Fixed Point Theory, 21(2020), No. 1, 309-318
DOI: 10.24193/fpt-ro.2020.1.22
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON SOME RESULTS OF KRASNOSELSKII THEOREM FOR WEAK TOPOLOGY IN BANACH SPACES

CESIM TEMEL

Department of Mathematics, Van Yuzuncu Yil University,
Faculty of Sciences, 65080, Van, Turkey
E-mail: cesimtemel@yyu.edu.tr

Abstract. In this study, we aim to present some new versions of classical Schauder and Banach fixed point theorems under weak topology in general Banach spaces. We have also another main goal which is to prove that Krasnoselskii theorem ensures in general Banach spaces using these new results. As an application, an illustrative example providing such results is given. Our results extend known results on the issue.
Key Words and Phrases: Fixed point theorem, Krasnoselskii fixed point theorem, sequentially weakly continuous operator, weakly compact operator.
2010 Mathematics Subject Classification: 47H10, 47H30.

1. Introduction

Recently many authors have been working on the solution of the neutral functional differential equations, hereditary systems, stationary linear (nonlinear) model and retarded functional differential equations. Especially using the combination of Schauder and Banach fixed theorems, they have been intensely interested in finding the solutions of these linear (nonlinear) differential equations [1, 3-5, 7- 12].

Krasnoselskii [8] combined the Schauder theorem and Banach fixed point theorem into a new result in which Krasnoselskii fixed point theorem requires an operator $Tu = Su + Ku$ on a convex set U such that K is a compact and continuous operator, the operator S is contraction and for each $u, v \in U$, $Su + Kv \in U$, then there exists a fixed point of operator T in U. After then Burton [4] improved this theorem and gave that if for each $v \in U$, $u = Su + Kv$, then $u \in U$. This condition provides significant convenience in the applications of functional differential equation, integral equation and stability theory. Hale et al. [7] tackled the hereditary system that determine present state by its past history. This system contains neutral functional differential equation, retarded functional differential equation and Volterra integral equation. Melvin [11] and Barroso [3] gave some results on the modifications and generalizations of Krasnoselskii fixed point theorem. These extensions create to be useful in the action of continuous dependence for difference equation and integral equation. Rotenberg [12] and Latrach [9] dealt with transport theory for the partial differential equation of the growing cell populations and improved a algorithm for the solution of transport equation. Latrach and Jeribi [10] gave a modified stationary model of Rotenberg to indicate the existence of solutions of this model. Arino et al. [2] introduced some results of sequentially continuous mappings on general Banach spaces.

Rotenberg [12], Latrach [9] and Latrach and Jeribi [10] have been interested in the solution of the transport equation

$$s\frac{\partial u}{\partial t}(t,s)+\sigma(t,s,u(t,s))+\lambda u(t,s)=\int_{-1}^{1}p(t,s,s',u(t,s'))\,ds', \tag{1.1}$$

where $t\in(0,\infty)$, $s\in[-1,1]$, λ is a complex. The scattering kernel p and the collision frequency σ are nonlinear functions of u. Here $u(t,s)$ is a unknown function that means the number density of gas molecules. These gas molecules have the position t and the direction cosine of spread s. This equation defines the transport of particles (the growing cell population, neutrons, photons, gas molecules, etc.). This equation is given with the following boundary conditions

$$u_{|\Omega_1}=Fu_{|\Omega_2} \tag{1.2}$$

where Ω_1 is the incoming of the boundary and Ω_2 is the outgoing of the boundary, $u_{|\Omega_1}$ (resp. $u_{|\Omega_2}$) is the restriction of u to Ω_1 (resp. Ω_2) and the operator F of a function space on Ω_2 into a function space on Ω_1 is linear.

Let (A,Ω,μ) be a measure space and $k\in L^{\infty}(A\times A,\Omega\times\Omega,\mu\times\mu)$. Suppose that an operator $S:L^1(\mu)\to L^1(\mu)$ is given by

$$(Su)(t)=\int k(t,s)\,u(s)\,d\mu(s).$$

Then the operator S of the space $L^1(\mu)$ into itself is weakly compact.

