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WEAK AND STRONG CONVERGENCE RESULTS FOR SUM OF TWO MONOTONE OPERATORS IN REFLEXIVE BANACH SPACES

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Abstract. The purpose of this paper is to study an inclusion problem which involves the sum of two monotone operators in a real reflexive Banach space. Using the technique of Bregman distance, we study the operator $Res_T^f \circ A^f$ which is the composition of the resolvent of a maximal monotone operator T and the antiresolvent of a Bregman inverse strongly monotone operator A and prove that $0 \in Tx + Ax$ if and only if x is a fixed point of the composite operator $Res_T^f \circ A^f$. Consequently, weak and strong convergence results are given for the inclusion problem under study in a real reflexive Banach space. We apply our results to convex optimization and mixed variational inequalities in a real reflexive Banach space. Our results are new, interesting and extend many related results on inclusion problems from both Hilbert spaces and uniformly smooth and uniformly convex Banach spaces to more general reflexive Banach spaces.
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1. Introduction

Let E be a real reflexive Banach space with norm $\|.\|$ and E^* its topological dual space. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x^*, x\rangle$. Throughout this paper, $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous, and convex function, and the Fenchel conjugate of f (see, e.g., [38, 50]) is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

We denote by dom$f$ the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$. For any $x \in$ intdom$f$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^o(x,y) := \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}. \tag{1.1}$$

The function f is said to be Gâteaux differentiable at x if the limit as $t \to 0^+$ in (1.1) exists for any $y \in E$. In this case, $f^o(x,y)$ coincides with $\langle \nabla f(x), y\rangle$ the value of the gradient ∇f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ intdom$f$. The function f is Fréchet differentiable at x if the limit in (1.1) is attained with $||y|| = 1$ and uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $||y|| = 1$.
The function f is said to be Legendre if it satisfies the following two conditions:

(L1) intdom$f \neq \emptyset$ and the subdifferential ∂f is single-valued in its domain;
(L2) intdom$f^* \neq \emptyset$ and ∂f^* is single-valued on its domain;

where the subdifferential of f is the mapping $\partial f : E \to 2^{E^*}$ defined by

$$\partial f(x) = \{x^* \in E^* : f(x) - f(u) \leq \langle x-u, x^*\rangle,\ \forall\, u \in E\}.$$

The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke, Borwein and Combettes in [5]. Their definition is equivalent to conditions (L1) and (L2) because the space E is assumed to be reflexive (see [5], Theorems 5.4 and 5.6, page 634). It is well known that in reflexive spaces $\nabla f = (\nabla f^*)^{-1}$ (see [6], page 83). When this fact is combined with conditions (L1) and (L2), we obtain

$$ran\nabla f = dom\nabla f^* = int(domf^*) \text{ and } ran\nabla f^* = dom\nabla f = int(domf).$$

It also follows that f is Legendre if and only if f^* is Legendre (see [5], Corollary 5.5, page 634) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains. Several interesting examples of the Legendre functions are presented in [3, 5]. Especially, the functions $\frac{1}{s}\|.\|^s$ with $s \in (1,\infty)$ are Legendre, where the Banach space E is smooth and strictly convex, and in particular, a Hilbert space.
Definition 1.1. Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function, the function D_f : dom$f\times$ intdom$f \to [0,\infty)$ which is defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y-x\rangle, \tag{1.2}$$

is called the Bregman distance ([11, 16]).

The Bregman distance has the following important property, which is called the three point identity: for any $x \in$ dom$f$ and $y, z \in$ intdom$f$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x-y\rangle. \tag{1.3}$$

Let $T : C \to C$ (C, a non-empty subset of $int(domf)$) be a mapping, a point $x \in C$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is denoted by $F(T)$. Also, a point $\widehat{x} \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to $\widehat{x}$ and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Definition 1.2. ([4, 7, 36]) Let C be a nonempty, closed and convex subset of E. A mapping $T : C \to \text{int}(\text{dom } f)$ is called

(i) Bregman Firmly Nonexpansive (BFNE) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle \qquad \forall x, y \in C.$$

(ii) Bregman Strongly Nonexpansive (BSNE) with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \leq D_f(p, x)$$

for all $p \in \widehat{F}(T)$ and $x \in C$ and if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n\to\infty} \Big(D_f(p, x_n) - D_f(p, Tx_n)\Big) = 0,$$

it follows that

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

Now, let $T : E \to 2^{E^*}$ be a set-valued mapping. Recall that the domain of the mapping T is defined by $\text{dom}T = \{x \in E : Tx \neq \emptyset\}$. Let $G(T)$ be the graph of T, that is, $G(T) := \{(x, x^*) \in E \times E^* : x^* \in Tx\}$. A set-valued mapping T is said to be *monotone* if $\langle u - v, x - y\rangle \geq 0$ whenever $(x, u), (y, v) \in G(T)$. It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operator on E. It is known that if T is maximal monotone, then the set $T^{-1}(0^*) := \{z \in E : 0^* \in Tz\}$ is closed and convex. We know that the resolvent of a maximal monotone operator T, denoted by $Res_T^f : E \to 2^E$, is defined as follows (see, e.g., [4]):

$$Res_T^f := (\nabla f + T)^{-1} \circ \nabla f.$$

Remark 1.3. It is known that Res_T^f is a BFNE operator, single-valued and $F(Res_T^f) = T^{-1}(0^*)$ (see, e.g., [4]). Also, if $f : E \to \mathbb{R}$ is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E, then Res_T^f is BSNE and $\widehat{F}(Res_T^f) = F(Res_T^f)$ (see, e.g., [47]).

Assume that the Legendre function f satisfies the following range condition:

$$ran(\nabla f - A) \subseteq ran\nabla f.$$

An operator $A : E \to 2^{E^*}$ is called Bregman Inverse Strongly Monotone (BISM) if (dom A)$\cap$ (dom f) $\neq \emptyset$ and for any $x, y \in \text{intdom } f$ and each $u \in Ax$ and $v \in Ay$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geq 0. \tag{1.4}$$

Observe that if A is single-valued in a real Hilbert space H, then (1.4) is equivalent to

$$\langle Ax - Ay, x - y\rangle \geq \|Ax - Ay\|^2, \ \forall x, y \in H.$$

Therefore, the class of BISM mappings in (1.4) is a generalization of the class of single-valued firmly nonexpansive mappings in real Hilbert spaces. For any operator $A : E \to 2^{E^*}$, the anti-resolvent operator $A^f : E \to 2^E$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A).$$

Observe that

$$\text{dom}A^f \subseteq (\text{dom}A) \cap (\text{intdom}f) \quad \text{and} \quad \text{ran}A^f \subseteq \text{intdom}f.$$

It is also known that the operator A is BISM if and only if the anti-resolvent A^f is a single-valued BFNE (see, e.g., [14], Lemma 3.2(c) and (d), p. 2109) and $F(A^f) = A^{-1}(0^*)$. For examples and further information on BISM, see [14].

