, $\{x_n\}$ is bounded and so are $\{y_n\}$ $\{T^n y_n\}$ and $\{f(y_n)\}$. From (3.1), we obtain

$$\begin{aligned}
d(x_n, T^n y_n) &= d(\alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n, T^n y_n) \\
&\leq \alpha_n d(f(y_n), T^n y_n) \to 0 \text{ as } n \to \infty,
\end{aligned} \tag{3.4}$$

From Lemma 2.2(2), we obtain

$$\begin{aligned}
d^2(x_n,p) &= d^2(\alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n, p) \\
&\leq \alpha_n d^2(f(y_n),p) + (1-\alpha_n)d^2(T^n y_n, p) \\
&\leq \alpha_n d^2(f(y_n),p) + (1-\alpha_n)[(1+u_n)d(y_n,p) + v_n]^2 \\
&= \alpha_n d^2(f(y_n),p) + (1-\alpha_n)(1+u_n)^2 d^2(y_n,p) \\
&+ (1-\alpha_n)v_n^2 + 2(1-\alpha_n)(1+u_n)d(y_n,p)v_n,
\end{aligned}$$

which implies

$$\begin{aligned}
-d^2(y_n,p) &\leq \frac{1}{(1-\alpha_n)(1+u_n)^2}\left(\alpha_n d^2(f(y_n),p) - d^2(x_n,p)\right) \\
&+ \frac{v_n^2}{(1+u_n)^2} + \frac{2}{1+u_n} d(y_n,p)v_n.
\end{aligned} \tag{3.5}$$

Since J_{λ_n} is firmly nonexpansive, we obtain from the definition of quasilinearization map that

$$\begin{aligned} d^2(J_{\lambda_n}x_n, p) &\leq \langle \overrightarrow{J_{\lambda_n}x_n\ p}, \overrightarrow{x_np}\rangle \\ &= \frac{1}{2}\left(d^2(J_{\lambda_n}x_n, p) + d^2(p, x_n) - d^2(J_{\lambda_n}x_n, x_n)\right), \end{aligned}$$

which implies from (3.5) that

$$\begin{aligned} d^2(J_{\lambda_n}x_n, x_n) &\leq d^2(p, x_n) + \frac{1}{(1-\alpha_n)(1+u_n)^2}\left(\alpha_n d^2(f(y_n), p) - d^2(x_n, p)\right) \\ &+ \frac{v_n^2}{(1+u_n)^2} + \frac{2}{(1+u_n)}d(y_n, p)v_n \to 0, \text{ as } n \to \infty. \end{aligned}$$

That is

$$\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d(J_{\lambda_n}x_n, x_n) = 0. \tag{3.6}$$

Since $0 < \lambda \leq \lambda_n$, we obtain from Lemma 2.9(iii) and (3.6) that

$$d(J_\lambda x_n, x_n) \leq 2d(J_{\lambda_n}x_n, x_n) \to 0, \text{ as } n \to \infty. \tag{3.7}$$

From (3.4) and (3.6), we obtain that

$$\lim_{n\to\infty} d(y_n, T^n y_n) = 0. \tag{3.8}$$

Using the asymptotic regularity of T, we obtain

$$\begin{aligned} d(y_n, Ty_n) &\leq d(y_n, T^n y_n) + d(T^n y_n, T^{n+1}y_n) + d(T^{n+1}y_n, Ty_n) \\ &\leq (1+L)d(y_n, T^n y_n) + d(T^{n+1}y_n, T^n y_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.9}$$

By the boundedness of $\{x_n\}$, we obtain from Lemma 2.3 that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which $\triangle-$converges to p. It then follows from the boundedness of $\{y_n\}$ and (3.6) that there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which $\triangle-$converges to p. Thus, from (3.7), (3.9), and Lemma 2.4, we obtain that $p \in \Gamma$.

Now, let $w_n := 2u_n + u_n^2$, then from Lemma 2.2, 2.8 and (3.1), we have

$$\begin{aligned} d^2(x_n, p) &= d^2(\alpha_n f(y_n) \oplus (1-\alpha)T^n y_n, p) \\ &\leq \alpha_n^2 d^2(f(y_n), p) + (1-\alpha_n)d^2(T^n y_n, p) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(y_n)p}, \overrightarrow{T^n y_n p}\rangle \\ &\leq \alpha_n^2 d^2(f(y_n), p) + (1-\alpha_n)[(1+u_n)d(y_n, p) + v_n]^2 \\ &+2\alpha_n(1-\alpha_n)[\langle \overrightarrow{f(y_n)p}, \overrightarrow{T^n y_n y_n}\rangle + \langle \overrightarrow{f(y_n)f(p)}, \overrightarrow{y_np}\rangle + \langle \overrightarrow{f(p)p}, \overrightarrow{y_np}\rangle \\ &\leq \alpha_n^2 d^2(f(y_n), p) + (1-\alpha_n)[(1+w_n)d^2(y_n, p) + 2(1+u_n)v_n d(y_n, p) + v_n^2] \\ &+2\alpha_n(1-\alpha_n)[\langle \overrightarrow{f(y_n)p}, \overrightarrow{T^n y_n y_n}\rangle + \gamma d^2(y_n, p)] + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(p)p}, \overrightarrow{y_np}\rangle \\ &\leq \Big[(1-\alpha_n)(1+w_m) + 2\gamma\alpha_n(1-\alpha_n)\Big]d^2(x_n, p) \\ &+\alpha_n\Big[\alpha_n d(f(y_n), p) + 2(1-\alpha_n)d(T^n x_n, y_n)\Big]d(f(y_n), p) \\ &+(1-\alpha_n)v_n[2(1+u_n)d(y_n, p) + v_n] + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{f(p)p}, \overrightarrow{y_np}\rangle. \end{aligned} \tag{3.10}$$