Specifically Krasnoselskii [8], Hale et al. [7], Melvin [11], Barroso [3] and Burton [4] have been concerned with the solution of sum of two linear difference equations in Banach spaces.

It is well known that a linear boundary value problem in the transport equation (or the growing cell population) may not be solved by the combination of the classic Schauder theorem and contraction mapping principle on general Banach spaces. For example, the transport equation (1.1) with boundary (1.2) may not has a solution in the space $L^1(\mu)$ with the sum of a compact operator and a contraction operator on Banach spaces. The thing that has led us to this study is under which conditions such equation has a fixed point on general Banach spaces. To solve such linear equations under the weak topology on general Banach spaces, Krasnoselskii's theorem is required for general Banach spaces with respect to weak topology case. For this, we need to the new version of the classic Schauder theorem and Banach fixed point theorem for weak topology.

In this paper, we aim to give some results of classical Schauder theorem and Banach fixed point theorem for weak topology. Then we also show that a new version of Krasnoselskii theorem ensure in general Banach spaces using these results of Schauder and Banach fixed point theorems. At the same time, we support the study with an application.

2. Preliminaries

Recall that an operator S of Banach space A into itself is said to be *weakly compact* if the closure of $S(U)$ is weakly compact for each bounded subset $U \subset A$. And an operator S of Banach space A into itself is said to be *sequentially weakly continuous* if for every weakly convergent sequence (u_n) in A, defined by $u_n \overset{w}{\to} u$, there exists $S(u_n) \overset{w}{\to} S(u)$.

Let U be a bounded closed convex subset of Banach space A. The closure of $S(U)$ is defined by $\text{cl}S(U)$ and the closed convex hull of $\text{cl}S(U)$ is defined by $\overline{co}(\text{cl}S(U))$.

Let A be a locally convex space. Suppose A^* denote the space of continuous linear functionals on space A. *The weak topology* on the space A, defined with $\sigma(A, A^*)$, is a topology for the class of seminorms $\{\rho_f : f \in A^*\}$ such that $\rho_f(u) = |f(u)|$.

Let $L(A)$ denote the space of linear operators from the Banach space A into itself. If $S \in L(A)$ is weakly continuous operator and $\|S\| < 1$, then $(I - S)^{-1} \in L(A)$. Hence the operator $(I - S)^{-1}$ is weakly continuous on A.

3. Schauder and Banach fixed point theorems for weak topology

In this section we aim to present new versions of the classic Schauder theorem and Banach fixed point theorem in Banach spaces relative to weak topology.

As the some main results of our work we give the following theorems.

Theorem 3.1. *Suppose A is a Banach space and assume the set $U \subset A$ is closed bounded convex. If S from the set U into itself is weakly compact operator and also if the operator S is sequentially weakly continuous, then there is a fixed point of the operator S in U.*

Proof. Since S is a weakly compact operator, $\text{cl}S(U)$ is a weakly compact set for every bounded convex subset U of A. Therefore if $\text{cl}S(U)$ is a weakly compact subset of A, then $\Psi = \overline{co}(\text{cl}S(U))$ is weakly compact by the Krein-Smulian Theorem [5, 13.4]. Because

$$S(\Psi) \subset \text{cl}S(U) \subset \Psi,$$

it turns out that $S(\Psi) \subset \Psi$. Hence we obtain that $S(U) \subset \Psi \subset U$. Therefore U has a weakly compact subset Ψ that contains $S(U)$. Since S is a sequentially weakly continuous operator, S is a weakly continuous operator by [2, Theorem 1]. Because Ψ is a weakly compact set and operator S is weakly continuous, $S(\Psi)$ is relatively weakly compact. Hence there is a fixed point of operator S in U.

