Fixed Point Theory, 21(2020), No. 1, 319-338
DOI: 10.24193/fpt-ro.2020.1.23
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON ψ-CONTRACTIONS AND COMMON FIXED POINT RESULTS IN PROBABILISTIC METRIC SPACES

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Abstract. In this paper, by weakening the conditions on the gauge function ψ, some new fixed point and common fixed point (common coupled fixed point, common tripled fixed point) theorems for nonlinear mappings with a gauge function ψ in Menger probabilistic metric spaces are established. An example is given to illustrate our theory.
Key Words and Phrases: Coupled fixed point, fixed point, metric space, probabilistic φ-contractions, gauge function.
2010 Mathematics Subject Classification: 54E70, 54H25, 47H10.

1. Introduction

Probabilistic metric spaces were initiated by Menger [19] in 1942 and fixed point theory in these space was presented by Sehgal and Bharucha-Reid [25] in 1972. A mapping $\mathcal{L} : X \to X$ is called a probabilistic ψ-contraction if it satisfies

$$\Phi_{\mathcal{L}x,\mathcal{L}y}(\psi(\xi)) \geq \Phi_{x,y}(\xi)$$

for all $x, y \in X$ and $\xi > 0$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a gauge function satisfying certain conditions.

Fixed point results for probabilistic ψ-contractions in Menger probabilistic metric space were investigated by many researchers (for example, see [2], [3], [13], [15], [18] and [21]). However, some of these results are obtained under the assumption that the function ψ is non-decreasing and $\sum_{n=1}^{\infty} \psi^n(\xi) < \infty$ for any $\xi > 0$ (see [7], [9], [10], [20], [27]). Ćirić pointed out that the condition "the gauge function ψ is non-decreasing and $\sum_{n=1}^{\infty} \psi^n(\xi) < \infty$ for any $\xi > 0$ " can be strong and difficult to check in practice. A natural question is whether the conditions can be improved. Jachymski [14] established the following.

Theorem of Jachymski. *Assume that*

(i) $(X, \mathcal{F}, \Gamma)$ *is a complete Menger probabilistic metric space such that* Γ *is a continuous* t*-norm of* H*-type;*

(ii) ψ*:* $\mathbb{R}^+ \to \mathbb{R}^+$ *is a gauge function satisfying* $\psi(\xi) < \xi$, $\psi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to\infty} \psi^n(\xi) = 0$ *for all* $\xi > 0$*;*

(iii) $\mathcal{L} : X \to X$ *is a mapping with the property that:*

$$\Phi_{\mathcal{L}x,\mathcal{L}y}(\psi(\xi)) \geq \Phi_{x,y}(\xi)$$

for all $x, y \in X$.

Then there is a unique $u \in X$ *such that* $u = \mathcal{L}u$.

In [27], using the properties of the pseudo-metric and the triangular norm, Xiao, Zhu and Cao gave the following common coupled fixed point theorem for probabilistic ψ-contractions in Menger probabilistic metric space.

Theorem of XZC. *Assume that*

(i) $(X, \mathcal{F}, \Gamma)$ *is a complete Menger probabilistic metric space, where* Γ *is a Hadžić type* t*-norm;*

(ii) ψ*:* $\mathbb{R}^+ \to \mathbb{R}^+$ *is a gauge function satisfying* $\psi(\xi) > \xi$, $\psi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to\infty} \psi^n(\xi) = +\infty$ *for all* $\xi > 0$*;*

(iii) $\mathcal{L} : X \times X \to X$ *and* $g : X \to X$ *are two mappings with the property that*

$$\Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\psi(\xi)) \geq \min\{\Phi_{Ax,Ap}(\xi), \Phi_{Ay,Aq}(\xi)\}$$

for all $x, y, p, q \in X$*, where* $\mathcal{L}(X \times X) \subset A(X)$*;*

(iv) A *is continuous and commutative with* $\mathcal{L}$.

Then there is a unique $u \in X$ *such that* $u = Au = \mathcal{L}(u, u)$.

In 2014, Luo, Zhu and Wu [17] gave a generalization of the above theorem.

Theorem of LZW. *Assume that*

(i) $(X, \mathcal{F}, \Gamma)$ *is a complete generalized Menger probabilistic metric space, where* Γ *is a Hadžić type* t*-norm;*

(ii) ψ*:* $\mathbb{R}^+ \to \mathbb{R}^+$ *is a gauge function satisfying* $\psi(\xi) > \xi$, $\psi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to\infty} \psi^n(\xi) = +\infty$ *for all* $\xi > 0$*;*

(iii) $\mathcal{L} : X \times X \times X \to X$ *and* $g : X \to X$ *are two mappings with the property that:*

$$\Phi_{\mathcal{L}(x,y,z),\mathcal{L}(p,q,r)}(\psi(\xi)) \geq \min\{\Phi_{Ax,Ap}(\xi), \Phi_{Ay,Aq}(\xi), \Phi_{Az,Ar}(\xi)\}$$

for all $x, y, z, p, q, r \in X$*, where* $\mathcal{L}(X \times X \times X) \subset A(X)$*;*

(iv) A *is continuous and commutative with* $\mathcal{L}$.

Then there is a unique $u \in X$ *such that* $u = Au = \mathcal{L}(u, u, u)$.

Remark 1.1 There are many gauge functions ψ that do not satisfy the conditions in the above theorems. For example, if the gauge function $\psi : [0, +\infty) \to [0, +\infty)$ is defined by

$$\psi(\xi) = \begin{cases} \frac{5}{2}, & \xi = 1, \\ \frac{t}{3}, & \xi \in [0, 1) \cup (1, +\infty), \end{cases}$$

then $\lim_{n\to+\infty}\psi^n(\xi)=0$ for any $\xi>0$. However, for $\xi=1$, $\psi(\xi)=\frac{5}{2}>1$ contrary to $\psi(\xi)<\xi$.

If the gauge function $\psi:[0,+\infty)\to[0,+\infty)$ is defined by

$$\psi(\xi)=\begin{cases}\frac{1}{3}, & \xi=1,\\ 2\xi, & \xi\in[0,1)\cup(1,+\infty),\end{cases}$$

then $\lim_{n\to+\infty}\psi^n(\xi)=+\infty$ for any $\xi>0$.
However, for $\xi=1$, $\psi(\xi)=\frac{1}{3}<1=\xi$ contrary to $\psi(\xi)>\xi$.

A natural question is whether the conditions " $\lim_{n\to\infty}\psi^n(\xi)=0$ and $\psi(\xi)<\xi$ for any $\xi>0$" and "$\psi(\xi)>\xi$ and $\lim_{n\to\infty}\psi^n(\xi)=+\infty$ for any $\xi>0$" can be weakened? We give an affirmative answer to this question.

In Section 2, we recall some concepts and results in Menger probabilistic metric spaces. In Section 3, by using methods similar to that in [26] we prove some fixed point and common fixed point (common coupled fixed point, common tripled fixed point) theorems for nonlinear mappings with a gauge function ψ in Menger probabilistic metric spaces. Our results improve and generalize the corresponding ones from [14, 17, 27]. Moreover, we use an example to illustrate the theory.

2. Preliminaries

Suppose that $\mathbb{R}$ denotes the real, $\mathbb{R}^+=[0,+\infty)$, and $\mathbb{Z}^+$ is the set of all positive integers. A function $\Phi:\mathbb{R}\to[0,1]$ is called a distribution function if it is left-continuous and nondecreasing with $\Phi(-\infty)=0,\Phi(+\infty)=1$. Let $\mathcal{D}_\infty$ be the set of all distribution functions. Write $\mathcal{D}=\{\Phi\in\mathcal{D}_\infty:\inf_{t\in\mathbb{R}}\Phi(t)=0,\sup_{t\in\mathbb{R}}\Phi(t)=1\}$, $\mathcal{D}_\infty^+=\{\Phi\in\mathcal{D}_\infty:\Phi(0)=0\}$, and $\mathcal{D}^+=\mathcal{D}\cap\mathcal{D}_\infty^+$.

Definition 2.1. ([23]) If a mapping $\Gamma:[0,1]\times[0,1]\to[0,1]$ satisfies the following conditions:

(Γ-1) $\Gamma(\xi,1)=\xi$;

(Γ-2) $\Gamma(\xi,\eta)=\Gamma(\eta,\xi)$;

(Γ-3) $\Gamma(\xi,\eta)\geq\Gamma(\mu,\nu)$, for $\xi\geq\mu,\eta\geq\nu$;

(Γ-4) $\Gamma(\Gamma(\xi,\eta),\mu)=\Gamma(\xi,\Gamma(\eta,\mu))$, then Γ is called a triangular norm (for short, a t-norm), where $\xi,\eta,\mu,\nu\in[0,1]$.

By the definition of Γ, it is easy to see that $\min\{\xi,\eta\}\geq\Gamma(\xi,\eta)$ for all $\xi,\eta\in[0,1]$.

Two typical examples of continuous t-norm are $\Gamma_M(\xi,\eta)=\min\{\xi,\eta\}$ and $\Gamma_p(\xi,\eta)=\xi\eta$ for all $\xi,\eta\in[0,1]$.