Suppose T is a maximal monotone operator in a real Hilbert space H. A basic problem that arises in several branches of applied mathematics (see, for instance, [19, 28, 32, 48, 49, 57, 62] and the references therein) is to

$$\text{find } x \in H \quad \text{such that} \quad 0 \in Tx. \tag{1.5}$$

One of the methods for solving this problem is the well-known Proximal Point Algorithm (PPA) introduced by Martinet [37]. The PPA generates for any starting point $x_1 = x \in H$, a sequence $\{x_n\}$ in H such that

$$x_{n+1} = (I + \lambda_n T)^{-1} x_n, \quad n = 1, 2, ... \tag{1.6}$$

where $\{\lambda_n\}$ is a given sequence of positive real numbers. This algorithm was further developed by Rockafellar (see, [49]), who proved that the sequence generated by (1.6) converges weakly to an element of $T^{-1}(0)$ (where $T^{-1}(0) := \{x \in H : 0 \in Tx\}$) when $T^{-1}(0)$ is nonempty and $\liminf\limits_{n\to\infty} \lambda_n > 0$.

Furthermore in [49], Rockafellar asked if the sequence generated by (1.6) converges strongly in general. This question was answered in the negative by Güler [24] who presented an example of a subdifferential for which the sequence generated by (1.6) converges weakly but not strongly. For more recent results on PPA, please see [8, 9, 10, 22, 39, 53] and the references contained therein.

In the synthetic formulation, the operator T in Problem (1.5) can be decomposed as sum of two monotone operators which leads to the problem of the form:

$$\text{find } x \in H \quad \text{such that} \quad 0 \in Tx + Ax, \tag{1.7}$$

where $A, T : H \to 2^H$ are nonlinear monotone operators. Problem (1.7) is a very general format for certain concrete problems in machine learning, linear inverse problem and many nonlinear problems such as convex programming, variational inequalities and split feasibility problem.

Example 1.4. A stationary solution to the initial value problem of the evolution equation

$$0 \in \frac{\partial u}{\partial t} + Ku, \quad u(0) = u_0,$$

can be rewritten as (1.7) where the governing monotone operator K is of the form $K = T + A$.

Example 1.5. Let $\phi, \varphi : H \to (-\infty, +\infty]$ be two proper, lower semi-continuous and convex functions and $B : H \to H$ be a bounded linear operator. The minimization problem

$$\min_{x \in H} \{\phi(x) + \varphi(Bx)\} \tag{1.8}$$

can be written in the form of (1.7), where ϕ and $\varphi \circ B$ have a common point of continuity with $T := \partial\phi$ and $A := B^* \circ \partial\varphi \circ B$. Here B^* is the adjoint of B. It is well-known (see, e.g., [12, 20, 54, 55]) that the minimization problem (1.8) is widely used in image recovery, signal processing and machine learning.

Example 1.6. Let $\phi : H \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function and $\partial\phi$ be the subdifferential of ϕ. If $T = \partial\phi$, then problem (1.7) is equivalent to the following mixed variational inequality problem: find $x \in H$ such that

$$\langle Ax, y - x^* \rangle + \phi(y) - \phi(x) \geq 0, \quad \forall\, y \in H. \tag{1.9}$$

In particular, if C is a nonempty, closed and convex subset of H and ϕ is the indicator function, that is

$$\phi(x) = \begin{cases} 0, & if\ x \in C, \\ \\ +\infty, & if\ x \notin C, \end{cases} \tag{1.10}$$

then problem (1.9) is equivalent to the classical variational inequality problem: find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall\, y \in C. \tag{1.11}$$

It is easy to see that (1.11) is equivalent to finding a point $x \in C$ such that

$$0 \in Tx + Ax,$$

where T is the subdifferential of the indicator function of C.

However, there is little work in the existing literature on the approximation of zeroes of sum of monotone operators in Banach spaces, though there are some works on finding zero of sum of accretive operators in Banach space (see [17, 18, 34, 52, 63, 60] and references contained therein). The main difficulties are due to the fact that the inner product structure of a Hilbert space fails to be true in a Banach space and that unlike the accretive operators the definition of monotone operators in Banach spaces does not involve the duality mapping.

In this paper, we shall use the technique of Bregman distance to carry out certain investigations on the approximation of zero of the sum of a maximal monotone operator T and a Bregman inverse strongly monotone operator A using the composition of the resolvent operator on T and the antiresolvent operator on A in a real reflexive Banach space. Our results complement and extend many corresponding results (in particular, [26, 27, 30, 33, 49, 61]) from both Hilbert spaces and uniformly smooth and uniformly convex Banach spaces to more general reflexive Banach spaces.

2. Preliminaries

In this section, we will give some basic definitions and results that are needed in the sequel. We denote the strong convergence of $\{x_n\}$ to a point $x \in E$ by $x_n \to x$ and the weak convergence of $\{x_n\}$ to x by $x_n \rightharpoonup x$.
The Bregman projection (see [11]) of $x \in \mathrm{int}(\mathrm{dom} f)$ onto the nonempty, closed and convex subset C $\subset \mathrm{int}(\mathrm{dom} f)$ is defined as the necessarily unique vector $\mathrm{Proj}_C^f(x) \in C$ satisfying

$$D_f(Proj_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{2.1}$$

Note that if E is a Hilbert space and $f(x) = \frac{1}{2}||x||^2$, then the Bregman projection of x onto C is the metric projection P_C.
Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function f is said to be totally convex at $x \in$ int dom f if its modulus of totally convexity at x, that is, the function $v_f : \mathrm{int}(\mathrm{dom} f) \times [0, +\infty) \to [0, +\infty]$ defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in dom f, ||y - x|| = t\} \tag{2.2}$$

is positive for any $t > 0$. The function f is said to be totally convex when it is totally convex at every point $x \in \mathrm{int}(\mathrm{dom}\ f)$. In addition, the function f is said to be totally convex on bounded sets if $v_f(B, t)$ is positive for any nonempty bounded subset B, where the modulus of total convexity of the function f on the set B is the function $v_f : \mathrm{int}(\mathrm{dom}\ f) \times [0, +\infty) \to [0, +\infty]$ which is defined by

$$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap dom f\}. \tag{2.3}$$

For further details and examples on totally convex functions see [7, 13, 15].
Lemma 2.1. ([15]) *Let f be totally convex on int(domf). Let C be a nonempty, closed and convex subset of int(domf) and $x \in$int(domf), if $\omega \in C$, then the following conditions are equivalent:*

i. *the vector ω is the Bregman projection of x onto C, with respect to f,*
ii. *the vector ω is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0 \qquad \forall y \in C,$$

iii. *the vector ω is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x) \qquad \forall y \in C.$$