Therefore

$$
\begin{aligned}
d^2(x_n,p) &\leq \frac{[\alpha_n d(f(y_n),p)+2(1-\alpha_n)d(T^n y_n,y_n)]d(f(y_n),p)}{[1-(1-\alpha_n)w_n/\alpha_n-2\gamma(1-\alpha_n)]}\\
&+ \frac{(1-\alpha_n)v_n/\alpha_n[2(1+u_n)d(y_n,p)+v_n]}{[1-(1-\alpha_n)w_n/\alpha_n-2\gamma(1-\alpha_n)]}\\
&+ \frac{2(1-\alpha_n)\langle\overrightarrow{f(p)p},\overrightarrow{y_np}\rangle}{[1-(1-\alpha_n)w_n/\alpha_n-2\gamma(1-\alpha_n)]}. \qquad (3.11)
\end{aligned}
$$

Since $\{y_{n_j}\}$ $\triangle-$converges to a point $p\in\Gamma$, by Lemma 2.6, we obtain

$$
\lim_{j\to\infty}\langle\overrightarrow{f(p)p},\overrightarrow{y_{n_j}p}\rangle\leq 0. \qquad (3.12)
$$

From (3.11) and (3.12), we get

$$
\lim_{j\to\infty} d^2(x_{n_j},p)=0.
$$

Hence, $\lim\limits_{j\to\infty} x_{n_j}=p$. Next, we show that $p\in\Gamma$ is a solution of variational inequality (3.2). From Lemma 2.2 and (3.1), and for all $q\in\Gamma$, letting $w_m:=u_m^2+2u_m$, we obtain

$$
\begin{aligned}
d^2(x_m,q) &\leq \alpha_m d^2(f(y_m),q)+(1-\alpha_m)d^2(T^m y_m,q)\\
&- \alpha_m(1-\alpha_m)d^2(f(y_m),T^m y_m)\\
&\leq \alpha_m d^2(f(y_m),q)+(1-\alpha_m)[(1+u_m)d(y_m,q)+v_m]^2\\
&- \alpha_m(1-\alpha_m)d^2(f(y_m),T^m y_m)\\
&\leq \alpha_m d^2(f(y_m),q)+(1-\alpha_m)(1+w_m)d^2(x_m,q)\\
&+ v_m(1-\alpha_m)[2(1+u_m)d(y_m,q)+v_m]\\
&- \alpha_m(1-\alpha_m)d^2(f(y_m),T^m y_m),
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d^2(x_m,q) &\leq \frac{(d^2(f(y_m),q)+v_m/\alpha_m[2(1+u_m)d(y_m,q)+v_m]}{[1-(1-\alpha_m)w_m/\alpha_m]}\\
&+ \frac{(1-\alpha_m)(1+w_m)v_m/\alpha_m[2(1+u_m)d(y_m,q)+v_m]}{[1-(1-\alpha_m)w_m/\alpha_m]}\\
&- \frac{(1-\alpha_m)d^2(f(y_m),T^m y_m)}{[1-(1-\alpha_m)w_m/\alpha_m]}. \qquad (3.13)
\end{aligned}
$$

Since $\lim\limits_{m\to\infty} x_m=p$, then taking limit through as $m\to\infty$ in (3.13), we obtain

$$
d^2(p,q)\leq d^2(f(p),q)-d^2(f(p),p).
$$

Hence

$$
\langle\overrightarrow{pf(p)},\overrightarrow{qp}\rangle=\frac{1}{2}\Big(d^2(p,p)+d^2(f(p),q)-d^2(p,q)-d^2(f(p),p)\Big)\geq 0,
$$

where $q \in \Gamma$, it implies that p solves the variational inequality (3.2). Assume there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which $\triangle-$converges to q by the same argument, we have that $q \in \Gamma$ which solve the variational inequality (3.2), that is

$$\langle \overrightarrow{qf(q)}, \overrightarrow{qp} \rangle \leq 0 \quad \text{also} \quad \langle \overrightarrow{pf(p)}, \overrightarrow{pq} \rangle \leq 0,$$