Definition 2.2. ([12]) A t-norm Γ is called a Hadžić type t-norm if the family $\{\Gamma^n(\xi)\}_{n=1}^{+\infty}$ of its iterates defined for each $\xi\in[0,1]$ by

$$\Gamma^1(\xi)=\Gamma(\xi,\xi),\Gamma^2(\xi)=\Gamma(\xi,\Gamma^1(\xi)),\cdots,\Gamma^n(\xi)=\Gamma(\xi,\Gamma^{n-1}(\xi)),\cdots.$$

is equi-continuous at $\xi=1$. Obviously, $\Gamma^n(\xi)\leq\xi$ for any $n\in\mathbb{Z}^+$ and $\xi\in[0,1]$, and Γ_M is a Hadžić type t-norm [11].

Definition 2.3. A function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is said to be a gauge function (a G-function) if $\psi(0)=0$.

In the paper, we assume that $\psi^n(\xi)$ denotes the nth iteration of $\psi(\xi)$, where $\xi\in\mathbb{R}^+$.

Definition 2.4. ([19]) A Menger probabilistic metric space (shortly, Menger PM-space) is a triple $(X,\mathcal{F},\Gamma)$, where X is a nonempty set, $\mathcal{F}$ is a mapping of $X\times X\to\mathcal{D}^+_\infty$ and Γ is a t-norm which satisfies the following conditions (we denote the distribution function $\mathcal{F}(x,y)$ by $\Phi_{x,y}$):

(PM-1) $\Phi_{x,y}(\xi)=1$ for all $\xi>0$ if and only if $x=y$;

(PM-2) $\Phi_{x,y}(\xi)=\Phi_{y,x}(\xi)$, $\forall x,y\in X$, $\xi>0$;

(PM-3) $\Phi_{x,z}(\xi_1+\xi_2)\geq\Gamma(\Phi_{x,y}(\xi_1),\Phi_{y,z}(\xi_2))$, $\forall x,y,z\in X$, $\xi_1>0$, $\xi_2>0$.

Remark 2.5. Schweizer et al. [23] have pointed out that if the t-norm Γ of a Menger probabilistic metric space satisfies the condition $\sup_{0<a<1}\Gamma(a,a)=1$, then $(X,\mathcal{F},\Gamma)$ is a first countable Hausdorff topological space in the (ε,λ)-topology τ, that is, the family

$$\{U_p(\varepsilon,\lambda):\varepsilon>0,\lambda\in(0,1],p\in X\}$$

is a base of neighborhoods of point p for τ, where

$$U_p(\varepsilon,\lambda)=\{x\in X:\Phi_{p,x}(\varepsilon)>1-\lambda\}.$$

Definition 2.6. ([23]) Assume that $(X,\mathcal{F},\Gamma)$ is a Menger probabilistic metric space.

(a) A sequence $\{x_n\}\subset X$ is convergent to x (we write $x_n\to x$ or $\lim_{n\to\infty}x_n=x$) if $\lim_{n\to\infty}\Phi_{x_n,x}(\xi)=1$ for all $\xi>0$.

(b) A sequence $\{x_n\}\subset X$ is a Cauchy sequence if for any given $\varepsilon>0$ and $\lambda\in(0,1]$, there is $N=N(\varepsilon,\lambda)\in\mathbb{Z}^+$ such that $\Phi_{x_n,x_m}(\varepsilon)>1-\lambda$ whenever $n,m\geq N$.

(c) A Menger probabilistic metric space $(X,\mathcal{F},\Gamma)$ is complete if every Cauchy sequence in X converges to an element in X.

In this paper, we shall always suppose that $(X,\mathcal{F},\Gamma)$ is a Menger space with the (ε,λ)-topology.

Definition 2.7. ([4]) An element $(x,y)\in X\times X$ is said to be a coupled coincidence point of the mappings $\mathcal{L}:X\times X\to X$ and $A:X\to X$ if

$$\mathcal{L}(x,y)=Ax,\ \mathcal{L}(y,x)=Ay.$$

Definition 2.8. ([24]) A mapping $A:X\to X$ is said to be commutative with a mapping $\mathcal{L}:X\times X\to X$ if $A\mathcal{L}(x,y)=\mathcal{L}(Ax,Ay)$ for all $x,y\in X$.

Definition 2.9. ([1]) The mappings $\mathcal{L}:X\times X\to X$ and $A:X\to X$ are called weakly compatible (or w-compatible) if $\mathcal{L}(x,y)=Ax$ and $\mathcal{L}(y,x)=Ay$, then

$$A\mathcal{L}(x,y)=\mathcal{L}(Ax,Ay)$$

and

$$A\mathcal{L}(y,x)=\mathcal{L}(Ay,Ax)$$

for all $x,y\in X$.

Definition 2.10. ([16]) An element $x\in X$ is said to be a common fixed point of the mappings $\mathcal{L}:X\times X\to X$ and $A:X\to X$ if

$$\mathcal{L}(x,x)=Ax=x.$$

Definition 2.11. ([5]) An element $(x,y,z)\in X\times X\times X$ is said to be a tripled fixed point of $\mathcal{L}:X\times X\times X\to X$ if $\mathcal{L}(x,y,z)=x,\mathcal{L}(y,x,y)=y$, and $\mathcal{L}(z,y,x)=z$.

Definition 2.12. ([6]) An element $(x, y, z) \in X \times X \times X$ is said to be a tripled coincidence point of the mappings $\mathcal{L} : X \times X \times X \to X$ and $A : X \to X$ if

$$\mathcal{L}(x, y, z) = Ax,\ \mathcal{L}(y, x, y) = Ay,\ \mathcal{L}(z, y, x) = Az.$$

Moreover, (x, y, z) is said to be a tripled common fixed point of $\mathcal{L}$ and A if $\mathcal{L}(x, y, z) = Ax = x, \mathcal{L}(y, x, y) = Ay = y$, and $\mathcal{L}(z, y, x) = Az = z$.

Definition 2.13. ([22]) A mapping $A : X \to X$ is said to be commutative with a mapping $\mathcal{L} : X \times X \times X \to X$ if $A\mathcal{L}(x, y, z) = \mathcal{L}(Ax, Ay, Az)$ for all $x, y, z \in X$.

Definition 2.14. ([22]) The mappings $\mathcal{L} : X \times X \times X \to X$ and $A : X \to X$ are called w-compatible if $A\mathcal{L}(x, y, z) = \mathcal{L}(Ax, Ay, Az)$ whenever $Ax = \mathcal{L}(x, y, z)$, $Ay = \mathcal{L}(y, x, y)$, and $Az = \mathcal{L}(z, y, x)$.

Lemma 2.15. ([14]) *Let $\Phi \in \mathcal{D}^+$, and let $\Phi_n : \mathbb{R} \to [0, 1]$ be nondecreasing for each $n \in \mathbb{Z}^+$. Suppose $g_n : (0, +\infty) \to (0, +\infty)$ satisfies $\lim_{n\to\infty} g_n(\xi) = 0$ for any $\xi > 0$. If*

$$\Phi_n(g_n(\xi)) \geq \Phi(\xi)$$

for any $\xi > 0$, then $\lim_{n\to\infty} \Phi_n(\xi) = 1$ for any $\xi > 0$.

3. Common fixed point results for nonlinear contractive mappings in Menger probabilistic metric spaces

Theorem 3.1. *Assume that*

(i) $(X, \mathcal{F}, \Gamma)$ is a Menger PM-space, where Γ is a Hadžić type t-norm;

(ii) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a G-function satisfying $\psi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to\infty} \psi^n(\xi) = +\infty$ for any $\xi > 0$;

(iii) The mappings $\mathcal{L} : X \times X \to X$ and $A : X \to X$ satisfy the property:

$$\Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\xi) \geq \min\{\Phi_{Ax,Ap}(\psi(\xi)), \Phi_{Ay,Aq}(\psi(\xi))\} \tag{3.1}$$

$\forall x, y, p, q \in X$, where $\mathcal{L}(X \times X) \subseteq A(X)$;

(iv) $\mathcal{L}(X \times X)$ is complete;

(v) A and $\mathcal{L}$ are ω-compatible.

Then there is a unique $u \in X$ such that $u = Au = \mathcal{L}(u, u)$.

Proof. Assume that x_0, y_0 are two arbitrary points of X. Since $\mathcal{L}(X \times X) \subset A(X)$, we can choose $x_1, y_1 \in X$ such that $Ax_1 = \mathcal{L}(x_0, y_0)$ and $Ay_1 = \mathcal{L}(y_0, x_0)$. Again from $\mathcal{L}(X \times X) \subset A(X)$, we can choose $x_2, y_2 \in X$ such that $Ax_2 = \mathcal{L}(x_1, y_1)$ and $Ay_2 = \mathcal{L}(y_1, x_1)$. Continue this process and we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{n+1} = \mathcal{L}(x_n, y_n)$$

and

$$Ay_{n+1} = \mathcal{L}(y_n, x_n)$$

for all $n \in \mathbb{N}$.