Recall that a function f is said to be sequentially consisted (see [15]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

The following lemma follows from [46].
Lemma 2.2. *If domf contains at least two points, then the function f is totally convex on bounded sets if and only if it is sequentially consistent.*

One powerful tool for deriving weak or strong convergence of iterative sequence is due to Opial [41]. A Banach space E is said to satisfy Opial property [41] if for any weakly convergent sequence $\{x_n\}$ in E with weak limit x, we have

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

for all y in E with $y \neq x$. It is well know that all Hilbert spaces, all finite dimensional Banach spaces and the Banach space l^p $(1 \leq p < \infty)$ satisfy the Opial property. However, it is well known that not every Banach space satisfies the Opial property, see, for example, [21, 23]. But, the following Bregman Opial-like inequality for every Banach space E has been proved in [25].

Lemma 2.3. ([25, 42]) *Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper strictly convex function and Gâteaux differentiable such that ∇f is weakly sequentially continuous and $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup u$ for some $u \in E$. Then*

$$\limsup_{n\to\infty} D_f(u, x_n) < \limsup_{n\to\infty} D_f(v, x_n),$$

for all v in the interior of $domf$ with $u \neq v$.

The following lemmas will be used in the convergence analysis in the sequel.

Lemma 2.4. ([29]) *Let $f : E \to (-\infty, +\infty]$ be a Legendre function and let $A : E \to 2^{E^*}$ be a BISM operator such that $A^{-1}(0^*) \neq \emptyset$. Then the following statements hold:*

(i) $A^{-1}(0^*) = F(A^f)$.

(ii) *For any $w \in A^{-1}(0^*)$ and $x \in$ dom A^f, we have*

$$D_f(w, A^f(x)) + D_f(A^f(x), x) \leq D_f(w, x).$$

Remark 2.5. If the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the anti-resolvent A^f is a single-valued BSNE operator which satisfies $F(A^f) = \widehat{F}(A^f)$ (cf. [47]).

Lemma 2.6. ([44]) *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.7. ([46]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_1 \in E$ and the sequence $\{D_f(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.8. ([29]) *Assume that $f : E \to \mathbb{R}$ is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of E. Let C be a nonempty closed and convex subset of E. Let $\{T_i : 1 \leq i \leq N\}$ be BSNE operators which satisfy $\widehat{F}(T_i) = F(T_i)$ for each $1 \leq i \leq N$ and let $T := T_N T_{N-1} \ldots T_1$. If*

$$\cap\{F(T_i) : 1 \leq i \leq N\}$$

and $F(T)$ are nonempty, then T is also BSNE with $F(T) = \widehat{F}(T)$.

Lemma 2.9. ([46]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_1 \in E$ and let C be a nonempty, closed and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak sequential limit of $\{x_n\}$ belongs to*

C. If $D_f(x_n, x_1) \leq D_f(Proj_C^f(x_1), x_1)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $Proj_C^f(x_1)$.

Lemma 2.10. ([58]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + b_n, \ \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(b) $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} b_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11. ([35]) *Let $\{x_n\}$ be a sequence of real numbers such that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} < x_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$x_{m_k} \leq x_{m_k+1} \text{ and } x_k \leq x_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \cdots, k\}$ such that the condition $x_n < x_{n+1}$ holds.

Let $f : E \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function and let $V_f : E \times E^* \to [0, \infty)$ (see [1, 16]) be defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \ \forall x \in E, x^* \in E^*. \tag{2.4}$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \ \forall x \in E, \ x^* \in E^*$.
Furthermore, by the subdifferential inequality, we have (see [31])

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*) \ \forall x \in E, x^*, y^* \in E^*. \tag{2.5}$$

In addition, if $f : E \to (-\infty; +\infty]$ is a proper lower semi-continuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper $weak^*$ lower semi-continuous and convex function (see [43]). Hence V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i). \tag{2.6}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Let E be a Banach space and let $B_r := \{z \in E : ||z|| \leq r\}$ for all $r > 0$. Then a function $f : E \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of E if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, \infty) \to [0, \infty]$ is defined by

$$\rho_r(t) = \inf_{x,y\in B_r, ||x-y||=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$

for all $t \geq 0$. ρ_r is called the gauge of uniform convexity of f (see, [59], pp. 203, 221).

Lemma 2.12. ([40]) *Let Let E be a Banach space, let $r > 0$ be a constant and let $f : E \to \mathbb{R}$ be a convex function which is uniformly convex function on bounded subsets of E. Then*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_r(||x-y||) \tag{2.7}$$

for all $x, y \in B_r$ and $\alpha \in (0,1)$, where ρ_r is the guage of uniform convexity of f.

For the rest of this paper, we define

$$(T+A)^{-1}(0) := \{x \in E : 0 \in Tx + Ax\},$$

where $T : E \to 2^{E^*}, A : E \to E^*$. Thus, $(T+A)^{-1}(0)$ is the set of solutions to the inclusion problem

$$0 \in Tx + Ax, x \in E. \tag{2.8}$$

3. Main results

We start our contributions in this paper with the following important proposition.

Proposition 3.1. *Assume that $f : E \to \mathbb{R}$ is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subsets of E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator and let $A : E \to E^*$ be a single valued BISM operator such that $(T+A)^{-1}0 \neq \emptyset$. Then,*

(a) $F(Res_T^f \circ A^f) = (T+A)^{-1}0$,
(b) $F(Res_T^f \circ A^f) = F(Res_T^f) \cap F(A^f)$,
(c) $(T+A)^{-1}0$ *is closed and convex,*
(d) $Res_T^f \circ A^f$ *is a BSNE operator and* $F(Res_T^f \circ A^f) = \widehat{F}(Res_T^f \circ A^f)$.