adding the two, we obtain

$$\begin{aligned} 0 &\geq \langle \overrightarrow{pf(p)}, \overrightarrow{pq} \rangle - \langle \overrightarrow{qf(q)}, \overrightarrow{pq} \rangle \\ &= \langle \overrightarrow{pf(q)}, \overrightarrow{pq} \rangle + \langle \overrightarrow{f(q)f(p)}, \overrightarrow{pq} \rangle \\ &- \langle \overrightarrow{qp}, \overrightarrow{pq} \rangle - \langle \overrightarrow{pf(q)}, \overrightarrow{pq} \rangle \\ &= \langle \overrightarrow{pq}, \overrightarrow{pq} \rangle - \langle \overrightarrow{f(q)f(p)}, \overrightarrow{qp} \rangle \\ &\geq \langle \overrightarrow{pq}, \overrightarrow{pq} \rangle - d(f(q)f(p))d(q,p) \\ &\geq d^2(p,q) - \gamma d^2(q,p) \\ &= (1-\gamma)d^2(p,q). \end{aligned}$$

Since $\gamma \in (0,1)$, we $d(p,q) = 0$, and so $p = q$. Hence, $\{x_n\}$ converges strongly to p, which is a solution of the variational inequality (3.2). $\square$

We now give the following remark which will be needed in what follows.

Remark 3.2. If X is a CAT(0) space and $A : X \to 2^{X^*}$ is a multivalued monotone mapping, then for $0 < \lambda \leq \mu$, we have that

$$d(J_\lambda x, J_\mu x) \leq \left(\sqrt{1 - \frac{\lambda}{\mu}} \right) d(x, J_\mu x), \ \forall x \in X.$$

Indeed, from Lemma 2 (iii), we obtain that

$$\frac{\mu + \lambda}{\mu} d^2(J_\lambda x, J_\mu x) \leq \frac{\mu - \lambda}{\mu} d^2(x, J_\mu x),$$

which implies that

$$d^2(J_\lambda x, J_\mu x) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2(x, J_\mu x).$$

That is,

$$d(J_\lambda x, J_\mu x) \leq \left(\sqrt{1 - \frac{\lambda}{\mu}} \right) d(x, J_\mu x).$$

Theorem 3.3. *Let X be a Hadamard space and X^* be its dual space. Let $T : X \to X$ be uniformly asymptotically regular and uniformly L-Lipschitzian generalized asymptotically nonexpansive mapping with sequences $\{u_n\}, \{v_n\} \subset [0, \infty)$ and $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} v_n = 0$. Let $A : X \to 2^{X^*}$ be a multivalued monotone mapping which satisfies the range condition and f be a contraction mapping on X with coefficient $\gamma \in (0,1)$. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$ and for arbitrary $x_1 = x \in C$,* *the sequence* $\{x_n\}_{n=1}^{\infty}$ *is generated by*

$$\begin{cases} w_n = J_{\lambda_n} x_n, \\ y_n = \alpha_n f(w_n) \oplus (1-\alpha_n) T^n w_n, \\ x_{n+1} = \beta_n w_n \oplus (1-\beta_n) T^n y_n, \ n \geq 1, \end{cases} \tag{3.14}$$

where $0 < \lambda \leq \lambda_n \ \forall n \geq 1$ *and* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset (0,1)$, *satisfying*

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \frac{u_n}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{v_n}{\alpha_n} = 0$,

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(c) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$,

(d) $L < (1-\alpha_n\gamma)/(1-\alpha_n)$,

(e) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Then, $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \Gamma$ *which solves the variational inequality*

$$\langle \overrightarrow{pf(p)}, \overrightarrow{qp} \rangle \geq 0, \quad q \in \Gamma. \tag{3.15}$$

Proof. From condition (a) in Theorem 3.3, we get that

$$\lim_{n\to\infty} (1-\alpha_n) u_n / \alpha_n = 0,$$

$$\lim_{n\to\infty} (1-\alpha_n) u_n / [\alpha_n (1+u_n)] = 0,$$

$$\lim_{n\to\infty} (1-\alpha_n) v_n / \alpha_n = 0$$

and

$$\lim_{n\to\infty} (1-\alpha_n) v_n / [\alpha_n (1+u_n)] = 0.$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$(1-\alpha_n) u_n / \alpha_n < (1-\gamma)/4,$$

$$(1-\alpha_n) u_n / [\alpha_n (1+u_n)] < (1-\gamma)/4,$$

$$(1-\alpha_n) v_n / \alpha_n < (1-\gamma)/4$$

and

$$(1-\alpha_n) v_n / [\alpha_n (1+u_n)] < (1-\gamma)/4$$

respectively for all $n \geq n_0$. Letting $p \in \Gamma$, we obtain from (3.14) that

$$\begin{aligned} d(y_n, p) &= d(\alpha_n f(w_n) \oplus (1-\alpha_n) T^n w_n, p) \\ &\leq \alpha_n d(f(w_n), p) + (1-\alpha_n) d(T^n w_n, p) \\ &\leq \alpha_n \gamma d(w_n, p) + \alpha_n d(f(p), p) + (1-\alpha_n)[(1+u_n) d(w_n, p) + v_n] \\ &= \Big(1 - \alpha_n[(1-\gamma) - (1-\alpha_n) u_n/\alpha_n]\Big) d(w_n, p) + \alpha_n d(f(p), p) \\ &+ (1-\alpha_n) v_n. \end{aligned} \tag{3.16}$$