Using condition (3.1) we get (for $\xi > 0$)

$$\begin{aligned} \Phi_{Ax_n,Ax_{n+1}}(\xi) &= \Phi_{\mathcal{L}(x_{n-1},y_{n-1}),\mathcal{L}(x_n,y_n)}(\xi) \\ &\geq \min\{\Phi_{Ax_{n-1},Ax_n}(\psi(\xi)), \Phi_{Ay_{n-1},Ay_n}(\psi(\xi))\} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\Phi_{Ay_n,Ay_{n+1}}(\xi) &= \Phi_{\mathcal{L}(y_{n-1},x_{n-1}),\mathcal{L}(y_n,x_n)}(\xi) \\ &\geq \min\{\Phi_{Ay_{n-1},Ay_n}(\psi(\xi)), \Phi_{Ax_{n-1},Ax_n}(\psi(\xi))\}.\end{aligned} \tag{3.3}$$

Let $D_n(\xi) = \min\{\Phi_{Ax_{n-1},Ax_n}(\xi), \Phi_{Ay_{n-1},Ay_n}(\xi)\}$. Using inequalities (3.2) and (3.3), we find that $D_{n+1}(\xi) \geq D_n(\psi(\xi))$. This implies that

$$D_{n+1}(\xi) \geq D_n(\psi(\xi)) \geq D_{n-1}(\psi^2(\xi)) \geq \cdots \geq D_1(\psi^n(\xi)). \tag{3.4}$$

Since

$$\lim_{\xi\to+\infty} D_1(\xi) = \lim_{\xi\to+\infty} \min\{\Phi_{Ax_0,Ax_1}(\xi), \Phi_{Ay_0,Ay_1}(\xi)\} = 1$$

and

$$\lim_{n\to+\infty} \psi^n(\xi) = +\infty$$

for each $\xi > 0$, we have $\lim_{n\to+\infty} D_1(\psi^n(\xi)) = 1$.

Also using (3.2)-(3.4), we get

$$\Phi_{Ax_n,Ax_{n+1}}(\xi) \geq D_{n+1}(\xi) \geq D_n(\psi(\xi)) \geq \ldots \geq D_1(\psi^n(\xi))$$

and

$$\Phi_{Ay_n,Ay_{n+1}}(\xi) \geq D_{n+1}(\xi) \geq D_n(\psi(\xi)) \geq \ldots \geq D_1(\psi^n(\xi)).$$

Hence, we have

$$\lim_{n\to+\infty} \Phi_{Ax_n,Ax_{n+1}}(\xi) = 1$$

and

$$\lim_{n\to+\infty} \Phi_{Ay_n,Ay_{n+1}}(\xi) = 1.$$

These imply that

$$\lim_{n\to+\infty} D_n(\xi) = 1 \quad \text{for all } \xi > 0. \tag{3.5}$$

Since $\lim_{n\to+\infty} \psi^n(\xi) = +\infty$, for any fixed $\xi > 0$, there is a $n_0 = n_0(\xi) \in \mathbb{N}$ such that

$$\psi^{n_0+1}(\xi) > \psi^{n_0}(\xi) > \xi.$$

Similarly, since $\lim_{n\to+\infty} \psi^n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)) = +\infty$, there exists a $m_0 = m_0(\xi) \in \mathbb{N}$ such that

$$\psi^{m_0}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)) > \psi^{n_0+1}(\xi) - \psi^{n_0}(\xi).$$

From (3.4), we get

$$\begin{aligned}&\Phi_{Ax_{n+m_0},A_{x_{n+m_0+1}}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)) \geq D_{n+m_0}(\psi(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi))) \\ &\geq \quad \ldots \geq D_n(\psi^{m_0}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi))) \geq D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)).\end{aligned} \tag{3.6}$$

Similarly, we find that

$$\Phi_{Ay_{n+m_0},A_{y_{n+m_0+1}}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)) \geq D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)). \tag{3.7}$$

Next we prove that for any $k \in \mathbb{Z}^+ \cup \{0\}$,

$$\Phi_{Ax_{n+m_0},Ax_{n+m_0+k}}(\psi^{n_0+1}(\xi)) \geq \Gamma^k(D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi))) \tag{3.8}$$

and

$$\Phi_{Ay_{n+m_0},Ay_{n+m_0+k}}(\psi^{n_0+1}(\xi)) \geq \Gamma^k(D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi))). \tag{3.9}$$

We use mathematical induction. It is obvious that (3.8) and (3.9) hold for $k = 0$ since

$$\Phi_{Ax_{n+m_0},Ax_{n+m_0+k}}(\psi^{n_0+1}(\xi)) = \Phi_{Ay_{n+m_0},Ay_{n+m_0+k}}(\psi^{n_0+1}(\xi)) = 1.$$

Now, assume that inequalities (3.8) and (3.9) are valid for some fixed $k \in \mathbb{Z}^+ \cup \{0\}$. From (PM-3), (3.1), (3.8), (3.9), (3.6), (3.7) and the monotonicity of Γ, we have

$$\begin{aligned}
&\Phi_{Ax_{n+m_0},Ax_{n+m_0+k+1}}(\psi^{n_0+1}(\xi)) \\
&\quad = \Phi_{Ax_{n+m_0},Ax_{n+m_0+k+1}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi) + \psi^{n_0}(\xi)) \\
&\quad \geq \Gamma(\Phi_{Ax_{n+m_0},Ax_{n+m_0+1}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)), \Phi_{Ax_{n+m_0+1},Ax_{n+m_0+k+1}}(\psi^{n_0}(\xi))) \\
&\quad = \Gamma(\Phi_{Ax_{n+m_0},Ax_{n+m_0+1}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)), \\
&\qquad\quad \Phi_{\mathcal{L}(x_{n+m_0},y_{n+m_0}),\mathcal{L}(x_{n+m_0+k},y_{n+m_0+k})}(\psi^{n_0}(\xi))) \\
&\quad \geq \Gamma(\Phi_{Ax_{n+m_0},Ax_{n+m_0+1}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)), \\
&\qquad\quad \min\{\Phi_{Ax_{n+m_0},Ax_{n+m_0+k}}(\psi^{n_0+1}(\xi)), \Phi_{Ay_{n+m_0},Ay_{n+m_0+k}}(\psi^{n_0+1}(\xi))\}) \\
&\quad \geq \Gamma(\Phi_{Ax_{n+m_0},Ax_{n+m_0+1}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)), \Gamma^k(D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)))) \\
&\quad \geq \Gamma(\min\{\Phi_{Ax_{n+m_0},Ax_{n+m_0+1}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)), \\
&\qquad\quad \Phi_{Ay_{n+m_0},Ay_{n+m_0+1}}(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi))\}, \Gamma^k(D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)))) \\
&\quad = \Gamma(D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)), \Gamma^k(D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)))) \\
&\quad = \Gamma^{k+1}(D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi))).
\end{aligned}$$

Similarly we obtain

$$\Phi_{Ay_{n+m_0},Ay_{n+m_0+k+1}}(\psi^{n_0+1}(\xi)) \geq \Gamma^{k+1}(D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi))).$$

Thus by mathematical induction, we find that (3.8) and (3.9) hold for all $k \in \mathbb{Z}^+ \cup \{0\}$. Assume that $\xi > 0$ and $\varepsilon > 0$ is given. By hypothesis, $\{\Gamma^n : n \in \mathbb{N}\}$ is equi-continuous at 1 and $\Gamma^n(1) = 1$, so there is a $\delta > 0$ such that, for any $\mu \in (1 - \delta, 1]$,

$$\Gamma^n(\mu) > 1 - \varepsilon \tag{3.10}$$

for all $n \in \mathbb{N}$. By (3.5), we have $\lim_{n\to+\infty} D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)) = 1$. Then there is a $N_0 \in \mathbb{N}$ such that $D_n(\psi^{n_0+1}(\xi) - \psi^{n_0}(\xi)) \in (1 - \delta, 1]$ for all $n > N_0$. Hence, by (3.8), (3.9) and (3.10) we have

$$\Phi_{Ax_{n+m_0},Ax_{n+m_0+k}}(\psi^{n_0+1}(\xi)) > 1 - \varepsilon$$

and

$$\Phi_{Ay_{n+m_0},Ay_{n+m_0+k}}(\psi^{n_0+1}(\xi)) > 1 - \varepsilon.$$

Thus, for any $k \in \mathbb{N} \cup \{0\}$ and all $n > N_0$ we have

$$\min\{\Phi_{Ax_{n+m_0},Ax_{n+m_0+k}}(\psi^{n_0+1}(\xi)), \Phi_{Ay_{n+m_0},Ay_{n+m_0+k}}(\psi^{n_0+1}(\xi))\} > 1 - \varepsilon.$$