Proof. (a) Let $x \in (Res_T^f \circ A^f)$, we have that

$$\begin{aligned} x = (Res_T^f \circ A^f)(x) &\Leftrightarrow x = ((\nabla f + T)^{-1} \circ \nabla f) \circ (\nabla f^* \circ (\nabla f - A))(x) \\ &\Leftrightarrow x = (\nabla f + T)^{-1} \circ (\nabla f - A)(x) \\ &\Leftrightarrow (\nabla f - A)x \in (\nabla f + T)(x) \\ &\Leftrightarrow 0 \in (T+A)(x). \end{aligned}$$

Thus, $F(Res_T^f \circ A^f) = (T+A)^{-1}0$.

(b) Since $F(Res_T^f) \cap F(A^f) \subseteq F(Res_T^f \circ A^f)$, it suffices to show that $F(Res_T^f \circ A^f) \subseteq F(Res_T^f) \cap F(A^f)$. Let $x \in F(Res_T^f \circ A^f)$ and $y \in F(Res_T^f) \cap F(A^f)$. Then

$$\begin{aligned} D_f(y,x) &= D_f(y, Res_T^f \circ A^f(x)) \\ &\leq D_f(y, A^f(x)). \end{aligned} \tag{3.1}$$

By Lemma 2.4, we obtain

$$D_f(y, A^f(x)) + D_f(A^f(x), x) \leq D_f(y,x),$$

then from (3.1), we have

$$\begin{aligned} D_f(A^f(x), x) &\leq D_f(y, x) - D_f(y, A^f(x)) \\ &\leq D_f(y, A^f(x)) - D_f(y, A^f(x)) \\ &= 0. \end{aligned}$$

This implies that $A^f(x) = x$, and thus $x \in F(A^f)$.
Moreover, $x \in F(Res_T^f \circ A^f)$ and $x \in F(A^f)$ implies that

$$x = Res_T^f \circ A^f(x) = Res_T^f(x),$$

thus $x \in F(Res_T^f)$. Hence $x \in F(Res_T^f) \cap F(A^f)$.
Therefore, $F(Res_T^f \circ A^f) \subseteq F(Res_T^f) \cap F(A^f)$ and the conclusion follows.
(c). From (a) and (b), we know that

$$(T + A)^{-1}0 = F(Res_T^f \circ A^f) = F(Res_T^f) \cap F(A^f)$$

and since $F(Res_T^f)$ and $F(A^f)$ are both closed and convex, we have that $(T + A)^{-1}0$ is closed and convex.
(d). Since Res_T^f and A^f are BSNE operators (see Remarks 1.3 and 2.5) and

$$F(Res_T^f) \cap F(A^f) = (T + A)^{-1}0 \neq \emptyset,$$

it then follows from Lemma 2.8 that $Res_T^f \circ A^f$ is BSNE and

$$F(Res_T^f \circ A^f) = \widehat{F}(Res_T^f \circ A^f). \qquad \square$$

Remark 3.2. As a passing remark, it should be noted that Proposition 3.1 also holds in a reflexive Banach space E with $f : E \to \mathbb{R}$ a strongly coercive Bregman function which is locally bounded, locally uniformly convex, and locally uniformly smooth on E.

We now present a weak convergence theorem for approximating solutions of inclusion problem (2.8) in a real reflexive Banach space.

Theorem 3.3. *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Bregman function which is locally bounded, locally uniformly convex, and locally uniformly smooth on E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a single valued BISM operator. Suppose $\Gamma = (T + A)^{-1}0 \neq \emptyset$. Define a sequence $\{x_n\}$ in E as follows:*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(Res_T^f \circ A^f(x_n))), \quad \forall n \geq 1, \tag{3.2}$$

where $\{\alpha_n\}$ is an arbitrary sequence in (0,1) and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Then the sequence $\{x_n\}$ converges weakly to a point in Γ.

Proof. Let $p \in \Gamma$, then we have

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res_T^f \circ A^f(x_n))) \\ &\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, Res_T^f \circ A^f(x_n)) \\ &\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, x_n) \\ &= D_f(p, x_n). \end{aligned} \tag{3.3}$$

This implies that $\{D_f(p, x_n)\}$ is bounded and nonincreasing, thus $\lim_{n\to\infty} D_f(p, x_n)$ exist and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, x_{n+1})) = 0. \tag{3.4}$$

Moreover, since $\{D_f(p, x_n)\}$ is bounded, it follows from Lemma 2.7 that $\{x_n\}$ is also bounded. Furthermore, since f is bounded on bounded subsets of E, then ∇f is also bounded on bounded subsets of E^*. This implies that the sequence $\{\nabla f(x_n)\}$ and $\{\nabla f(Res_T^f \circ A^f(x_n))\}$ are bounded in E^*.

Let $s = \sup\{||\nabla f(x_n)||, ||\nabla f(Res_T^f \circ A^f(x_n))||\}$ and $\rho_s^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function f^*. Using Lemma 2.12, (2.4), (3.2), we have

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res_T^f \circ A^f(x_n)))) \\ &= V_f(p, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res_T^f \circ A^f(x_n))) \\ &= f(p) - \langle p, \alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res_T^f \circ A^f(x_n))\rangle \\ &+ f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Res_T^f \circ A^f(x_n))) \\ &\leq \alpha_n f(p) + (1-\alpha_n) f(p) - \alpha_n \langle p, \nabla f(x_n)\rangle \\ &+ (1-\alpha_n)\langle p, \nabla f(Res_T^f \circ A^f(x_n))\rangle \\ &+ \alpha_n f^*(\nabla f(x_n)) + (1-\alpha_n) f^*(\nabla f(Res_T^f \circ A^f(x_n))) \\ &- \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(Res_T^f \circ A^f(x_n))||) \\ &= \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, Res_T^f \circ A^f(x_n)) \\ &- \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(Res_T^f \circ A^f(x_n))||) \\ &\leq \alpha_n D_f(p, x_n) + (1-\alpha_n) D_f(p, x_n) \\ &- \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(Res_T^f \circ A^f(x_n))||) \\ &= D_f(p, x_n) - \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(Res_T^f \circ A^f(x_n))||). \end{aligned} \tag{3.5}$$

Thus

$$\alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(Res_T^f \circ A^f(x_n))||) \leq D_f(p, x_n) - D_f(p, x_{n+1}).$$

Therefore,

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(Res_T^f \circ A^f(x_n))||) \leq D_f(p, x_1) < \infty.$$