Theorem 3.2. *Suppose A is a Banach space and assume the set $U \subset A$ is weakly compact convex. If operator $S : U \to U$ is a sequentially weakly continuous satisfying $\|S\| < 1$, then there is a fixed point of S in U.*

Proof. Note that S is a sequentially weakly continuous and contraction. Hence operator $(I - S)^{-1}$ exists such that this operator of the set U into itself is defined by

$$y \to (I - S)^{-1} u = v. \tag{3.1}$$

By the Eberlein-Smulian Theorem [5, 13.1], operator $(I-S)^{-1}$ is weakly continuous on A. Since U is a weakly compact set which has a convergent subsequence (u_{n_k}) such that $u_{n_k} \overset{w}{\to} u$ in U and the operator $(I-S)^{-1}$ is weakly continuous, then $(I-S)^{-1}(U)$ is relatively weakly compact. Now also put

$$u_{n_k} = Su_{n_k} + v_{n_k} \tag{3.2}$$

for all $(v_{n_k}) \subset U$. If the equality (3.2) is applied in the equality (3.1), then we obtain

$$u_{n_k} = (I-S)^{-1} v_{n_k} \overset{w}{\to} u$$

by the weak compactness of $(I-S)^{-1}(U)$ and U. At the same time, if we use the Krein-Smulian Theorem [5, 13.4] in the weak compact set $(I-S)^{-1}(U)$, then $\overline{co}\left(\mathrm{cl}\,(I-S)^{-1}(U)\right)$ is a weakly compact set. And then we have inclusion

$$\begin{aligned}(I-S)^{-1}\left(\overline{co}\left(\mathrm{cl}\,(I-S)^{-1}(U)\right)\right) &\subset \mathrm{cl}\,(I-S)^{-1}(U) \\ &\subset \overline{co}\left(\mathrm{cl}\,(I-S)^{-1}(U)\right).\end{aligned}$$

Consequently, the operator $(I-S)^{-1}$ has a fixed point in U by Theorem 3.1. Thus the proof is complete.

Theorem 3.3. *Suppose A is a Banach space and assume the set $U \subset A$ is weakly compact convex. If the operator $S : U \to A$ is sequentially weakly continuous satisfying $\|S^m\| < 1$ for some positive integer m, then there is a fixed point of S in U.*

Proof. Suppose that $K = S^m$. Note that S is a sequentially weakly continuous operator and $\|K\| = \|S^m\| < 1$, then $(I-K)^{-1}$ is a sequentially weakly continuous operator. Therefore the operator $(I-K)^{-1}$ is weakly continuous on U by [2, Theorem 1]. Hence Theorem 3.2 implies that K has a fixed point such that $Ku = u$ for some $u \in U$. Thus for $n = 0, 1, 2, \ldots$, put $u_{n+1} = K^{n+1}u_0 = K(K^n u_0) = K(u_n)$ and for each weakly convergent sequence (u_n) in weak compact set U with $u_n \overset{w}{\to} u$, then

$$K^{n+1}u_0 = K(u_n) \overset{w}{\to} Ku = u$$

by the weak continuity of K. Because S is a sequentially weakly continuous operator, the operator S is weakly continuous. And therefore if K has a fixed point and T is weakly continuous, then for each weakly convergent sequence (x_n) in weak compact set U

$$K^{n+1}Sx_0 = S(Kx_n) \overset{w}{\to} S(Kx) = Sx.$$

Hence for a weakly compact convex subset U of A, $S(U)$ is a relatively weakly compact set. Thus S has a fixed point in U by Theorem 3.1.

4. Krasnoselskii fixed point theorem for weak topology

In this section our main goal is to prove Krasnoselskii's theorem for weak topology by using these new results of Schauder theorem and Banach fixed point theorem.

As the other fundamental result of our work we give the following theorems.

Theorem 4.1. *Let U be a weakly compact convex subset of Banach space A. If the operator $S : U \to A$ is a sequentially weakly continuous satisfying $\|S\| < 1$, the operator $K : U \to A$ is sequentially weakly continuous and weakly compact such that*

$$S\left(U\right) + K\left(U\right) := \{z : \ z = Su + Kv, \quad u, v \in U\} \subset U.$$

Then $S + K$ has a fixed point in U.