From (3.14) and (3.16), we obtain

$$
\begin{aligned}
&d(x_{n+1},p) = d(\beta_n w_n \oplus (1-\beta_n)T^n y_n, p) \\
&\le \beta_n d(w_n,p) + (1-\beta_n)(1+u_n)d(y_n,p) + (1-\beta_n)v_n \\
&\le \Big(1-\alpha_n(1-\beta_n)(1+u_n)[(1-\gamma)-(1-\alpha_n)u_n/\alpha_n - u_n/[\alpha_n(1+u_n)]]\Big)d(w_n,p) \\
&+ \alpha_n(1-\beta_n)(1+u_n)[d(f(p),p) + (1-\alpha_n)v_n/\alpha_n + v_n/[\alpha_n(1+u_n)] \\
&\le \Big[1-\alpha_n(1-\beta_n)(1+u_n)\Big((1-\gamma)-(1-\alpha_n)u_n/\alpha_n - u_n/[\alpha_n(1+u_n)]\Big)\Big]d(x_n,p) \\
&+ \alpha_n(1-\beta_n)(1+u_n)\Big((1-\gamma)-(1-\alpha_n)u_n/\alpha_n - u_n/[\alpha_n(1+u_n)]\Big) \\
&\times \frac{2[d(f(p),p) + (1-\alpha_n)v_n/\alpha_n + v_n/[\alpha_n(1+u_n)]]}{1-\gamma} \\
&\le \max\Big\{d(x_n,p), \frac{2d(f(p),p)}{1-\gamma}+1\Big\}.
\end{aligned}
$$

By induction, we have

$$
d(x_n,p) \le \max\left\{d(x_{n_0},p), \frac{2d(f(p),p)}{1-\gamma}+1\right\},\ \forall n \ge n_0.
$$

Thus, $\{x_n\}$ is bounded and so $\{y_n\}, \{T^n w_n\}$ and $\{f(w_n)\}$ are all bounded. Furthermore, from (3.14), Lemma 2.1 and 2.2(5), we obtain

$$
\begin{aligned}
d(y_{n+1},y_n) &= d(\alpha_{n+1}f(w_{n+1}) \oplus (1-\alpha_{n+1})T^{n+1}w_{n+1}, \alpha_n f(w_n) \oplus (1-\alpha_n)T^n w_n) \\
\le\ & d(\alpha_{n+1}f(w_{n+1}) \oplus (1-\alpha_{n+1})T^{n+1}w_{n+1}, \alpha_{n+1}f(w_{n+1}) \oplus (1-\alpha_{n+1})T^{n+1}w_n) \\
& + d(\alpha_{n+1}f(w_{n+1}) \oplus (1-\alpha_{n+1})T^{n+1}w_n, \alpha_{n+1}f(w_n) \oplus (1-\alpha_{n+1})T^n w_n) \\
& + d(\alpha_{n+1}f(w_n) \oplus (1-\alpha_{n+1})T^n w_n, \alpha_n f(w_n) \oplus (1-\alpha_n)T^n w_n) \\
\le\ & (1-\alpha_{n+1})d(T^{n+1}w_{n+1}, T^{n+1}w_n) + \alpha_{n+1}d(f(w_{n+1}), f(w_n)) \\
& + (1-\alpha_{n+1})d(T^{n+1}w_n, T^n w_n) + |\alpha_{n+1}-\alpha_n| d(f(w_n), T^n w_n) \\
\le\ & (1-\alpha_{n+1})[(1+u_{n+1})d(w_{n+1},w_n) + v_{n+1}] + \alpha_{n+1}\gamma d(w_{n+1},w_n) \\
& + (1-\alpha_{n+1})d(T^{n+1}w_n, T^n w_n) + |\alpha_{n+1}-\alpha_n| d(f(w_n), T^n w_n) \\
=\ & [1-\alpha_{n+1}(1-\gamma) + (1-\alpha_{n+1})u_{n+1}]d(w_{n+1},w_n) \\
& + (1-\alpha_{n+1})d(T^{n+1}w_n, T^n w_n) \\
& + |\alpha_{n+1}-\alpha_n| d(f(w_n), T^n w_n) + (1-\alpha_{n+1})v_{n+1}.
\end{aligned} \tag{3.17}
$$

Without loss of generality, we may assume that $0 < \lambda_n \le \lambda_{n+1}$ $\forall n \ge 1$. Now, from Remark 3.2, we obtain

$$
\begin{aligned}
d(w_{n+1},w_n) &= d(J_{\lambda_{n+1}}x_{n+1}, J_{\lambda_n}x_n) \\
&\le d(J_{\lambda_{n+1}}x_{n+1}, J_{\lambda_{n+1}}x_n) + d(J_{\lambda_{n+1}}x_n, J_{\lambda_n x_n)} \\
&\le d(x_{n+1},x_n) + \left(\sqrt{1-\frac{\lambda_n}{\lambda_{n+1}}}\right) d(J_{\lambda_{n+1}}x_n, x_n).
\end{aligned} \tag{3.18}
$$