By the control condition on $\{\alpha_n\}$, we have

$$\liminf_{n\to\infty} \rho_s^*(||\nabla f(x_n) - \nabla f(Res_T^f \circ A^f(x_n))||) = 0.$$

Hence, by the property of ρ_s^*, we have that

$$\liminf_{n\to\infty} ||\nabla f(x_n) - \nabla f(Res_T^f \circ A^f(x_n))|| = 0. \tag{3.6}$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we deduce that

$$\liminf_{n\to\infty} ||x_n - Res_T^f \circ A^f(x_n)|| = 0.$$

Moreover, since E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x} \in E$ as $i \to \infty$. Since $Res_T^f \circ A^f$ is BSNE, we have that $\bar{x} \in F(Res_T^f \circ A^f)$. Next, we show that $\bar{x}$ is unique. Suppose there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $x \in C$ with $\bar{x} \neq x$. This implies that $x \in F(Res_T^f \circ A^f)$. Since $\lim_{n\to\infty} D_f(\bar{x}, x_n)$ exists for all $\bar{x} \in F(Res_T^f \circ A^f)$. It follows from the Bregman Opial-like property of E that (more precisely Lemma 2.3)

$$\begin{aligned} \lim_{n\to\infty} D_f(\bar{x}, x_n) &= \lim_{i\to\infty} D_f(\bar{x}, x_{n_i}) < \lim_{i\to\infty} D_f(x, x_{n_i}) \\ &= \lim_{n\to\infty} D_f(x, x_n) = \lim_{j\to\infty} D_f(x, x_{n_j}) \\ &< \lim_{j\to\infty} D_f(\bar{x}, x_{n_j}) = \lim_{n\to\infty} D_f(\bar{x}, x_n), \end{aligned} \tag{3.7}$$

which is a contradiction. Thus we have that $\bar{x} = x$ and the desired result follows. This completes the proof of Theorem 3.3. □

Remark 3.4. Our Theorem 3.3 extends Theorem 3.2 of [26] from uniformly smooth Banach space which is also uniformly convex to a more general reflexive Banach space. Consequently, Theorem 3.3 extends the results in [27, 49] from Hilbert spaces to reflexive Banach spaces.

Next, we state and prove the following strong convergence theorems for approximating solution of inclusion problem (2.8) in a real reflexive Banach space. The first strong convergence result involves an additional projection onto the intersection of two half-spaces.

Theorem 3.5. *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a single valued BISM operator. Suppose $\Gamma = (T + A)^{-1}0 \neq \emptyset$. Define a sequence $\{x_n\}$ in E as follows:*

$$\begin{cases} x_1 \in E, \\ y_n = Res_T^f \circ A^f(x_n), \\ C_n = \{u \in E : D_f(u, y_n) \leq D_f(u, x_n)\}, \\ Q_n = \{u \in E : \langle \nabla f(x_1) - \nabla f(x_n), u - x_n\rangle \leq 0\}, \\ x_{n+1} = Proj_{C_n \cap Q_n}^f(x_1), \quad \forall n \geq 1. \end{cases} \tag{3.8}$$

Then the sequence $\{x_n\}$ converges strongly to a point $p = Proj_\Gamma^f(x_1)$.

Proof. First, we show that (3.8) is well defined and $\Gamma \subset C_n \cap Q_n$ for every $n \geq 1$.
It is obvious that C_n and Q_n are closed and convex for every $n \geq 1$. Now let $w \in \Gamma$. Then we have

$$D_f(w, y_n) = D_f(w, Res_T^f \circ A^f(x_n)) \leq D_f(w, x_n),$$

which implies that $w \in C_n$ for $n \geq 1$. So, we have $\Gamma \subset C_n$ for all $n \geq 1$.
For $n = 1$, we have $Q_1 = E$ and thus, we have $\Gamma \subset C_1 \cap Q_1$. Suppose that x_k is given and $\Gamma \subset C_k \cap Q_k$ for some $k > 1$. There exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = Proj^f_{C_k \cap Q_k}(x_1)$. From Lemma 2.1(ii), we have

$$\langle \nabla f(x_1) - \nabla f(x_{k+1}), x_{k+1} - u \rangle \geq 0,$$

for any $u \in C_k \cap Q_k$. Since $\Gamma \subset C_k \cap Q_k$, we get $\Gamma \subset Q_{k+1}$. Therefore $\Gamma \subset C_{k+1} \cap Q_{k+1}$. Thus $\{x_n\}$ is well defined.
We now show that $\{x_n\}$ is bounded.
By the definition of Q_n, we have that $x_n = Proj^f_{Q_n} x_1$. Therefore,

$$\begin{aligned} D_f(x_n, x_1) &= D_f(Proj^f_{Q_n}(x_1), x_1) \\ &\leq D_f(w, x_1) - D_f(w, Proj^f_{Q_n}(x_1)) \\ &\leq D_f(w, x_1). \end{aligned} \tag{3.9}$$

Hence, the sequence $\{D_f(x_n, x_1)\}$ is bounded by $D_f(w, x_1)$ for any $w \in \Gamma$. Therefore, by Lemma 2.7, we have that $\{x_n\}$ is bounded. Since f is bounded on bounded subsets of E, therefore, ∇f is also bounded on bounded subsets of E (see [13], Proposition 1.1.11). This implies that the sequence $\{\nabla f(x_n)\}$ and $\{\nabla f(Res_T^f \circ A^f(x_n))\}$ are bounded in E.
Moreover, since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = Proj^f_{Q_n}(x_1)$, we have

$$D_f(x_{n+1}, Proj^f_{Q_n}(x_1)) + D_f(Proj^f_{Q_n}(x_1), x_1) \leq D_f(x_{n+1}, x_1).$$

Thus

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1). \tag{3.10}$$

Therefore the sequence $\{D_f(x_n, x_1)\}$ is increasing and since it is also bounded, $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Thus, it follows from (3.10) that $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$, and by Lemma 2.2, we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.11}$$

Since $x_{n+1} \in C_n$, we have

$$D_f(x_{n+1}, y_n) = D_f(x_{n+1}, Res_T^f \circ A^f(x_n)) \leq D_f(x_{n+1}, x_n) \to 0, \ n \to \infty,$$

and by Lemma 2.2, we have $\lim_{n\to\infty} ||x_{n+1} - Res_T^f \circ A^f(x_n)|| = 0$.
It follows from (3.11) that

$$||Res_T^f \circ A^f(x_n) - x_n|| \leq ||Res_T^f \circ A^f(x_n) - x_{n+1}|| + ||x_{n+1} - x_n|| \to 0, \ n \to \infty. \tag{3.12}$$

Now, since $\{x_n\}$ is bounded and E is a reflexive Banach space, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $q \in E$. Therefore by (3.12), we have that $q \in F(Res_T^f \circ A^f) = \Gamma$.