Noting (3.2), (3.3) and (3.4), we get

$$\begin{aligned}
&\Phi_{Ax_{n+m_0+n_0+1},Ax_{n+m_0+n_0+1+k}}(\xi) \\
&\quad \geq \min\{\Phi_{Ax_{n+m_0},Ax_{n+m_0+k}}(\psi^{n_0+1}(\xi)), \Phi_{Ay_{n+m_0},Ay_{n+m_0+k}}(\psi^{n_0+1}(\xi)) \\
&\quad > 1 - \varepsilon
\end{aligned}$$

and

$$\begin{aligned}&\Phi_{Ay_{n+m_0+n_0+1},Ay_{n+m_0+n_0+1+k}}(\xi)\\&\geq \min\{\Phi_{Ax_{n+m_0},Ax_{n+m_0+k}}(\psi^{n_0+1}(\xi)),\Phi_{Ay_{n+m_0},Ay_{n+m_0+k}}(\psi^{n_0+1}(\xi))\\&>1-\varepsilon.\end{aligned}$$

These imply that for all $k\in\mathbb{N}$,

$$\Phi_{Ax_m,Ax_{m+k}}(\xi)>1-\varepsilon$$

and

$$\Phi_{Ay_m,Ay_{m+k}}(\xi)>1-\varepsilon,$$

where $m>N_0+n_0+m_0+1$. Thus $\{Ax_n\}$ and $\{Ay_n\}$, that is, $\{\mathcal{L}(x_n,y_n)\}$ and $\{\mathcal{L}(y_n,x_n)\}$ are Cauchy sequences. Since $\mathcal{L}(X\times X)$ is complete and $\mathcal{L}(X\times X)\subseteq A(X)$, there exists $(\widehat{x},\widehat{y})\in X\times X$ such that

$$\lim_{n\to+\infty}\mathcal{L}(x_n,y_n)=A\widehat{x}$$

and

$$\lim_{n\to+\infty}\mathcal{L}(y_n,x_n)=A\widehat{y}.$$

Next we shall prove that $A\widehat{x}=\mathcal{L}(\widehat{x},\widehat{y})$ and $A\widehat{y}=\mathcal{L}(\widehat{y},\widehat{x})$. Using condition (3.1), we obtain that

$$\Phi_{\mathcal{L}(\widehat{x},\widehat{y}),\mathcal{L}(x_n,y_n)}(\xi)\geq\min\{\Phi_{A\widehat{x},Ax_n}(\psi(\xi)),\Phi_{A\widehat{y},Ay_n}(\psi(\xi))\}\tag{3.11}$$

for any $\xi>0$.

Taking the limit as $n\to+\infty$ in (3.11), since $\lim\limits_{n\to+\infty}Ax_n=A\widehat{x}$ and $\lim\limits_{n\to+\infty}Ay_n=A\widehat{y}$, we get

$$\lim_{n\to+\infty}\mathcal{L}(x_n,y_n)=\mathcal{L}(\widehat{x},\widehat{y}),$$

from which it follows that

$$\mathcal{L}(\widehat{x},\widehat{y})=A\widehat{x}.$$

Similarly, we have

$$\mathcal{L}(\widehat{y},\widehat{x})=A\widehat{y}.$$

Let $u=A\widehat{x}$ and $v=A\widehat{y}$. Since A and $\mathcal{L}$ are w-compatible, we obtain

$$Au=A(A\widehat{x})=A(\mathcal{L}(\widehat{x},\widehat{y}))=\mathcal{L}(A\widehat{x},A\widehat{y})=\mathcal{L}(u,v)\tag{3.12}$$

and

$$Av=A(A\widehat{y})=A(\mathcal{L}(\widehat{y},\widehat{x}))=\mathcal{L}(A\widehat{y},A\widehat{x})=\mathcal{L}(v,u).\tag{3.13}$$

These imply that the mappings A and $\mathcal{L}$ have a coupled coincidence point (u,v).
The next step is to show that $Au=A\widehat{x}$ and $Av=A\widehat{y}$. Using condition (3.1), we have

$$\begin{aligned}\Phi_{Au,Ax_n}(\xi)&=\Phi_{\mathcal{L}(u,v),\mathcal{L}(x_{n-1},y_{n-1})}(\xi)\\&\geq\min\{\Phi_{Au,Ax_{n-1}}(\psi(\xi)),\Phi_{Av,Ay_{n-1}}(\psi(\xi))\}\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}\Phi_{Av,Ay_n}(\xi)&=\Phi_{\mathcal{L}(v,u),\mathcal{L}(y_{n-1},x_{n-1})}(\xi)\\&\geq\min\{\Phi_{Av,Ay_{n-1}}(\psi(\xi)),\Phi_{Au,Ax_{n-1}}(\psi(\xi))\}.\end{aligned}\tag{3.15}$$

Let us define $E_n(\xi) := \min\{\Phi_{Au,Ax_n}(\xi), \Phi_{Av,Ay_n}(\xi)\}$. Then, from (3.14) and (3.15) we obtain that $E_n(\xi) \geq E_{n-1}(\psi(\xi))$. This implies that

$$E_n(\xi) \geq E_{n-1}(\psi(\xi)) \geq \cdots \geq E_0(\psi^n(\xi)).$$

Since $\lim_{n\to+\infty} \psi^n(\xi) = +\infty$, we have

$$E_0(\psi^n(\xi)) = \min\{\Phi_{Av,Ax_0}(\psi^n(\xi)), \Phi_{Au,Ay_0}(\psi^n(\xi))\} \to 1$$

as $n \to +\infty$. This shows that $E_n(\xi) \to 1$ as $n \to +\infty$, and so we have $Au = A\widehat{x}$ and $Av = A\widehat{y}$. Therefore, we obtain that $Au = u$ and $Av = v$. From (3.12) and (3.13) we have $u = Au = \mathcal{L}(u, v)$ and $v = Av = \mathcal{L}(v, u)$.

Now, we show that $u = v$. In fact, using condition (3.1) we get, for any $\xi > 0$,

$$\Phi_{u,v}(\xi) = \Phi_{\mathcal{L}(u,v),\mathcal{L}(v,u)}(\xi) \geq \min\{\Phi_{Au,Av}(\psi(\xi)), \Phi_{Av,Au}(\psi(\xi))\} = \Phi_{u,v}(\psi(\xi)).$$

By induction we obtain that $\Phi_{u,v}(\xi) \geq \Phi_{u,v}(\psi^n(\xi))$. Passing to the limit when $n \to +\infty$ and since $\psi^n(\xi) \to +\infty$ as $n \to +\infty$, we have $\Phi_{u,v}(\xi) = 1$ for any $\xi > 0$, that is, $u = v$. Hence, u is a common fixed point of A and $\mathcal{L}$.

Finally, we show that u is a unique common fixed point of A and $\mathcal{L}$. Let $\widehat{u} \in X$ be another common fixed point of A and $\mathcal{L}$. Using condition (3.1) we have

$$\Phi_{u,\widehat{u}}(\xi) = \Phi_{\mathcal{L}(u,u),\mathcal{L}(\widehat{u},\widehat{u})}(\xi) \geq \min\{\Phi_{Au,A\widehat{u}}(\psi(\xi)), \Phi_{Au,A\widehat{u}}(\psi(\xi))\} = \Phi_{u,\widehat{u}}(\psi(\xi)),$$

and then we have

$$\Phi_{u,\widehat{u}}(\xi) \geq \Phi_{u,\widehat{u}}(\psi^n(\xi)).$$

Applying Lemma 2.15 it follows that $u = \widehat{u}$, i.e. the mappings A and $\mathcal{L}$ have a unique common fixed point. The proof of Theorem 3.1 is complete. □

With $A = I$ (I is the identity mapping) in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let $(X, \mathcal{F}, \Gamma)$ be a Menger PM-space such that Γ is a Hadžić type t-norm. Suppose $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a G-function satisfying $\psi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to+\infty} \psi^n(\xi) = +\infty$ for any $\xi > 0$. If $\mathcal{L} : X \times X \to X$ is a mapping with*

$$\Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\xi) \geq \min\{\Phi_{x,p}(\psi(\xi)), \Phi_{y,q}(\psi(\xi))\},$$

$\forall x, y, p, q \in X$, $\xi > 0$, and if $\mathcal{L}(X \times X)$ is complete, then there is a unique $u \in X$ such that $u = \mathcal{L}(u, u)$.

Theorem 3.3. *Assume that*

(i) $(X, \mathcal{F}, \Gamma)$ is a Menger PM-space, where Γ is a Hadžić type t-norm;

(ii) ψ: $\mathbb{R}^+ \to \mathbb{R}^+$ is a G-function satisfying $\psi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to+\infty} \psi^n(\xi) = 0$ for any $\xi > 0$;

(iii) The mappings $\mathcal{L}$: $X \times X \to X$ and $A : X \to X$ satisfy the property:

$$\Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\psi(\xi)) \geq \min\{\Phi_{Ax,Ap}(\xi), \Phi_{Ay,Aq}(\xi)\}, \tag{3.16}$$

$\forall x, y, p, q \in X$, where $\mathcal{L}(X \times X) \subseteq A(X)$;

(iv) $\mathcal{L}(X \times X)$ is complete;

(v) A and $\mathcal{L}$ are ω-compatible.

Then there is a unique $u \in X$ such that $u = Au = \mathcal{L}(u, u)$.