Proof. Since S is sequentially weakly continuous and contraction, for every $v \in U$ there is an $u = u(v) \in A$ such that $u = Su + Kv$ by Theorem 3.2. Besides

$$\{u : u = Su + Kv, \quad u, v \in U\} \subset U$$

and hence $u \in U$. For every $y \in K\left(U\right)$, $n = 0, 1, 2, ...$ and any $u_0 \in U$, let define the sequence of iterates by $u_{n+1} = Su_n + y$. Because of $S\left(U\right) + K\left(U\right) \subset U$, we obtain $u_n \in U$ for $n = 0, 1, 2, ...$ By the weakly sequential continuity and contraction of operator S, for every $y \in U$, $u_{n+1} = Su_n + y \overset{w}{\to} u = Su + y$. Therefore $u \to Su + y$ defines a contraction operator of set U into itself. Hence $u = Su + y$ has a solution $u \in U$. This shows that (u_n) converges weakly to $u \in U$. Thus u provides $\left(I - S\right) u = y$. Therefore, $K\left(U\right) \subset \left(I - S\right) U$. To find the fixed point of operator $S + K$, it is enough to show that the operator $\left(I - S\right)^{-1} K$ has a fixed point in U. Let operator $\left(I - S\right)^{-1} K$ of space U into itself is defined by

$$y \to \left(I - S\right)^{-1} Ky = u.$$

And then the operator $\left(I - S\right)^{-1} K$ is weakly continuous on U by [2, Theorem 1] and the Eberlein-Smulian Theorem [5, 13.1]. Since the operator K from subset U to itself is weakly compact and also K is sequentially weakly continuous, $K(U)$ is relatively weakly compact by Theorem 3.1. Hence $\left(I - S\right)^{-1} K\left(U\right)$ is also relatively weakly compact. $\Psi = \overline{co}\left(\mathrm{cl}\left(I - S\right)^{-1} K\left(U\right)\right)$ is a weakly compact set by the Krein-Smulian Theorem [5, 13.4]. And then we have inclusion

$$\left(I - S\right)^{-1} K\left(\Psi\right) \subset \mathrm{cl}\left(I - S\right)^{-1} K\left(U\right) \subset \Psi,$$

in this way we get,

$$\mathrm{cl}\left(I - S\right)^{-1} K\left(U\right) \subset \Psi \subset U$$

and U has a weakly compact subset Ψ that contains $\left(I - S\right)^{-1} K\left(U\right)$. Therefore we take

$$\left(I - S\right)^{-1} K\left(\Psi\right) \subset \Psi$$

and hence Theorem 3.1 implies that the operator $\left(I - S\right)^{-1} K$ has a fixed point in $\Psi \subset U$. Thus $S + K$ has a fixed point in U.

If Theorem 3.3 and Theorem 4.1 are combined, the following result is obtained for the weak topology in general Banach spaces.

Corollary 4.2. *Let U be a weakly compact convex subset of Banach space A. If the operator $S : U \to A$ is a weakly sequentially continuous satisfying $\|S^m\| < 1$ for some positive integer m, $K : U \to A$ is a sequentially weakly continuous and weakly compact operator such that*

$$S\left(U\right) + K\left(U\right) := \{z : \ z = Su + Kv, \quad u, v \in U\} \subset U.$$

Then $S + K$ has a fixed point in U.

Example 4.3. We are going to show that the following equation has at least one solution in Banach space for weak topology.

$$s\frac{\partial u}{\partial t}(t,s)+\sigma(t,s,u(t,s))+\lambda u(t,s)=\int_a^b p(t,s,s',u(t,s'))\,ds' \tag{4.1}$$

with boundary $u_{|\Omega_1}=Fu_{|\Omega_2}$, where $t\in(0,\infty)$, $s\in[a,b]$, $0\le a<b<\infty$, $\lambda\in\mathbb{C}$, $\Omega_1=\{0\}\times[a,b]$, $\Omega_2=\{1\}\times[a,b]$ and F from space Ω_2 into space Ω_1 is a bounded linear operator.