We now prove that $\{x_n\}$ converges strongly to $Proj^f_\Gamma(x_1)$. Let $p = Proj^f_\Gamma(x_1)$, then since $x_{n+1} = Proj^f_{C_n\cap Q_n}x_1$ and $\Gamma \subset C_n \cap Q_n$, we have

$$D_f(x_{n+1}, x_1) \leq D_f(p, x_1).$$

Therefore by Lemma 2.9, we have that $\{x_n\}$ converges strongly to $Proj^f_\Gamma(x_1)$. This completes the proof of Theorem 3.5. □

In the next result, we give strong convergence theorem for the inclusion problem (2.8) based on the shrinking projection method.

Theorem 3.6. *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a single valued BISM operator. Suppose $\Gamma = (T + A)^{-1}0 \neq \emptyset$. Define a sequence $\{x_n\}$ in E as follows:*

$$\begin{cases} x_1 \in C_1 = E, \\ y_n = Res^f_T \circ A^f(x_n), \\ C_{n+1} = \{z \in C_n : D_f(z, y_n) \leq D_f(z, x_n)\}, \\ x_{n+1} = Proj^f_{C_{n+1}}(x_1), \quad \forall n \geq 1. \end{cases} \tag{3.13}$$

Then the sequence $\{x_n\}$ converges strongly to a point $q = Proj^f_\Gamma(x_1)$.

Proof. We first show that (3.13) is well defined. By Proposition 3.1,

$$F(Res^f_T \circ A^f) = F(Res^f_T) \cap F(A^f)$$

is nonempty, closed and convex and also $C_1 = E$ is closed and convex. Suppose C_k is closed and convex for some $k \in \mathbb{N}$. For each $z \in C_k$, we see that $D_f(z, y_k) \leq D_f(z, x_k)$ is equivalent to

$$\langle \nabla f(x_k) - \nabla f(y_k), z\rangle \leq f(y_k) - f(x_k) + \langle f(x_k), x_k\rangle - \langle \nabla f(y_k), y_k\rangle.$$

By the construction of the set C_{k+1}, we see that C_{k+1} is also closed and convex. Therefore $\{x_n\}$ is well defined.

We now show that $\{x_n\}$ is bounded. Let $w \in \Gamma$, then from Lemma 2.1, we have

$$\begin{aligned} D_f(x_n, x_1) &= D_f(Proj^f_{C_n}(x_1), x_1) \\ &\leq D_f(w, x_1) - D_f(w, Proj^f_{C_n}(x_1)) \\ &\leq D_f(w, x_1). \end{aligned} \tag{3.14}$$

Hence, the sequence $\{D_f(x_n, x_1)\}$ is bounded by $D_f(w, x_1)$. Therefore, by Lemma 2.7, the sequence $\{x_n\}$ is bounded too.

Moreover, since $x_{n+1} \in C_{n+1} \subset C_n$, it follows from Lemma 2.1(iii) that

$$D_f(x_{n+1}, Proj^f_{C_n}(x_1)) + D_f(Proj^f_{C_n}(x_1), x_1) \leq D_f(x_{n+1}, x_1)$$

and hence

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_1) \leq D_f(x_{n+1}, x_1). \tag{3.15}$$

Therefore, the sequence $\{D_f(x_n, x_1)\}$ is increasing and since it is bounded, $\lim_{n\to\infty} D_f(x_n, x_1)$ exists. Thus, it follows from (3.15) that

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0. \tag{3.16}$$

By Lemma 2.2 and (3.16), we have that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.17}$$

Also, since $x_{n+1} = Proj^f_{C_{n+1}} x_1 \in C_n$, we have

$$D_f(x_{n+1}, y_n) = D_f(x_{n+1}, Res^f_T \circ A^f(x_n)) \leq D_f(x_{n+1}, x_n) \to 0, \quad n \to \infty.$$

Thus, by Lemma 2.2, we have

$$\lim_{n\to\infty} ||x_{n+1} - Res^f_T \circ A^f(x_n)|| = 0. \tag{3.18}$$

Therefore, we have

$$\begin{aligned} ||Res^f_T \circ A^f(x_n) - x_n|| &\leq ||Res^f_T \circ A^f(x_n) - x_{n+1}|| \\ &\quad + ||x_{n+1} - x_n|| \to 0,\ n \to \infty. \end{aligned} \tag{3.19}$$

Next, we show that $\{x_n\}$ is Cauchy. Since $x_m = Proj^f_{C_m} x_1 \in C_m \subset C_n$ for $m > n \geq 1$, by Lemma 2.1, we have that

$$\begin{aligned} D_f(x_m, x_n) &= D_f(x_m, Proj^f_{C_n} x_1) \\ &\leq D_f(x_m, x_1) - D_f(Proj^f_{C_n} x_1, x_1) \\ &= D_f(x_m, x_1) - D_f(x_n, x_1). \end{aligned} \tag{3.20}$$

Letting $m, n \to \infty$ in (3.20), we have $D_f(x_m, x_n) \to 0$. Since f is totally convex on bounded subsets on E, by Lemma 2.2, f is sequentially consistent.
Thus $||x_m - x_n|| \to 0$ as $m, n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of the space E, we can assume that $x_n \to q \in E$ as $n \to \infty$. Clearly, it follows from (3.19) that $q \in F(Res^f_T \circ A^f) = \Gamma$.
We now show that $q = Proj^f_\Gamma(x_1)$. From $x_n = Proj^f_{C_n} x_1$, we have

$$\langle \nabla f(x_1) - \nabla f(x_n), x_n - z\rangle \geq 0, \quad \forall z \in C_n.$$

Since $\Gamma \subset C_n$, we also have

$$\langle \nabla f(x_1) - \nabla f(x_n), x_n - z\rangle \geq 0, \quad \forall z \in \Gamma.$$

Taking the limit of the above inequality as $n \to \infty$, we obtain

$$\langle \nabla f(x_1) - \nabla f(q), q - z\rangle \geq 0, \quad \forall z \in \Gamma.$$

Hence, we have $q = Proj^f_\Gamma(x_1)$. This completes the proof. □

Remark 3.7. We emphasize here that there is a distinction between Theorems 3.5, 3.6 above and the results of [45, 51]. Here in Theorems 3.5 and 3.6, we focus on finding zero of sum of two monotone operators (in which one of them is a maximal monotone operator) using an auxiliary composite operator, while in [45, 51], the focus was on finding common zero of maximal monotone operators. Similarly, our algorithms in both (3.8) and (3.13) are different from the algorithms (3.1) and (3.11) studied in [45].

