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SOME FIXED POINT RESULTS ARE FUTILE FOR THE METRIC SPACES WITH FINITE DIAMETER

EHSAN POURHADI

International Center for Mathematical Modelling in Physics and Cognitive Sciences,
Linnaeus University, SE-351 95, Växjö, Sweden
and
Department of Mathematics, Iran University of Science and Technology,
Narmak, Tehran 16846-13114, Iran
E-mail: epourhadi@alumni.iust.ac.ir

Abstract. In this short note, we state some crucial facts which have to be accompanied in a series of results in the literature. Furthermore, the aim of this paper is to warn researchers to be cautious in modeling different problems governed by expansive or contractive type mappings, in the setting of a bounded metric space.
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1. Introduction

In the last decades, several fixed point results related to whether expansive or contractive mappings with the surjection property have been presented.

The goal of this paper is to include an important fact concerning with the main results of numerous papers in literature which should be considered in mathematical modeling (see for instance [2], [7]).

Let us first recall some auxiliary concepts which will be utilized further on.

Definition 1.1. Let (X, d) be a metric space and $\mathcal{M} \subseteq X$. The mapping $T : \mathcal{M} \to X$ is said to be *expansive*, if there exists a constant $\alpha > 1$ such that

$$d(Tx, Ty) \geq \alpha d(x, y), \quad x, y \in \mathcal{M}$$

and it is called *contractive* if we consider the inequality with reverse direction for a constant $\alpha < 1$.

In 1982, in a classic result, Wang et al. ([8], [9]) proved the following theorem.

Theorem 1.2. ([8]) *Let (X, d) be a complete metric space. If T is an onto self-mapping on X and if there exists a constant $k > 1$ such that $d(Fx, Fy) \geq kd(x, y)$ for all $x, y \in X$, then T has a unique fixed point in X.*

Based on the recent result, Xiang and Yuan [11] derived the following which also helped them to obtain a Krasnosel'skii type fixed point theorem.

Theorem 1.3. ([11]) *Assume $\mathcal{M} \subseteq X$ is a closed subset. Let the mapping $T : \mathcal{M} \to X$ be expansive mapping and $T(\mathcal{M}) \supseteq \mathcal{M}$, then there exists a unique point $x^* \in \mathcal{M}$ such that $Tx^* = x^*$.*

In 2015, Ouahab [5] presented an analogous result in the setting of generalized metric which is called Perov type fixed point theorem and also considered as a vectorial version of the above result.

Some recent results play a crucial role to some Krasnosel'skii type fixed point theorems (see Corollary 2.3 and Theorem 2.2 of [11]).

Considering expansive mappings on cone metric spaces (with a solid cone P) endowed with the partial ordering generated by the cone, some fixed point theorems are proved. For instance, we recall some of them.

Theorem 1.4. ([6]) *Let (X, d) be a complete cone metric space with a solid cone P. Let $T : X \to X$ be a surjective mapping satisfying*

$$d(Tx, Ty) \succeq \alpha d(x, y), \quad x, y \in X$$

where $\alpha > 1$. Then T has a fixed point.

Theorem 1.5. ([3]) *Let (X, d) be a complete cone metric space. Suppose mappings $f, g : X \to X$ are onto and satisfy*

$$d(fx, gy) \geq kd(x, y), \quad x, y \in X$$

where $k > 1$ is a constant. Then f and g have a unique common fixed point.

Furthermore, the expansive mappings have been considered in generalized form in order to infer fixed point theorems. For example, see the following result.

Theorem 1.6. ([4]) *Let (X, d) be a complete cone metric space and $T : X \to X$ be a surjection. Suppose that there exist $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 > 1$ such that*

$$d(Tx, Ty) \geq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty), \quad \text{for all } x, y \in X, \ x \neq y.$$

Then T has a fixed point in X.

In 2012, using a control function ϕ, Wang and Wang [10] presented the following result.

Definition 1.7. ([10]) Let (X, d) be a metric space, and let $\mathcal{M}$ be a certain subset of X. The mapping $T : \mathcal{M} \to X$ is said to be a *nonlinear expansion*, if there exists a function $\phi : [0, \infty) \to [0, \infty)$ satisfying

(i) ϕ is nondecreasing,
(ii) $\phi(t) > t$ for each $t > 0$,
(iii) ϕ is right-continuous, such that

$$d(Tx, Ty) \geq \phi(d(x, y)), \quad \text{for all } x, y \in X.$$

Theorem 1.8. ([10]) *Let $\mathcal{M}$ be a closed subset of complete metric space (X, d). Assume that mapping $T : \mathcal{M} \to X$ is a nonlinear expansion and $T(\mathcal{M}) \supseteq \mathcal{M}$, then T has a unique fixed point $u \in \mathcal{M}$.*

There are many papers in which the control functions ϕ are used for the nonlinear contraction condition. See, for example, Corollary 3.17 in [1].

2. Some facts with application of expansive and contractive mappings

Suppose (X, d) is a metric space. Loosely speaking, if $\operatorname{diam}(X) < \infty$ then

- for any expansive self-mapping T on X with constant $\alpha > 1$ and satisfying $TX \subseteq X$ we have
$$\operatorname{diam}(X) \geq \operatorname{diam}(TX) \geq \alpha \operatorname{diam}(X)$$
which is a contradiction;
- for any contractive self-mapping T on X with constant $\alpha < 1$ and satisfying $TX \supseteq X$ we have
$$\operatorname{diam}(X) \leq \operatorname{diam}(TX) \leq \alpha \operatorname{diam}(X)$$
which is again a contradiction.

Therefore, for the metric space (X, d) with $\operatorname{diam}(X) < \infty$

- there is no expansive mapping $T : X \to X$ with $TX \subseteq X$;
- there is no contractive mapping $T : X \to X$ with $TX \supseteq X$.

Now, considering these facts for the case $\operatorname{diam}(X) < \infty$ we see that

(a) the hypotheses of main result of Wang et al. ([8], [9]) do not happen all together (see Theorem 1.2);
(b) for the special case $\mathcal{M} = X$, the conditions of main result of Xiang and Yuan [11] are never going to happen all together (see Theorem 1.3);
(c) by considering a partial order in inequalities as above the assumptions of main result of Shatanawi and Awawdeh [6], and Han and Xu [3] are not satisfied all together (resp. see Theorems 1.4 and 1.5);
(d) for the case $a_2 = a_3 = 0$, the hypotheses of Theorem 1.6 do not hold all together;
(e) to check the assumptions of Theorem 1.8, taking the special case $\mathcal{M} = X$, one can see that $TX = X$ and
$$\operatorname{diam}(X) \geq \operatorname{diam}(TX) \geq \phi(\operatorname{diam}(X)) > \operatorname{diam}(X)$$
which shows that the assumptions of Theorem 1.8 are not satisfied at the same time.

Remark 2.1. We note that the author of current paper selected the subjected papers typically and also only expansive mappings. One can see more related papers in literature and attach our remark to them.

Remark 2.2. Considering the applications of results in references together with numerous subsequent papers we observe that likely all researchers fortunately applied these theorems in the setting of metric spaces with infinite diameter. The problem arises whenever one utilises these results in spaces with finite diameter and the researchers may misapply them in their works. Hence, the researchers are asked to be cautious.

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