Assume for $A=[0,1]\times[a,b]$, $W:=L^1(A,dtds)$. Let

$$W^1:=L^1(\Omega_1,dtds) \text{ and } W^2:=L^1(\Omega_2,dtds)$$

be boundary spaces. Let Ψ be a space defined by

$$\Psi=\left\{u\in W: s\frac{\partial u}{\partial t}\in W\right\}.$$

Define the operator T_F by

$$\begin{cases} T_F: D(T_F)\subseteq W\to W \\ u(t,s)\to T_F u(t,s)=-s\dfrac{\partial u}{\partial t}(t,s) \\ D(T_F)=\left\{u\in\Psi: u_{|\Omega_1}=Fu_{|\Omega_2}\right\}. \end{cases}$$

The map $\sigma: A\times\mathbb{C}\to\mathbb{C}$ is a weakly Carathéodory function if σ is measurable and sequentially weakly continuous. Then operator L_σ can be defined by

$$(L_\sigma u)(t,s)=\sigma(t,s,u(t,s))$$

for each $(t,s)\in A$. Noting that a continuous linear operator P is defined by

$$\begin{cases} P: W\to W \\ u\to Pu(t,s)=\displaystyle\int_a^b k(t,s,s')u(t,s')\,ds' \end{cases}$$

and

$$dt\otimes ds-\operatorname*{ess\text{-}sup}_{(t,s)\in[0,1]\times[a,b]}\int_a^b |k(t,s,s')|\,ds'=\|P\|<\infty,$$

where the mapping $k(t,s,s')$ is measurable of $A\times[a,b]$ into $\mathbb{R}^+$. Then the operator P is called regular if the set $\{k(t,.,s'):(t,s')\in A\}$ is relatively weakly compact in $L^1([a,b],dt)$, Hence it is clear that for all $t\in[0,1]$ and any $u\in L^1([a,b])$,

$$Pu(t)=\int_a^b k(t,s,s')u(s')\,ds',$$

then the operator P of the space $L^1([a,b])$ into itself is weakly compact. Thus P holds in $[a,b]$. Let's think that for $r>0$, $U_r:=\{u\in W:\|u\|_\infty\le r\}$ is a weakly compact convex set.

Under the following conditions the equation (4.1) with boundary $u_{|\Omega_1} = Fu_{|\Omega_2}$ has at least one solution for weak topology:

(i) Set $R = \{\lambda \in \mathbb{C} : Re\lambda > \max\{0, b\log(\|F\|)\}\}$ is included in the resolvent set of T_F. And also given the following inequality

$$\left\|(\lambda - T_F)^{-1}\right\| \leq \frac{1}{Re\lambda}\left(1 + \frac{\|F\|}{1 - \|F\| e^{-\frac{Re\lambda}{b}}}\right)$$

(ii) Suppose that $F \in L\left(W^2, W^1\right)$,

$$|\sigma(t, s, u_1) - \sigma(t, s, u_2)| \leq |H(t, s)| |u_1 - u_2|,$$

where $H \in W$.

(iii) σ is a weak Carathéodory function such that

$$|\sigma(t, s, u(t, s))| \leq K(t, s) g(|u(t, s)|),$$

where $K \in W$ and $g \in L^\infty(\mathbb{R}^+)$.

(iv) Let h be a measurable function and $S : L^1(A) \to L^\infty(A)$ be a continuous linear operator such that

$$p(t, s, s', u(t, s')) = k(t, s, s') h(t, s', Su(t, s')),$$

where $k(t, s, s')$ holds the relatively weak compactness of set$\{k(t, ., s') : (t, s') \in A\}$.

(v) L_h is a operator from U_r into itself and $L_h S : W \to W$ is weakly compact and sequentially weakly continuous.

The following theorem is proved using some results and techniques in study [10].