Finally, we give a strong convergence analysis for solving (2.8) where the proposed method is Halpern-type method.

Theorem 3.8. *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a single valued BISM operator. Suppose $\Gamma := (T + A)^{-1}(0) \neq \emptyset$. Suppose that $u \in E$ and define the sequence x_n as follows: $x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(Res_T^f \circ A^f(x_n))), \forall n \geq 1, \tag{3.21}$$

where $\alpha_n \subset (0, 1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $Proj_\Gamma^f(u)$.

Proof. First we note that $\Gamma = F(Res_T^f \circ A^f)$ is closed and convex. Let

$$p = Proj_\Gamma^f(u) \in F(Res_T^f \circ A^f) = \widehat{F}(Res_T^f \circ A^f).$$

From (2.6), we have

$$\begin{aligned} D_f(p, x_{n+1}) &\leq \alpha_n D_f(p, u) + (1 - \alpha_n)D_f(p, Res_T^f \circ A^f(x_n)) \\ &\leq \alpha_n D_f(p, u) + (1 - \alpha_n)D_f(p, x_n) \\ &\leq \max\{D_f(p, u), D_f(p, x_n)\}. \end{aligned}$$

Thus by induction, we have that the sequence $D_f(p, x_n)$ is bounded and by Lemma 2.7, we have that $\{x_n\}$ is also bounded. Furthermore, since f is bounded on bounded subsets of $E, \nabla f$ is also bounded on bounded subsets of E (see [13], Proposition 1.1.11). Therefore $\nabla f(Res_T^f \circ A^f(x_n))$ is bounded.

We next show that if there exists a subsequence x_{n_k} of x_n such that

$$\lim_{k\to\infty}(D_f(p, x_{n_k+1}) - D_f(p, x_{n_k})) = 0,$$

then

$$\lim_{k\to\infty}(D_f(p, Res_T^f \circ A^f(x_{n_k})) - D_f(p, x_{n_k})) = 0.$$

Since $\nabla f(Res_T^f \circ A^f(x_{n_k}))$ is bounded and $\alpha_{n_k} \to 0$, we obtain

$$\begin{aligned} &\lim_{k\to\infty} ||\nabla f(x_{n_k+1}) - \nabla f(Res_T^f \circ A^f(x_{n_k}))|| \\ &= \lim_{k\to\infty} \alpha_{n_k}||\nabla f(u) - \nabla f(Res_T^f \circ A^f(x_{n_k}))|| = 0. \end{aligned} \tag{3.22}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, f^* is uniformly Fréchet differentiable on bounded subsets of E (see [59]). Moreover, f^* is bounded on bounded sets (see [5, 59]). Since f is Legendre, applying Lemma 2.6, we have

$$\begin{aligned}&\lim_{k\to\infty}||x_{n_k+1}-Res_T^f\circ A^f(x_{n_k})||\\ =&\lim_{k\to\infty}||\nabla f^*(\nabla f(x_{n_k+1}))-\nabla f^*(\nabla f(Res_T^f\circ A^f(x_{n_k})))||=0. \quad (3.23)\end{aligned}$$

On the other hand, if f is uniformly Fréchet differentiable on bounded subsets of E, then f is uniformly continuous on bounded subsets of E (see [2]). It follows that

$$\lim_{k\to\infty}||f(x_{n_k+1})-f(Res_T^f\circ A^f(x_{n_k}))||=0. \quad (3.24)$$

We now consider the following equality.

$$\begin{aligned}&D_f(p,Res_T^f\circ A^f(x_{n_k}))-D_f(p,x_{n_k})=f(p)-f(Res_T^f\circ A^f(x_{n_k}))\\ &-\langle\nabla f(Res_T^f\circ A^f(x_{n_k})),p-Res_T^f\circ A^f(x_{n_k})\rangle-D_f(p,x_{n_k})\\ &=f(p)-f(x_{n_k+1})+f(x_{n_k+1})-f(Res_T^f\circ A^f(x_{n_k}))\\ &-\langle\nabla f(x_{n_k+1}),p-x_{n_k+1}\rangle+\langle\nabla f(x_{n_k+1}),p-x_{n_k+1}\rangle\\ &-\langle\nabla f(Res_T^f\circ A^f(x_{n_k})),p-Res_T^f\circ A^f(x_{n_k})\rangle-D_f(p,x_{n_k})\\ &=D_f(p,x_{n_k+1})+(f(x_{n_k+1})-f(Res_T^f\circ A^f(x_{n_k})))+\langle\nabla f(x_{n_k+1}),p-x_{n_k+1}\rangle\\ &-\langle\nabla f(Res_T^f\circ A^f(x_{n_k})),p-Res_T^f\circ A^f(x_{n_k})\rangle-D_f(p,x_{n_k})\\ &=(D_f(p,x_{n_k+1})-D_f(p,x_{n_k}))+(f(x_{n_k+1})-f(Res_T^f\circ A^f(x_{n_k})))\\ &+\langle\nabla f(x_{n_k+1})-\nabla f(Res_T^f\circ A^f(x_{n_k})),p-x_{n_k+1}\rangle\\ &-\langle\nabla f(Res_T^f\circ A^f(x_{n_k})),x_{n_k+1}-Res_T^f\circ A^f(x_{n_k})\rangle.\end{aligned}$$

It follows from (3.22),(3.23) and (3.24) that

$$\lim_{k\to\infty}(D_f(p,Res_T^f\circ A^f(x_{n_k}))-D_f(p,x_{n_k}))=0.$$

We divide the remaining proof of the theorem into two cases.