Proof. As in the proof of Theorem 3.1, we can construct two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in X such that $Ax_{n+1} = \mathcal{L}(x_n, y_n)$ and $Ay_{n+1} = \mathcal{L}(y_n, x_n)$. Using condition (3.16), we get

$$\begin{aligned} \Phi_{Ax_n,Ax_{n+1}}(\psi(\xi)) &= \Phi_{\mathcal{L}(x_{n-1},y_{n-1}),\mathcal{L}(x_n,y_n)}(\psi(\xi)) \\ &\geq \min\{\Phi_{Ax_{n-1},Ax_n}(\xi), \Phi_{Ay_{n-1},Ay_n}(\xi)\} \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \Phi_{Ay_n,Ay_{n+1}}(\psi(\xi)) &= \Phi_{\mathcal{L}(y_{n-1},x_{n-1}),\mathcal{L}(y_n,x_n)}(\psi(\xi)) \\ &\geq \min\{\Phi_{Ay_{n-1},Ay_n}(\xi), \Phi_{Ax_{n-1},Ax_n}(\xi)\} \end{aligned} \tag{3.18}$$

for any $\xi > 0$.

To simplify let $P_n(\xi) = \min\{\Phi_{Ax_n,Ax_{n+1}}(\xi), \Phi_{Ay_n,Ay_{n+1}}(\xi)\}$. It follows from (3.17) and (3.18) that $P_{n+1}(\psi(\xi)) \geq P_n(\xi)$. Thus

$$P_n(\psi^n(\xi)) \geq P_{n-1}(\psi^{n-1}(\xi)) \geq \cdots \geq P_0(\xi)$$

for $n \geq 1$. Since $P_0(\xi) = \min\{\Phi_{Ax_0,Ax_1}(\xi), \Phi_{Ay_0,Ay_1}(\xi)\} \in \mathcal{D}^+$ and $\lim_{n\to+\infty} \psi^n(\xi) = 0$ for any $\xi > 0$, by Lemma 2.15 we get

$$\lim_{n\to+\infty} P_n(\xi) = 1.$$

Noting that $\Phi_{Ax_n,Ax_{n+1}}(\xi) \geq P_n(\xi)$, we get for any $\xi > 0$ that

$$\lim_{n\to+\infty} \Phi_{Ax_n,Ax_{n+1}}(\xi) = 1. \tag{3.19}$$

Similarly, we have

$$\lim_{n\to+\infty} \Phi_{Ay_n,Ay_{n+1}}(\xi) = 1. \tag{3.20}$$

Since $\lim_{n\to+\infty} \psi^n(\xi) = 0$, for any fixed $\xi > 0$, there is a $n_0 = n_0(\xi) \in \mathbb{N}$ such that $\psi^{n_0+1}(\xi) < \psi^{n_0}(\xi) < \xi$. Next, by mathematical induction we prove that for any $k \in \mathbb{N} \cup \{0\}$,

$$\Phi_{Ax_n,Ax_{n+k}}(\psi^{n_0}(\xi)) \geq \Gamma^k(P_n(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi))) \tag{3.21}$$

and

$$\Phi_{Ay_n,Ay_{n+k}}(\psi^{n_0}(\xi)) \geq \Gamma^k(P_n(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi))). \tag{3.22}$$

It is obvious that (3.21) and (3.22) hold for $k = 0$ since

$$\Phi_{Ax_n,Ax_{n+k}}(\psi^{n_0}(\xi)) = \Phi_{Ay_n,Ay_{n+k}}(\psi^{n_0}(\xi)) = 1.$$

Now assume that inequalities (3.21) and (3.22) are valid for some fixed $k \in \mathbb{N} \cup \{0\}$. Noting that $\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi) > 0$, by (PM-3) and the monotonicity of Γ we get

$$\begin{aligned} &\Phi_{Ax_n,Ax_{n+k+1}}(\psi^{n_0}(\xi)) \\ &= \Phi_{Ax_n,Ax_{n+k+1}}(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi) + \psi^{n_0+1}(\xi)) \\ &\geq \Gamma(\Phi_{Ax_n,Ax_{n+1}}(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi)), \Phi_{Ax_{n+1},Ax_{n+k+1}}(\psi^{n_0+1}(\xi))) \\ &\geq \Gamma(\min\{\Phi_{Ax_n,Ax_{n+1}}(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi)), \\ &\qquad \Phi_{Ay_n,Ay_{n+1}}(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi))\}, \Phi_{Ax_{n+1},Ax_{n+k+1}}(\psi^{n_0+1}(\xi))) \\ &= \Gamma(P_n(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi)), \Phi_{Ax_{n+1},Ax_{n+k+1}}(\psi^{n_0+1}(\xi))). \end{aligned} \tag{3.23}$$

Using inequalities (3.16), (3.21) and (3.22), we get

$$\begin{aligned}\Phi_{Ax_{n+1},Ax_{n+k+1}}(\psi^{n_0+1}(\xi)) &= \Phi_{\mathcal{L}(x_n,y_n),\mathcal{L}(x_{n+k},y_{n+k})}(\psi^{n_0+1}(\xi)) \\ &\geq \min\{\Phi_{Ax_n,Ax_{n+k}}(\psi^{n_0}(\xi)), \Phi_{Ay_n,Ay_{n+k}}(\psi^{n_0}(\xi))\} \\ &\geq \Gamma^k(P_n(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi))). \end{aligned} \tag{3.24}$$

Therefore, combining (3.23) and (3.24), and using the monotonicity of Γ, we obtain that

$$\begin{aligned}\Phi_{Ax_n,Ax_{n+k+1}}(\psi^{n_0}(\xi)) &\geq \Gamma(P_n(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi)), \Gamma^k(P_n(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi)))) \\ &= \Gamma^{k+1}(P_n(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi))).\end{aligned}$$

Similarly, we have $\Phi_{Ay_n,Ay_{n+k+1}}(\psi^{n_0}(\xi)) \geq \Gamma^{k+1}(P_n(\psi^{n_0}(\xi) - \psi^{n_0+1}(\xi)))$.

Thus if (3.21) and (3.22) hold for some fixed $k \in \mathbb{N} \cup \{0\}$, then (3.21) and (3.22) hold for $k+1$. Then by mathematical induction we conclude that (3.21) and (3.22) hold for all $k \in \mathbb{N} \cup \{0\}$.

Now we prove that $\{Ax_n\}$ and $\{Ay_n\}$, that is, $\{\mathcal{L}(x_n, y_n)\}$ and $\{\mathcal{L}(y_n, x_n)\}$ are Cauchy sequences. Suppose that $\xi > 0$ and $\varepsilon > 0$ is given. Since $\lim_{n\to+\infty} \psi^n(\xi) = 0$, there is a $n_1 = n_1(\xi) \in \mathbb{N}$ such that $\psi^{n_1+1}(\xi) < \psi^{n_1}(\xi) < \xi$.
By hypothesis, $\{\Gamma^n : n \in \mathbb{N}\}$ is equi-continuous at 1 and $\Gamma(1) = 1$, so there exists a $\delta > 0$ such that, for any $\mu \in (1-\delta, 1]$,

$$\Gamma^n(\mu) > 1 - \varepsilon \tag{3.25}$$

for all $n \in \mathbb{N}$. It follows from (3.19) and (3.20) that

$$\lim_{n\to+\infty} \Phi_{Ax_n,Ax_{n+1}}(\psi^{n_1}(\xi) - \psi^{n_1+1}(\xi)) = \lim_{n\to+\infty} \Phi_{Ay_n,Ay_{n+1}}(\psi^{n_1}(\xi) - \psi^{n_1+1}(\xi)) = 1.$$

Then, there is a $N \in \mathbb{N}$ such that

$$\Phi_{Ax_n,Ax_{n+1}}(\psi^{n_1}(\xi) - \psi^{n_1+1}(\xi)) > 1 - \delta$$

and

$$\Phi_{Ay_n,Ay_{n+1}}(\psi^{n_1}(\xi) - \psi^{n_1+1}(\xi)) > 1 - \delta$$

for all $n > N$.
Hence, from (3.21) and (3.22) (replacing n_0 with n_1) and (3.25), we have

$$\Phi_{Ax_n,Ax_{n+k}}(\psi^{n_1}(\xi)) > 1 - \varepsilon$$

and

$$\Phi_{Ay_n,Ay_{n+k}}(\psi^{n_1}(\xi)) > 1 - \varepsilon$$

for any $k \in \mathbb{N} \cup \{0\}$.
Noting that $\xi > \psi^{n_1}(\xi)$, and using the monotonicity of Φ, we have for any $k \in \mathbb{N}\cup\{0\}$,

$$\Phi_{Ax_n,Ax_{n+k}}(\xi) \geq \Phi_{Ax_n,Ax_{n+k}}(\psi^{n_1}(\xi)) > 1 - \varepsilon$$

and

$$\Phi_{Ay_n,Ay_{n+k}}(\xi) \geq \Phi_{Ay_n,Ay_{n+k}}(\psi^{n_1}(\xi)) > 1 - \varepsilon.$$

Thus $\{Ax_n\}$ and $\{Ay_n\}$, that is, $\{\mathcal{L}(x_n, y_n)\}$ and $\{\mathcal{L}(y_n, x_n)\}$ are Cauchy sequences. Since $\mathcal{L}(X \times X)$ is complete and $\mathcal{L}(X \times X) \subseteq A(X)$, there are $\widehat{x}, \widehat{y} \in X$ such that

$$\lim_{n\to+\infty} \mathcal{L}(x_n, y_n) = A\widehat{x}$$

and

$$\lim_{n\to+\infty}\mathcal{L}(y_n,x_n)=A\widehat{y}.$$

Next we show that $A\widehat{x}=\mathcal{L}(\widehat{x},\widehat{y})$ and $A\widehat{y}=\mathcal{L}(\widehat{y},\widehat{x})$.