Thus, from (3.17) and (3.18), we obtain

$$
\begin{aligned}
& d(y_{n+1}, y_n) - d(x_{n+1}, x_n) \le [(1-\alpha_{n+1})u_{n+1} - \alpha_{n+1}(1-\gamma)]d(x_{n+1}, x_n) \\
& + [1-\alpha_{n+1}(1-\gamma) + (1-\alpha_{n+1})u_{n+1}] \left(\sqrt{1-\frac{\lambda_n}{\lambda_{n+1}}}\right) d(J_{\lambda_{n+1}}x_n, x_n) \\
& + (1-\alpha_{n+1})d(T^{n+1}w_n, T^n w_n) + |\alpha_{n+1} - \alpha_n| d(f(w_n), T^n w_n) \\
& + (1-\alpha_{n+1})v_{n+1},
\end{aligned}
$$

which implies that

$$\limsup_{n\to\infty}(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \le 0.$$

Therefore, it follows from Lemma 2.7 that

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{3.19}$$

Also, from (3.14), we obtain

$$\lim_{n\to\infty} d(y_n, T^n w_n) \le \lim_{n\to\infty} \alpha_n d(f(w_n), T^n w_n) = 0. \tag{3.20}$$

By repeating similar arguments in (3.4)-(3.8), we can easily show that

$$\lim_{n\to\infty} d(w_n, x_n) = \lim_{n\to\infty} d(J_{\lambda_n}x_n, x_n) = 0, \tag{3.21}$$

and

$$\lim_{n\to\infty} d(J_\lambda x_n, x_n) = 0. \tag{3.22}$$

From (3.19) and (3.21), we obtain

$$\lim_{n\to\infty} d(w_n, y_n) = 0. \tag{3.23}$$

Also, from (3.20) and (3.23), we have

$$d(y_n, T^n y_n) \le d(y_n, T^n w_n) + Ld(w_n, y_n) \to 0 \text{ as } n \to \infty. \tag{3.24}$$

Furthermore, from (3.23) and (3.24), we obtain

$$d(x_{n+1}, w_n) \le (1-\beta_n)[d(T^n y_n, y_n) + d(y_n, w_n)] \to 0 \text{ as } n \to \infty. \tag{3.25}$$

Since

$$
\begin{array}{rcl}
d(w_n, T^n w_n) & \le & d(w_n, x_{n+1}) + d(x_{n+1}, T^n w_n) \\
 & \le & d(w_n, x_{n+1}) + \beta_n d(w_n, T^n w_n) \\
 & + & (1-\beta_n)Ld(y_n, w_n),
\end{array}
$$

then, from (3.23) and (3.25), we obtain

$$d(w_n, T^n w_n) = 1/(1-\beta_n)d(x_{n+1}, w_n) + Ld(y_n, x_n) \to 0 \text{ as } n \to \infty. \tag{3.26}$$

Thus,

$$d(x_n, T^n x_n) \le (1+L)d(x_n, w_n) + d(w_n, T^n w_n) \to 0 \text{ as } n \to \infty. \tag{3.27}$$

Also, from (3.21) and (3.25), we obtain

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.28}$$

Again, from (3.27) and (3.28), we get

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &+ d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\leq (1+L)d(x_{n+1}, x_n) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &+ Ld(T^n x_n, x_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.29}$$

Similarly, we obtain

$$\begin{aligned} \lim_{n\to\infty} d(y_n, Ty_n) &\leq d(y_n, x_n) + d(x_n, Tx_n) + d(Tx_n, Ty_n) \\ &\leq (1+L)d(x_n, y_n) + d(x_n, Tx_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.30}$$

For each $m \geq 0$, let $z_m \in X$ be the unique fixed point of the contraction mapping such that $z_m = \beta_m z_m \oplus (1-\beta_m)T^m y_m$, where $y_m = \alpha_m f(z_m) \oplus (1-\alpha_m)T^m z_m$ (see Theorem 3.1), we obtain

$$\begin{aligned} d(z_m, y_n) &= d(\beta_n z_m \oplus (1-\beta_m)T^m y_m, y_m) \\ &\leq \beta_m d(z_m, y_n) + (1-\beta_m)d(T^m y_m, y_n) \\ &\leq \beta_n d(z_m, y_n) + (1-\beta_m)[d(T^m y_m, T^m y_n) + d(T^m y_n, y_n)] \\ &\leq \beta_n d(z_m, y_n) + (1-\beta_m)[(1+u_n)d(y_m, y_n) + v_n + d(T^m y_n, y_n)], \end{aligned}$$

which implies

$$d(z_m, y_n) \leq (1+u_n)d(y_m, y_n) + v_n + d(T^m y_n, y_n). \tag{3.31}$$

Now, let $t_m := 2u_m + u_m^2$, then from (3.31), Lemma 2.2(4) and $\lim_{m\to\infty} z_m = p$, we obtain