Theorem 4.4. *Let the conditions (i)-(v) be provide and assume that P is a regular operator on space W. Then there is a λ_2 such that for all λ with $Re\lambda > \lambda_2$, the Eq.* (4.1) *has a solution in U_r.*

Proof. Because of the linearity of operator F, T_F is a linear operator. Since $\lambda \in R$ as in assumption (i), there exists a linear operator $(\lambda - T_F)^{-1}$. Therefore the equation (4.1) can be written in the form

$$u = (\lambda - T_F)^{-1} L_\sigma u + (\lambda - T_F)^{-1} P L_h S u,$$

for $u \in D(T_F)$. For u_1, $u_2 \in W$,

$$\left\|(\lambda - T_F)^{-1} L_\sigma(u_1) - (\lambda - T_F)^{-1} L_\sigma(u_2)\right\| \leq \left\|(\lambda - T_F)^{-1}\right\| \|L_\sigma(u_1) - L_\sigma(u_2)\|.$$

Since as in assumption (i)

$$\left\|(\lambda - T_F)^{-1}\right\| \leq \frac{1}{Re\lambda}\left(1 + \frac{\|F\|}{1 - \|F\| e^{-\frac{Re\lambda}{b}}}\right) \tag{4.2}$$

and by assumption (ii), we have

$$\|L_\sigma(u_1) - L_\sigma(u_2)\| \leq \|H(t, s)\| \|u_1 - u_2\|,$$

then we take

$$\left\| (\lambda - T_F)^{-1} L_\sigma (u_1) - (\lambda - T_F)^{-1} L_\sigma (u_2) \right\|$$
$$\leq \frac{\|H(t,s)\|}{Re\lambda} \left(1 + \frac{\|F\|}{1 - \|F\| e^{-\frac{Re\lambda}{b}}} \right) \|u_1 - u_2\| . \tag{4.3}$$

The right side of this inequality is a strictly decreasing continuous function and identically zero on $(0, \infty)$. Hence there exists a $\lambda_0 \in (\max(0, b\log(\|F\|)), \infty)$ with $Re\lambda > \lambda_0$ such that for λ_0, the right side of inequality (4.3) is less then 1. Then for all λ with $Re\lambda > \lambda_0$, the operator $(\lambda - T_F)^{-1} L_\sigma$ is contraction.

Suppose that for $\psi_1, \psi_2 \in U_r$,

$$\left\| (\lambda - T_F)^{-1} L_\sigma (\psi_1) - (\lambda - T_F)^{-1} PL_hS(\psi_2) \right\|$$
$$\leq \left\| (\lambda - T_F)^{-1} PL_hS(\psi_2) \right\| + \left\| (\lambda - T_F)^{-1} L_\sigma (\psi_1) \right\|$$
$$\leq \left\| (\lambda - T_F)^{-1} \right\| (\|P\| \|L_hS(\psi_2)\| + \|L_\sigma(\psi_1)\|) . \tag{4.4}$$

Here L_σ has a upper bound C on U_r. And also by assumption (iii) we can write

$$\|L_hS(\psi_2)\| \leq K(t,s)\, g(|S(\psi_2)|) . \tag{4.5}$$

Then using inequalities (4.2) and (4.5) the inequality (4.4) can be given in the form

$$\left\| (\lambda - T_F)^{-1} L_\sigma (\psi_1) - (\lambda - T_F)^{-1} PL_hS(\psi_2) \right\|$$
$$\leq \frac{1}{Re\lambda} (\|P\| \|K\| \|g\|_\infty + \|C\|) \left(1 + \frac{\|F\|}{1 - \|F\| e^{-\frac{Re\lambda}{b}}} \right) . \tag{4.6}$$

And also the right side of this inequality is a strictly decreasing continuous function and identically zero. Therefore we obtain a $\lambda_1 \in (\max(0, b\log(\|F\|)), \infty)$ with $Re\lambda > \lambda_1$ such that for λ_1, the right side of inequality (4.6) is less then r. Thus for all λ with $Re\lambda > \lambda_1$ and $\psi_1, \psi_2 \in U_r$, we have

$$(\lambda - T_F)^{-1} L_\sigma (\psi_1) - (\lambda - T_F)^{-1} PL_hS(\psi_2) \in U_r.$$