Suppose that $\xi>0$. Since $\lim\limits_{n\to+\infty}\psi^n(\xi)=0$, there exists a $n_1=n_1(\xi)\in\mathbb{N}$ such that $\psi^{n_1+1}(\xi)<\psi^{n_1}(\xi)<\xi$. Using condition (3.16) and (PM-3), we have

$$\begin{aligned}&\Phi_{\mathcal{L}(\widehat{x},\widehat{y}),A\widehat{x}}(\xi)\geq\Phi_{\mathcal{L}(\widehat{x},\widehat{y}),A\widehat{x}}(\psi^{n_1}(\xi))\\&\geq\Gamma(\Phi_{\mathcal{L}(\widehat{x},\widehat{y}),\mathcal{L}(x_{n+n_1},y_{n+n_1})}(\psi^{n_1+1}(\xi)),\Phi_{\mathcal{L}(x_{n+n_1},y_{n+n_1}),A\widehat{x}}(\psi^{n_1}(\xi)-\psi^{n_1+1}(\xi)))\\&\geq\Gamma(\min\{\Phi_{A\widehat{x},Ax_{n+n_1}}(\psi^{n_1}(\xi)),\Phi_{A\widehat{y},Ay_{n+n_1}}(\psi^{n_1}(\xi))\},\\&\qquad\Phi_{\mathcal{L}(x_{n+n_1},y_{n+n_1}),A\widehat{x}}(\psi^{n_1}(\xi)-\psi^{n_1+1}(\xi))).\end{aligned}\tag{3.26}$$

Since $\lim\limits_{n\to+\infty}Ax_n=A\widehat{x}$, $\lim\limits_{n\to+\infty}Ay_n=A\widehat{y}$ and $\lim\limits_{n\to+\infty}\mathcal{L}(x_{n+n_1},y_{n+n_1})=A\widehat{x}$, taking the limit as $n\to+\infty$ in (3.26), we obtain

$$\Phi_{\mathcal{L}(\widehat{x},\widehat{y}),A\widehat{x}}(\xi)\geq\Gamma(1,1)=1.$$

Hence $\mathcal{L}(\widehat{x},\widehat{y})=A\widehat{x}$. Similarly, we have $\mathcal{L}(\widehat{y},\widehat{x})=A\widehat{y}$.

Now we show that if $(x^*,y^*)\in X\times X$ is another coupled coincidence point of A and $\mathcal{L}$, then $A\widehat{x}=Ax^*$ and $A\widehat{y}=Ay^*$.

Since $\lim\limits_{n\to+\infty}\psi^n(\xi)=0$, there is a $n_2=n_2(\xi)\in\mathbb{N}$ such that $\psi^{n_2}(\psi(\xi))<\psi(\xi)$. Using condition (3.16) we have

$$\begin{aligned}\Phi_{A\widehat{x},Ax^*}(\psi^{n_2+1}(\xi))&=\Phi_{\mathcal{L}(\widehat{x},\widehat{y}),\Phi(x^*,y^*)}(\psi^{n_2+1}(\xi))\\&\geq\min\{\Phi_{A\widehat{x},Ax^*}(\psi^{n_2}(\xi)),\Phi_{A\widehat{y},Ay^*}(\psi^{n_2}(\xi))\}\end{aligned}\tag{3.27}$$

and

$$\begin{aligned}\Phi_{A\widehat{y},Ay^*}(\psi^{n_2+1}(\xi))&=\Phi_{\mathcal{L}(\widehat{y},\widehat{x}),\mathcal{L}(y^*,x^*)}(\psi^{n_2+1}(\xi))\\&\geq\min\{\Phi_{A\widehat{y},Ay^*}(\psi^{n_2}(\xi)),\Phi_{A\widehat{x},Ax^*}(\psi^{n_2}(\xi))\}.\end{aligned}\tag{3.28}$$

It follows from (3.27) and (3.28) that

$$\begin{aligned}&\min\{\Phi_{A\widehat{x},Ax^*}(\psi(\psi^{n_2}(\xi))),\Phi_{A\widehat{y},Ay^*}(\psi(\psi^{n_2}(\xi)))\}\\&\quad=\min\{\Phi_{A\widehat{x},Ax^*}(\psi^{n_2+1}(\xi)),\Phi_{A\widehat{y},Ay^*}(\psi^{n_2+1}(\xi))\}\\&\quad\geq\min\{\Phi_{A\widehat{x},Ax^*}(\psi^{n_2}(\xi)),\Phi_{A\widehat{y},Ay^*}(\psi^{n_2}(\xi))\}.\end{aligned}$$

By induction we get

$$\begin{aligned}&\min\{\Phi_{A\widehat{x},Ax^*}(\psi^n(\psi^{n_2}(\xi))),\Phi_{A\widehat{y},Ay^*}(\psi^n(\psi^{n_2}(\xi)))\}\\&\quad\geq\min\{\Phi_{A\widehat{x},Ax^*}(\psi^{n_2}(\xi)),\Phi_{A\widehat{y},Ay^*}(\psi^{n_2}(\xi))\}.\end{aligned}\tag{3.29}$$

Applying Lemma 2.15 and from (3.29) we have $A\widehat{x}=Ax^*$ and $A\widehat{y}=Ay^*$. These show that A and $\mathcal{L}$ have a unique coupled coincidence point.

Now we prove that $A\widehat{x}=A\widehat{y}$. In fact, using condition (3.16) we get

$$\begin{aligned}\Phi_{A\widehat{x},Ay_n}(\psi(\xi))&=\Phi_{\mathcal{L}(\widehat{x},\widehat{y}),\mathcal{L}(y_{n-1},x_{n-1})}(\psi(\xi))\\&\geq\min\{\Phi_{A\widehat{x},Ay_{n-1}}(\xi),\Phi_{A\widehat{y},Ax_{n-1}}(\xi)\}\end{aligned}\tag{3.30}$$

and

$$\begin{aligned}\Phi_{A\widehat{y},Ax_n}(\psi(\xi)) &= \Phi_{\mathcal{L}(\widehat{y},\widehat{x}),\mathcal{L}(x_{n-1},y_{n-1})}(\psi(\xi)) \\ &\geq \min\{\Phi_{A\widehat{y},Ax_{n-1}}(\xi), \Phi_{A\widehat{x},Ay_{n-1}}(\xi)\}.\end{aligned} \tag{3.31}$$

Let us define $Q_n(\xi) := \min\{\Phi_{A\widehat{y},Ax_n}(\xi), \Phi_{A\widehat{x},Ay_n}(\xi)\}$. From inequalities (3.30) and (3.31), we find that

$$Q_n(\psi^n(\xi)) \geq Q_{n-1}(\psi^{n-1}(\xi)) \geq \cdots \geq Q_0(\xi).$$

From Lemma 2.15 we have $\lim_{n\to+\infty} Q_n(\xi) = 1$, which implies that

$$\lim_{n\to+\infty} \Phi_{A\widehat{y},Ax_n}(\xi) = \lim_{n\to+\infty} \Phi_{A\widehat{x},Ay_n}(\xi) = 1.$$

Since $\{Ax_n\}$ converges to $A\widehat{x}$ and $\{Ay_n\}$ converges to $A\widehat{y}$, we see that $A\widehat{y} = A\widehat{x}$.

Suppose now that $u = A\widehat{x}$. Then we get $u = A\widehat{y}$ (because $A\widehat{x} = A\widehat{y}$). In view of condition (v), we obtain

$$Au = A(A\widehat{x}) = A(\mathcal{L}(\widehat{x},\widehat{y})) = \mathcal{L}(A\widehat{x}, A\widehat{y}) = \mathcal{L}(u,u),$$

and then we obtain that the mappings A and $\mathcal{L}$ have a coupled coincidence point (u,u). Since A and $\mathcal{L}$ have a unique coupled coincidence point, we obtain that $Au = A\widehat{x}$, that is, $Au = u$. Hence, we get $u = Au = \mathcal{L}(u,u)$. The uniqueness of the common fixed point of A and $\mathcal{L}$ is similar to that in the proof of Theorem 3.1, and then the proof is complete. □

With $A = I$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4. *Let $(X,\mathcal{F},\Gamma)$ be a Menger PM-space such that Γ is a Hadžić type t-norm. Suppose $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a G-function satisfying $\psi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to+\infty} \psi^n(\xi) = 0$ for any $\xi > 0$. If the mapping $\mathcal{L}\colon X \times X \to X$ satisfies the property:*

$$\Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\psi(\xi)) \geq \min\{\Phi_{x,p}(\xi), \Phi_{y,q}(\xi)\},$$

$\forall x,y,p,q \in X$, $\xi > 0$, and if $\mathcal{L}(X \times X)$ is complete. Then there is a unique $u \in X$ such that $u = \mathcal{L}(u,u)$.