$$\begin{aligned} &d^2(y_m, y_n) = \langle \overrightarrow{y_m y_n}, \overrightarrow{y_m y_n} \rangle \\ &= \langle \overrightarrow{y_m T^m z_m}, \overrightarrow{y_m y_n} \rangle + \langle \overrightarrow{T^m z_m y_n}, \overrightarrow{y_m y_n} \rangle \\ &\leq \alpha_m \langle \overrightarrow{f(z_m) T^m z_m}, \overrightarrow{y_m y_n} \rangle + \langle \overrightarrow{T^m z_m y_n}, \overrightarrow{y_m y_n} \rangle \\ &= \alpha_m \langle \overrightarrow{f(z_m) T^m z_m}, \overrightarrow{y_m z_m} \rangle + \alpha_m \langle \overrightarrow{f(z_m) y_n}, \overrightarrow{z_m y_n} \rangle + \alpha_m \langle \overrightarrow{y_n T^m z_m}, \overrightarrow{z_m y_n} \rangle \\ &+ \langle \overrightarrow{T^m z_m T^m y_n}, \overrightarrow{y_m y_n} \rangle + \langle \overrightarrow{T^m y_n y_n}, \overrightarrow{y_m y_n} \rangle \\ &\leq \alpha_m d(f(z_m), T^m z_m)d(y_m, z_m) + \alpha_m \langle \overrightarrow{f(z_m) z_m}, \overrightarrow{z_m y_n} \rangle + \alpha_m \langle \overrightarrow{z_m T^m z_m}, \overrightarrow{z_m y_n} \rangle \\ &+ d(T^m z_m, T^m y_n)d(y_m, y_n) + d(T^m y_n, y_n)d(y_m, y_n) \\ &\leq \alpha_m d(f(z_m), T^m z_m)d(y_m, z_m) + \alpha_m \langle \overrightarrow{f(z_m) z_m}, \overrightarrow{z_m y_n} \rangle + \alpha_m \langle \overrightarrow{z_m T^m z_m}, \overrightarrow{z_m y_n} \rangle \\ &+ [(1+u_m)d(z_m, y_n) + v_m]d(y_m, y_n) + d(T^m y_n, y_n)d(y_m, y_n) \\ &\leq \alpha_m d(f(z_m), T^m z_m)d(y_m, z_m) + \alpha_m \langle \overrightarrow{f(z_m) z_m}, \overrightarrow{z_m y_n} \rangle + \alpha_m d(z_m, T^m z_m)d(z_m, y_n) \\ &+ (1+t_m)d^2(y_m, y_n) + v_m(2+u_m)d(y_m, y_n) + (2+u_m)d(T^m y_n, y_n)d(y_m, y_n). \end{aligned}$$

Therefore,

$$\begin{aligned}\langle \overrightarrow{f(z_m)z_m}, \overrightarrow{y_n z_m}\rangle &\leq d(f(z_m), T^m z_m)d(y_m, z_m) + d(z_m, T^m z_m)d(z_m, y_n)\\ &+ w_m/\alpha_m d^2(y_m, y_n) + v_m \alpha_m (2+u_m)d(y_m, y_n)\\ &+ (2+u_m)/\alpha_m d(T^m y_n, y_n)d(y_m, y_n).\end{aligned}$$

Thus, taking the upper limit as $n \to \infty$ first, and then as $m \to \infty$, we obtain from (3.19), (3.27) and (3.30) that

$$\limsup_{m\to\infty}\ \limsup_{n\to\infty}\langle \overrightarrow{f(z_m)z_m}, \overrightarrow{y_n z_m}\rangle \leq 0. \tag{3.32}$$

Furthermore

$$\begin{aligned}\langle \overrightarrow{f(p)p}, \overrightarrow{w_n p}\rangle &= \langle \overrightarrow{f(p)f(z_m)}, \overrightarrow{w_n p}\rangle + \langle \overrightarrow{f(z_m)z_m}, \overrightarrow{w_n y_n}\rangle\\ &+ \langle \overrightarrow{f(z_m)z_m}, \overrightarrow{y_n z_m}\rangle + \langle \overrightarrow{f(z_m)z_m}, \overrightarrow{z_m p}\rangle + \langle \overrightarrow{z_m p}, \overrightarrow{w_n p}\rangle\\ &\leq d(f(p), f(z_m))d(w_n, p) + d(f(z_m), z_m)d(w_n, y_n)\\ &+ \langle \overrightarrow{f(z_m)z_m}, \overrightarrow{y_n z_m}\rangle + d(f(z_m), z_m)d(z_m, p)\\ &+ d(z_m, p)d(w_n, p)\\ &\leq (1+\gamma)d(z_m, p)d(w_n, p) + \langle \overrightarrow{f(z_m)z_m}, \overrightarrow{y_n z_m}\rangle\\ &+ [d(w_n, y_n) + d(z_m, p)]d(f(z_m), z_m),\end{aligned}$$

which implies from (3.23) and $\lim_{m\to\infty} z_m = p$ that

$$\begin{aligned}\limsup_{n\to\infty}\langle \overrightarrow{f(p)p}, \overrightarrow{w_n p}\rangle &= \limsup_{m\to\infty}\ \limsup_{n\to\infty}\langle \overrightarrow{f(p)p}, \overrightarrow{w_n p}\rangle\\ &\leq \limsup_{m\to\infty}\ \limsup_{n\to\infty}\langle \overrightarrow{f(z_m)z_m}, \overrightarrow{y_n z_m}\rangle \leq 0.\end{aligned}$$