If the conditions (iii), (iv) and (v) are taken into account, then the operator

$$\left(I - (\lambda - T_F)^{-1} L_\sigma \right)^{-1} (\lambda - T_F)^{-1} PL_hS$$

is sequentially weakly continuous on W. To do end, it is sufficient to show that

$$\left(I - (\lambda - T_F)^{-1} L_\sigma \right)^{-1} (\lambda - T_F)^{-1} PL_hS(U_r)$$

is relatively weakly compact. Because the operator P is regular and L_hS is weakly compact and sequentially weakly continuous by assumption (v), then the operator

$$\left(I - (\lambda - T_F)^{-1} L_\sigma \right)^{-1} (\lambda - T_F)^{-1} PL_hS$$

is weakly continuous by [2, Theorem 1]. So, if the Krein-Smulian Theorem [5, Theorem 13.4] is applied this operator as in Theorem 4.1, then the set

$$\left(I - (\lambda - T_F)^{-1} L_\sigma\right)^{-1} (\lambda - T_F)^{-1} PL_h S\,(U_r)$$

is relatively weakly compact. Thus the Eq. (4.1) has a solution in U_r.

Remark 4.5. For every λ with $Re\lambda > \lambda_2$ ($\lambda_2 = \max(\lambda_0, \lambda_1)$), the operators $(\lambda - T_F)^{-1} L_\sigma$ and $(\lambda - T_F)^{-1} PL_h S$ provide the conditions of Theorem 4.1.

5. Conclusions

It is clear that the transport equation (1.1) with boundary (1.2) has not a solution in the space $L^1(\mu)$ with the sum of compact operators and contraction operators on Banach spaces. In particular, a linear boundary value problem in the transport equation (in the growing cell population or the hereditary systems) may not be solved in the weak topology on Banach spaces. Therefore, Krasnoselskii's fixed point theorem may not provide for the weak topology on general Banach spaces. Because of this reason, we introduced some results of Schauder fixed point theorem and Banach fixed point theorem for weak topology to ensure Krasnoselskii fixed point theorem in general Banach spaces. And also to solve such equations on Banach spaces with respect to weak topology, we gave a suitable type of Krasnoselskii fixed point theorem for weak topology. Finally, an illustrative example has been given to show the existence of solutions of these type equations.

Acknowledgement. The author wishes to thank the referees for their comments and suggestions which helped improve the quality of manuscript. This research has been supported by Van Yuzuncu Yil University, BAP, with project no. KONGRE-2015/141.

References

[1] R.P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2004.

[2] O. Arino, S. Gautier, J.P. Pento, *A fixed point theorem for sequentially continuous mapping with application to ordinary differential equations*, Functional Ekvac., **27**(1984), no. 3, 273-279.

[3] C.S. Barroso, *Krasnoselskii's fixed point theorem for weakly continuous maps*, Nonlinear Anal., **55**(2003), 25-31.

[4] T.A. Burton, *A fixed point theorem of Krasnoselskii*, Appl. Math. Lett., **11**(1998), 85-88.

[5] J.B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1990.

[6] N. Dunford, J.T. Schwartz, *Linear Operators, Part I: General Theory*, Vol. VII, Interscience Publishers Inc., New York, 1958.

[7] J.K. Hale, M.A. Cruz, *Existence, uniqueness and continuous dependence for hereditary systems*, Ann. Math. Pura Appl., **83**(1970), 63-82.

[8] M.A. Krasnoselskii, *Some problems of nonlinear analysis*, Amer. Math. Soc. Trans., **10**(1958), no. 2, 345-409.

[9] K. Latrach, *On a nonlinear stationary problem arising in transport theory*, J. Math. Phys., **37**(1996), 1336-1348.

[10] K. Latrach, A. Jeribi, *A nonlinear boundary value problem arising in growing sell populations*, Nonlinear Anal. T.M.A., **36**(1999), 843-862.

[11] W.R. Melvin, *Some extensions of the Krasnoselskii fixed point theorems*, J. Diff. Eq., **11**(1972), 335-345.

[12] M. Rotenberg, *Transport theory for growing cell populations*, J. Theor. Biol., **103**(1983), 181-199.

Received: January 9, 2018; Accepted: June 7, 2018.