By a similar argument to the above we can prove the following results.

Theorem 3.5. *Assume that*

(i) $(X,\mathcal{F},\Gamma)$ is a Menger PM-space, where Γ is a Hadžić type t-norm;

(ii) $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a G-function satisfying $\psi^{-1}(\{0\}) = \{0\}$ and

$$\lim_{n\to+\infty} \psi^n(\xi) = +\infty$$

for any $\xi > 0$;

(iii) The mappings $\mathcal{L} : X \to X$ and $g : X \to X$ satisfy the property:

$$\Phi_{\mathcal{L}x,\mathcal{L}y}(\xi) \geq \Phi_{Ax,Ay}(\psi(\xi)),$$

$\forall x,y \in X$, where $\mathcal{L}(X) \subseteq A(X)$;

(iv) $\mathcal{L}(X)$ is complete;

(v) A and $\mathcal{L}$ are ω-compatible.

Then there is a unique $u \in X$ such that $Au = \mathcal{L}u = u$.

Theorem 3.6. *Assume that*

(i) $(X, \mathcal{F}, \Gamma)$ *is a Menger PM-space, where* Γ *is a Hadžić type* t*-norm;*

(ii) ψ*:* $\mathbb{R}^+ \to \mathbb{R}^+$ *is a G-function satisfying* $\psi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to+\infty} \psi^n(\xi) = 0$ *for any* $\xi > 0$*;*

(iii) The mappings $\mathcal{L} : X \to X$ *and* $g : X \to X$ *satisfy the property:*

$$\Phi_{\mathcal{L}x,\mathcal{L}y}(\psi(\xi)) \geq \Phi_{Ax,Ay}(\xi),$$

$\forall x, y \in X$*, where* $\mathcal{L}(X) \subseteq A(X)$*;*

(iv) $\mathcal{L}(X)$ *is complete;*

(v) A *and* $\mathcal{L}$ *are* ω*-compatible.*

Then there is a unique $u \in X$ *such that* $Au = \mathcal{L}u = u$*.*

In Theorem 3.6 and Theorem 3.5, if we let $A = I$, then the following corollaries can be obtained.

Corollary 3.7. *Let* $(X, \mathcal{F}, \Gamma)$ *be a Menger PM-space such that* Γ *is a Hadžić type* t*-norm. Suppose* ψ*:* $\mathbb{R}^+ \to \mathbb{R}^+$ *is a G-function satisfying* $\psi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to+\infty} \psi^n(\xi) = +\infty$ *for any* $\xi > 0$*. If* $\mathcal{L} : X \to X$ *is a mapping with*

$$\Phi_{\mathcal{L}x,\mathcal{L}y}(\xi) \geq \Phi_{x,y}(\psi(\xi)),$$

$\forall x, y \in X$*, and if* $\mathcal{L}(X)$ *is complete, then there is a unique* $u \in X$ *such that* $u = \mathcal{L}u$*.*

Corollary 3.8. *Let* $(X, \mathcal{F}, \Gamma)$ *be a Menger PM-space such that* Γ *is a Hadžić type* t*-norm. Suppose* ψ*:* $\mathbb{R}^+ \to \mathbb{R}^+$ *is a G-function satisfying* $\psi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to+\infty} \psi^n(\xi) = 0$ *for any* $\xi > 0$*. If* $\mathcal{L} : X \to X$ *is a mapping with*

$$\Phi_{\mathcal{L}x,\mathcal{L}y}(\psi(\xi)) \geq \Phi_{x,y}(\xi),$$

$\forall x, y \in X$*, and if* $\mathcal{L}(X)$ *is complete, then there is a unique* $u \in X$ *such that* $u = \mathcal{L}u$*.*

Using the same methods as in Theorem 3.1 and Theorem 3.3, we can obtain the following common tripled fixed point theorems in generalized Menger probabilistic metric spaces proposed by Luo, Zhu and Wu [17].

Theorem 3.9. *Assume that*

(i) $(X, \mathcal{F}, \Gamma)$ *is a generalized Menger probabilistic metric space, where* Γ *is a Hadžić type* t*-norm;*

(ii) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a G-function satisfying* $\psi^{-1}(\{0\}) = \{0\}$ *and*

$$\lim_{n\to+\infty} \psi^n(\xi) = +\infty$$

for any $\xi > 0$*;*

(iii) The mappings $\mathcal{L} : X \times X \times X \to X$ *and* $A : X \to X$ *satisfy the property:*

$$\Phi_{\mathcal{L}(x,y,z),\mathcal{L}(p,q,r)}(\xi) \geq \min\{\Phi_{Ax,Ap}(\psi(\xi)), \Phi_{Ay,Aq}(\psi(\xi)), \Phi_{Az,Ar}(\psi(\xi))\}$$

$\forall x, y, z, p, q, r \in X$*, where* $\mathcal{L}(X \times X \times X) \subseteq A(X)$*;*

(iv) $\mathcal{L}(X \times X \times X)$ *is complete;*

(v) A *and* $\mathcal{L}$ *are* ω*-compatible.*

Then there is a unique $u \in X$ *such that* $u = Au = \mathcal{L}(u, u, u)$*.*

Theorem 3.10. *Assume that*

(i) $(X, \mathcal{F}, \Gamma)$ *is a generalized Menger probabilistic metric space, where* Γ *is a Hadžić type t-norm;*

(ii) $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is a G-function satisfying* $\psi^{-1}(\{0\}) = \{0\}$ *and*

$$\lim_{n\to+\infty} \psi^n(\xi) = 0$$

for any $\xi > 0$*;*

(iii) The mappings $\mathcal{L}\colon X \times X \times X \to X$ *and* $A : X \to X$ *satisfy the property:*

$$\Phi_{\mathcal{L}(x,y,z),\mathcal{L}(p,q,r)}(\psi(\xi)) \geq \min\{\Phi_{Ax,Ap}(\xi), \Phi_{Ay,Aq}(\xi), \Phi_{Az,Ar}(\xi)\},$$

$\forall x, y, z, p, q, r \in X$*, where* $\mathcal{L}(X \times X \times X) \subseteq A(X)$*;*

(iv) $\mathcal{L}(X \times X \times X)$ *is complete;*

(v) A *and* $\mathcal{L}$ *are* ω*-compatible.*

Then there is a unique $u \in X$ *such that* $u = Au = \mathcal{L}(u, u, u)$*.*

In Theorem 3.9 and Theorem 3.10, if we let $A = I$, then the following results can be obtained.

Corollary 3.11. *Let* $(X, \mathcal{F}, \Gamma)$ *be a generalized Menger probabilistic metric space such that* Γ *is a Hadžić type t-norm. Suppose* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a G-function satisfying* $\psi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to+\infty} \psi^n(\xi) = +\infty$ *for any* $\xi > 0$*. If* $\mathcal{L} : X \times X \times X \to X$ *is a mapping with*

$$\Phi_{\mathcal{L}(x,y,z),\mathcal{L}(p,q,r)}(\xi) \geq \min\{\Phi_{x,p}(\psi(\xi)), \Phi_{y,q}(\psi(\xi)), \Phi_{z,r}(\psi(\xi))\},$$

$\forall x, y, z, p, q, r \in X$*,* $\xi > 0$*, and if* $\mathcal{L}(X \times X \times X)$ *is complete, then there is a unique* $u \in X$ *such that* $u = \mathcal{L}(u, u, u)$*.*

Corollary 3.12. *Let* $(X, \mathcal{F}, \Gamma)$ *be a generalized Menger probabilistic metric space such that* Γ *is a Hadžić type t-norm. Suppose* $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *is a G-function satisfying* $\psi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to+\infty} \psi^n(\xi) = 0$ *for any* $\xi > 0$*. If* $\mathcal{L}\colon X \times X \times X \to X$ *is a mapping with*

$$\Phi_{\mathcal{L}(x,y,z),\mathcal{L}(p,q,r)}(\psi(\xi)) \geq \min\{\Phi_{x,p}(\xi), \Phi_{y,q}(\xi), \Phi_{z,r}(\xi)\}$$

$\forall x, y, z, p, q, r \in X$*,* $\xi > 0$*, and if* $\mathcal{L}(X \times X \times X)$ *is complete, then there is a unique* $u \in X$ *such that* $u = \mathcal{L}(u, u, u)$*.*

Finally, we provide an example to illustrate our theory.