Finally, we show that $x_n \to p$ as $n \to \infty$.
Since

$$\lim_{n\to\infty}(1-\alpha_n)u_n/\alpha_n = 0$$

and

$$\lim_{n\to\infty}(1-\alpha_n)v_n/\alpha_n = 0,$$

then there exists $n_0 \in \mathbb{N}$ such that

$$(1-\alpha_n)u_n/\alpha_n < (1-\gamma)/4$$

and

$$\lim_{n\to\infty}(1-\alpha_n)v_n/\alpha_n < (1-\gamma)/4$$

respectively for all $n \geq n_0$. Thus, using Lemma 2.2, we obtain

$$\begin{aligned}d^2(x_{n+1}, p) &= d^2(\beta_n w_n \oplus (1-\beta_n)T^n y_n, p)\\ &\leq \beta_n d^2(w_n, p) + (1-\beta_n)d^2(T^n y_n, p)\\ &= \beta_n d^2(w_n, p) + (1-\beta_n)\langle \overrightarrow{T^n y_n p}, \overrightarrow{T^n y_n p}\rangle\\ &= \beta_n d^2(w_n, p) + (1-\beta_n)[\langle \overrightarrow{T^n y_n p}, \overrightarrow{T^n y_n w_n}\rangle\\ &+ \langle \overrightarrow{T^n y_n y_n}, \overrightarrow{T^n y_n w_n}\rangle + \langle \overrightarrow{y_n p}, \overrightarrow{w_n p}\rangle],\end{aligned} \tag{3.33}$$

and

$$
\begin{aligned}
\langle \overrightarrow{y_n p}, \overrightarrow{w_n p} \rangle &\leq \alpha_n \langle \overrightarrow{f(w_n)p}, \overrightarrow{w_n p} \rangle + (1-\alpha_n)\langle \overrightarrow{T^n w_n p}, \overrightarrow{w_n p} \rangle \\
&\leq \alpha_n \langle \overrightarrow{f(w_n)f(p)}, \overrightarrow{w_n p} \rangle + \alpha_n \langle \overrightarrow{f(p)p}, \overrightarrow{w_n p} \rangle \\
&+ (1-\alpha_n) d(T^n w_n, p) d(w_n, p) \\
&\leq \alpha_n d(f(w_n), f(p)) d(w_n, p) + \alpha_n \langle \overrightarrow{f(p)p}, \overrightarrow{w_n p} \rangle \\
&+ (1-\alpha_n)[(1+u_n) d(w_n, p) + v_n] d(w_n, p) \\
&\leq [\alpha_n \gamma + (1-\alpha_n)(1+u_n)] d^2 d(w_n, p) + \alpha_n \langle \overrightarrow{f(p)p}, \overrightarrow{w_n p} \rangle \\
&+ (1-\alpha_n) v_n d(w_n, p).
\end{aligned} \tag{3.34}
$$

Since $\{w_n\}$ and $\{T^n y_n\}$ are bounded, there exists $M > 0$ such that

$$M := \sup_{n \geq 1} \{d(T^n y_n, p), d(w_n, p)\}.$$

Thus, we obtain from (3.33) and (3.34) that

$$
\begin{aligned}
d^2(x_{n+1}, p) &\leq \Big[\beta_n + (1-\beta_n)[\alpha_n \gamma + (1-\alpha_n)(1+u_n)]\Big] d^2(w_n, p) \\
&\quad + (1-\beta_n)(1-\alpha_n) v_n d(w_n, p) + (1-\beta_n)\langle \overrightarrow{T^n y_n p}, \overrightarrow{T^n y_n w_n} \rangle \\
&\quad + (1-\beta_n)\langle \overrightarrow{T^n y_n y_n}, \overrightarrow{w_n p} \rangle + \alpha_n (1-\beta_n) \langle \overrightarrow{f(p)p}, \overrightarrow{w_n p} \rangle \\
&\leq \Big[1 - \alpha_n(1-\beta_n)[1-\gamma-(1-\alpha_n)u_n/\alpha_n]\Big] d^2(w_n, p) \\
&\quad + (1-\beta_n)(1-\alpha_n) v_n d(w_n, p) + (1-\beta_n) d(T^n y_n, p) d(T^n w_n, y_n) \\
&\quad + (1-\beta_n) d(T^n y_n, y_n) d(w_n, p) + \alpha_n (1-\beta_n) \langle \overrightarrow{f(p)p}, \overrightarrow{w_n p} \rangle \\
&\leq \Big[1 - \alpha_n(1-\beta_n)[1-\gamma-(1-\alpha_n)u_n/\alpha_n]\Big] d^2(x_n, p) \\
&\quad + \alpha_n(1-\beta_n)[1-\gamma-(1-\alpha_n)u_n/\alpha_n] \\
&\quad \times \frac{[\langle \overrightarrow{f(p)p}, \overrightarrow{w_n p} \rangle + (1-\alpha_n) v_n/\alpha_n]}{(1-\gamma-(1-\alpha_n)u_n/\alpha_n)} \\
&\quad + (1-\beta_n) M [d(T^n y_n, w_n) + d(T^n y_n, y_n)] \\
&\leq \Big[1 - \alpha_n(1-\beta_n)[1-\gamma-(1-\alpha_n)u_n/\alpha_n]\Big] d^2(x_n, p) \\
&\quad + \alpha_n(1-\beta_n)[1-\gamma-(1-\alpha_n)u_n/\alpha_n] \\
&\quad \times \frac{[\langle \overrightarrow{f(p)p}, \overrightarrow{w_n p} \rangle + (1-\alpha_n) v_n/\alpha_n]}{(1-\gamma-(1-\alpha_n)u_n/\alpha_n)} \\
&\quad + (1-\beta_n) M [d(y_n, w_n) + 2d(T^n y_n, y_n)],
\end{aligned}
$$

that is,

$$d^2(x_{n+1}, p) \leq (1-\delta_n) d^2(x_n, p) + \delta_n \theta_n + \sigma_n, \tag{3.35}$$

where

$$\delta_n := \alpha_n(1-\beta_n)[1-\gamma-(1-\alpha_n)u_n/\alpha_n],$$

$$\theta_n := \frac{[\langle \overrightarrow{f(p)p}, \overrightarrow{x_n p} \rangle + (1-\alpha_n)v_n/\alpha_n]}{[1-\gamma-(1-\alpha_n)u_n/\alpha_n]}$$

and

$$\sigma_n := (1-\beta_n)M[d(y_n, w_n) + 2d(T^n y_n, y_n)].$$

Thus, applying Lemma 2.11 in (3.35), we obtain that $d(x_n, p) \to 0$ as $n \to \infty$. Hence $\{x_n\}$ converges strongly to p which solves the variational inequality (3.15). $\square$

The following corollaries follows from Theorem 3.1 and Theorem 3.3 respectively.

Corollary 3.4. *Let X be a Hadamard space and X^* be its dual space. Let $T : X \to X$ be uniformly asymptotically regular and uniformly L-Lipschitzian asymptotically nonexpansive mapping with sequence $\{u_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} u_n = 0$.*

Let $A : X \to 2^{X^*}$ be a multivalued monotone mapping which satisfies the range condition and f be a contraction mapping on X with coefficient $\gamma \in (0,1)$. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$ and for arbitrary $x_1 = x \in C$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by

$$\begin{cases} y_n = J_{\lambda_n}(x_n), \\ x_n = \alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n \;\; n \geq 1, \end{cases} \tag{3.36}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ satisfying

$$\lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} \frac{u_n}{\alpha_n} = 0,$$

assuming that $L < (1-\alpha_n\gamma)/(1-\alpha_n)$ and $0 < \lambda \leq \lambda_n \; \forall n \geq 1$.
Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in \Gamma$ which solves the variational inequality

$$\langle \overrightarrow{pf(p)}, \overrightarrow{qp} \rangle \geq 0, \quad q \in \Gamma. \tag{3.37}$$

Corollary 3.5. *Let X be a Hadamard space and X^* be its dual space.*
Let $T : X \to X$ be uniformly asymptotically regular and uniformly L-Lipschitzian asymptotically nonexpansive mapping with sequence $\{u_n\} \subset [0, \infty)$ such that

$$\lim_{n\to\infty} u_n = 0.$$

Let $A : X \to 2^{X^*}$ be a multivalued monotone mapping which satisfies the range condition and f be a contraction mapping on X with coefficient $\gamma \in (0,1)$. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$ and for arbitrary $x_1 = x \in C$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by

$$\begin{cases} w_n = J_{\lambda_n} x_n, \\ y_n = \alpha_n f(w_n) \oplus (1-\alpha_n)T^n w_n, \\ x_{n+1} = \beta_n w_n \oplus (1-\beta_n)T^n y_n, \; n \geq 1, \end{cases} \tag{3.38}$$

where $0 < \lambda \leq \lambda_n \; \forall n \geq 1$ and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset (0,1)$, satisfying

(a) $\lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \frac{u_n}{\alpha_n} = 0,$

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$

(c) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$,

(d) $L < (1 - \alpha_n \gamma)/(1 - \alpha_n)$,

(e) $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1$.

Then, $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in \Gamma$ *which solves the variational inequality*

$$\langle \overrightarrow{pf(p)}, \overrightarrow{qp} \rangle \geq 0, \quad q \in \Gamma. \tag{3.39}$$

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