Example 3.13. Let $X = [0, \frac{1}{2}) \cup \{1\}$ and define $\Phi : X \times X \to \mathcal{D}^+$ as follows:

$$\Phi(x, y)(\xi) = \Phi_{x,y}(\xi) = \frac{\xi}{\xi + |x - y|}$$

for all $\xi > 0$. Then $(X, \mathcal{F}, \Gamma_M)$ is a Menger PM-space, but it is not complete. Let $A : X \to X$ be defined by

$$A(x) = \begin{cases} \frac{x}{4} & \text{if } x \in [0, \frac{1}{4}], \\ \frac{x}{2} & \text{if } x \in (\frac{1}{4}, \frac{1}{2}), \\ \frac{1}{4} & \text{if } x = 1, \end{cases}$$

and $\mathcal{L} : X \times X \to X$ be defined by

$$\mathcal{L}(x,y) = \begin{cases} \frac{x}{16} & \text{if } x \in [0, \frac{1}{2}), \\ \frac{1}{64} & \text{if } x = 1. \end{cases}$$

Clearly, $\mathcal{L}(X \times X) \subseteq A(X)$ and $\mathcal{L}(X \times X)$ is complete.
Noting that $A(\mathcal{L}(1,1)) \neq \mathcal{L}(A(1), A(1))$, we immediately obtain that A and $\mathcal{L}$ do not commute.

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\psi(\xi) = \begin{cases} \frac{1}{3} & \text{if } \xi = 1, \\ 2\xi & \text{if } \xi \neq 1. \end{cases}$$

Then $\lim_{n \to +\infty} \psi^n(\xi) = +\infty$ for any $\xi > 0$.

Now, we prove that the mappings A and $\mathcal{L}$ satisfy condition (3.1) of Theorem 3.1. We consider the case $\xi \neq 1$ and $\xi = 1$ separately.

(I). Let $\xi \neq 1$. We consider four cases.

CASE 1. When $x \neq 1$ and $p \neq 1$.

Case 1.1. If $0 \leq x \leq \frac{1}{4}$ and $0 \leq p \leq \frac{1}{4}$, then for any $y, q \in X$, we have

$$\begin{aligned} \Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\xi) &= \frac{\xi}{\xi + |\frac{x}{16} - \frac{p}{16}|} = \frac{4\xi}{4\xi + |\frac{x}{4} - \frac{p}{4}|} \geq \frac{2\xi}{2\xi + |\frac{x}{4} - \frac{p}{4}|} \\ &= \Phi_{A(x),A(p)}(\psi(\xi)) \geq \min\{\Phi_{A(x),A(p)}(\psi(\xi)), \Phi_{A(y),A(q)}(\psi(\xi))\}. \end{aligned}$$

Case 1.2. If $0 \leq x \leq \frac{1}{4}$ and $\frac{1}{4} < p < \frac{1}{2}$, then for any $y, q \in X$ we have

$$\begin{aligned} \Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\xi) &= \frac{\xi}{\xi + |\frac{x}{16} - \frac{p}{16}|} = \frac{4\xi}{4\xi + (\frac{p}{4} - \frac{x}{4})} \geq \frac{4\xi}{4\xi + (\frac{p}{2} - \frac{x}{4})} \\ &\geq \frac{2\xi}{2\xi + |\frac{p}{2} - \frac{x}{4}|} = \Phi_{A(x),A(p)}(\psi(\xi)) \geq \min\{\Phi_{A(x),A(p)}(\psi(\xi)), \Phi_{A(y),A(q)}(\psi(\xi))\}. \end{aligned}$$

Case 1.3. If $\frac{1}{4} < x < \frac{1}{2}$ and $0 \leq p \leq \frac{1}{4}$. This case is similar to Case 1.2.

Case 1.4. If $\frac{1}{4} < x < \frac{1}{2}$ and $\frac{1}{4} < p < \frac{1}{2}$, then for any $y, q \in X$ we have

$$\begin{aligned} \Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\xi) &= \frac{\xi}{\xi + |\frac{x}{16} - \frac{p}{16}|} = \frac{4\xi}{4\xi + |\frac{x}{4} - \frac{p}{4}|} \geq \frac{2\xi}{2\xi + |\frac{x}{4} - \frac{p}{4}|} \\ &\geq \frac{2\xi}{2\xi + |\frac{x}{2} - \frac{p}{2}|} = \Phi_{A(x),A(p)}(\psi(\xi)) \geq \min\{\Phi_{A(x),A(p)}(\psi(\xi)), \Phi_{A(y),A(q)}(\psi(\xi))\}. \end{aligned}$$

CASE 2. If $x = 1$ and $p = 1$, then for any $y, q \in X$ we get

$$\begin{aligned} \Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\xi) &= \Phi_{\mathcal{L}(1,y),\mathcal{L}(1,q)}(\xi) = \frac{\xi}{\xi + |\frac{1}{64} - \frac{1}{64}|} \\ &= 1 \geq \min\{\Phi_{A(1),A(1)}(\psi(\xi)), \Phi_{A(y),A(q)}(\psi(\xi))\}. \end{aligned}$$

CASE 3. When $x = 1$ and $p \neq 1$.

Case 3.1. Suppose that $0 \le p \le \frac{1}{4}$. Then for any $y, q \in X$ we obtain

$$\begin{aligned}
&\Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\xi) = \Phi_{\mathcal{L}(1,y),\mathcal{L}(p,q)}(\xi) = \frac{\xi}{\xi + |\frac{1}{64} - \frac{p}{16}|} = \frac{2\xi}{2\xi + |\frac{1}{32} - \frac{p}{8}|} \\
&\ge \frac{2\xi}{2\xi + |\frac{1}{8} - \frac{p}{2}|} \ge \frac{2\xi}{2\xi + |\frac{1}{4} - \frac{p}{4}|} \ge \min\{\Phi_{A(1),A(p)}(\psi(\xi)), \Phi_{A(y),A(q)}(\psi(\xi))\}.
\end{aligned}$$

Case 3.2. If $\frac{1}{4} < p < \frac{1}{2}$, then for any $y, q \in X$ we have

$$\begin{aligned}
&\Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(\xi) = \Phi_{\mathcal{L}(1,y),\mathcal{L}(p,q)}(\xi) = \frac{\xi}{\xi + |\frac{1}{64} - \frac{p}{16}|} = \frac{2\xi}{2\xi + |\frac{1}{32} - \frac{p}{8}|} \\
&\ge \frac{2\xi}{2\xi + |\frac{1}{4} - \frac{p}{2}|} \ge \min\{\Phi_{A(1),A(p)}(\psi(\xi)), \Phi_{A(y),A(q)}(\psi(\xi))\}.
\end{aligned}$$

CASE 4. When $x \neq 1$ and $p = 1$. This case is similar to Case 3.

(II). Let $\xi = 1$. From Cases 1-4 above, we get

$$\Phi_{\mathcal{L}(x,y),\mathcal{L}(p,q)}(1) \ge \min\{\Phi_{Ax,Ap}(\psi(1)), \Phi_{Ay,Aq}(\psi(1))\}$$

for all $x, y, p, q \in X$.

Moreover, it can be seen that mappings A and $\mathcal{L}$ have a coupled coincidence point. Here $(0,0)$ is a coupled coincidence point of A and $\mathcal{L}$ in X. Also, the mappings A and $\mathcal{L}$ are weakly compatible at $(0,0)$. Thus all the required hypotheses of Theorem 3.1 hold. Therefore, we deduce the existence of a unique common fixed point of A and $\mathcal{L}$. Indeed, a point 0 is the unique common fixed point of A and $\mathcal{L}$.

Note that $A(x)$ is not continuous at $x = \frac{1}{4}$ and $(X, \mathcal{F}, \Gamma)$ is not complete, and therefore the unique common fixed point of A and $\mathcal{L}$ cannot be obtained from Theorem of XZC.

Remark 3.14. (a) Note in Corollary 3.8, the function ψ is only required to satisfy the condition $\psi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to+\infty} \psi^n(\xi) = 0$ for any $\xi > 0$. Note, ψ in Jachymski's result is required to satisfy the condition $\psi^{-1}(\{0\}) = \{0\}$, $\psi(\xi) < \xi$ and $\lim_{n\to+\infty} \psi^n(\xi) = 0$ for any $\xi > 0$.

(b) Note in some of our results, the function ψ is only required to satisfy the condition $\psi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to+\infty} \psi^n(\xi) = +\infty$ for any $\xi > 0$. Note, ψ in the results of Luo et al. [17] and Xiao et al. [27] are required to satisfy the conditions $\psi^{-1}(\{0\}) = \{0\}$, $\psi(\xi) > \xi$ and $\lim_{n\to+\infty} \psi^n(\xi) = +\infty$ for any $\xi > 0$;

(c) Note in our results, the Menger PM-space $(X, \mathcal{F}, \Gamma)$ is not required to be complete.

(d) Note in our results, the operator A is not necessarily continuous, while the operator A in the results of Luo et al. [17] and Xiao et al. [27] is required to be continuous;

(e) Note in our results, the function $\mathcal{L}$ and A are only required to be weakly compatible, but in the results of Xiao et al. [27] and Luo et al. [17] they are required to be commutable.

Acknowledgements. The authors would like to thank the anonymous referees. This work was supported by the Fundamental Research Funds for the Central Universities (No. 2015ZD29) and the Higher School Science Research Funds of Hebei Province of China (No. Z2015137).

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Received: September 12, 2017; Accepted: January 11, 2018.