Case 1. Suppose $D_f(p,x_{n+1})\le D_f(p,x_n)$ for all sufficiently large n. Hence the sequence $D_f(p,x_n)$ is bounded and non-increasing. Thus, we have that $\lim_{n\to\infty}D_f(p,x_n)$ exists. Therefore,

$$\lim_{n\to\infty}(D_f(p,x_{n+1})-D_f(p,x_n))=0$$

and hence

$$\lim_{n\to\infty}(D_f(p,Res_T^f\circ A^f(x_n))-D_f(p,x_n))=0.$$

Since $Res_T^f\circ A^f$ is a Bregman strongly nonexpansive mapping, we have that

$$\lim_{n\to\infty}D_f(x_n,Res_T^f\circ A^f(x_n))=0.$$

Moreover, since f is totally convex on bounded subsets of E, it follows from Lemma 2.2 that

$$\lim_{n\to\infty}||x_n-Res_T^f\circ A^f(x_n)||=0.$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q \in E$ and

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), x_n - p\rangle = \langle \nabla f(u) - \nabla f(p), q - p\rangle.$$

Again, since $||x_{n_k} - Res_T^f \circ A^f(x_{n_k})|| \to 0$ as $k \to \infty$, we have that

$$q \in \widehat{F}(Res_T^f \circ A^f) = F(Res_T^f \circ A^f).$$

From Lemma 2.1, we obtain that

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), x_n - p\rangle = \langle \nabla f(u) - \nabla f(p), q - p\rangle \leq 0.$$

Finally, we show that $x_n \to p$. From (2.5), we obtain

$$\begin{aligned}
D_f(p, x_{n+1}) &= V_f(p, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(Res_T^f \circ A^f(x_n))) \\
&\leq V_f(p, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(Res_T^f \circ A^f(x_n)) - \alpha_n(\nabla f(u) - \nabla f(p))) \\
&+ \langle \alpha_n(\nabla f(u) - \nabla f(p)), x_{n+1} - p\rangle \\
&= V_f(p, \alpha_n \nabla f(p) + (1-\alpha_n)\nabla f(Res_T^f \circ A^f(x_n))) \\
&+ \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\
&\leq \alpha_n V_f(p, \nabla f(p)) + (1-\alpha_n)V_f(p, \nabla f(Res_T^f \circ A^f(x_n))) \\
&+ \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\
&= (1-\alpha_n)D_f(p, Res_T^f \circ A^f(x_n)) + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle \\
&\leq (1-\alpha_n)D_f(p, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(p), x_{n+1} - p\rangle.
\end{aligned}$$

By Lemma 2.10, we can conclude that $\lim_{n\to\infty} D_f(p, x_n) = 0$. Therefore, by Lemma 2.2, $x_n \to p$ since f is totally convex on bounded subsets of E.

Case 2. Suppose there exists a subsequence $D_f(p, x_{n_j})$ of $D_f(p, x_n)$ such that $D_f(p, x_{n_j}) < D_f(p, x_{n_j+1})$ for all $j \in \mathbb{N}$. Then by Lemma 2.11, there exists a strictly increasing sequence m_k of positive integers such that the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1}) \text{ and } D_f(p, x_k) \leq D_f(p, x_{m_k+1}).$$

So, we have

$$\begin{aligned}
0 &\le \lim_{k\to\infty}(D_f(p,x_{m_k+1}) - D_f(p,x_{m_k}))\\
&\le \limsup_{n\to\infty}(D_f(p,x_{n+1}) - D_f(p,x_n))\\
&\le \limsup_{n\to\infty}\Big(\alpha_n D_f(p,u) + (1-\alpha_n)D_f(p, Res_T^f \circ A^f(x_n)) - D_f(p,x_n)\Big)\\
&= \limsup_{n\to\infty}\Big(\alpha_n(D_f(p,u) - D_f(p, Res_T^f \circ A^f(x_n)))\\
&\quad +(D_f(p, Res_T^f \circ A^f(x_n)) - D_f(p,x_n))\Big)\\
&\le \limsup_{n\to\infty}\alpha_n(D_f(p,u) - D_f(p, Res_T^f \circ A^f(x_n))) = 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}(D_f(p,x_{m_k+1}) - D_f(p,x_{m_k})) = 0. \tag{3.25}$$

Following the the same line of argument as in Case 1, we can have

$$\limsup_{k\to\infty}\langle \nabla f(u) - \nabla f(p), x_{m_k} - p\rangle \le 0,$$

and

$$D_f(p,x_{m_k+1}) \le (1-\alpha_{m_k})D_f(p,x_{m_k}) + \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p\rangle.$$

Therefore,

$$\begin{aligned}
\alpha_{m_k} D_f(p,x_{m_k}) &\le D_f(p,x_{m_k}) - D_f(p,x_{m_k+1})\\
&\quad + \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p\rangle\\
&\le \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p\rangle,
\end{aligned}$$

that is

$$D_f(p,x_{m_k}) \le \langle \nabla f(u) - \nabla f(p), x_{m_k+1} - p\rangle.$$

Hence $\limsup_{k\to\infty} D_f(p,x_{m_k}) = 0$. Using this and (3.25) together, we conclude that

$$\limsup_{k\to\infty} D_f(p,x_k) \le \lim_{k\to\infty} D_f(p,x_{m_k+1}) = 0.$$

This completes the proof. □

Corollary 3.9. ([56], Thm 5.1) *Let E be a real reflexive Banach space and $f : E \to \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $T : E \to 2^{E^*}$ be a maximal monotone operator. Suppose $\Gamma := (T)^{-1}(0) \neq \emptyset$. Suppose that $u \in E$ and define the sequence x_n as follows: $x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(Res_T^f x_n)), \forall n \ge 1, \tag{3.26}$$

where $\alpha_n \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $Proj_\Gamma^f(u)$.

Remark 3.10. Our Theorem 3.8 extends the results of Kamimura and Takahashi [27] from inclusion problem involving a maximal monotone operator in Hilbert spaces to inclusion problem (2.8) reflexive Banach spaces. Furthermore, our Theorem 3.8 extends the results of [30] and [61] from uniformly convex Banach space which is also uniformly smooth to reflexive Banach space.

4. Applications

4.1. **Application to convex minimization problem.** Let E be a real reflexive Banach space and $\phi : E \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semi-continuous functional. We consider the following Convex Minimization Problem (CMP): find $x^* \in E$ such that

$$x^* = \underset{x\in E}{\operatorname{argmin}}\ \phi(x). \tag{4.1}$$

We know that the subdifferential $\partial\phi$ is maximal monotone and

$$0 \in \partial\phi x (x = Res^f_{\partial\phi}(x))$$

if and only if x solves CMP (4.1). Here, the resolvent operator $Res^f_{\partial\phi} = prox_\phi$ where

$$\operatorname{prox}_\phi x = \underset{y\in E}{\operatorname{argmin}}\{\phi(y) + \frac{1}{2}D_f(y,x)\},$$

for each $x \in E$ (see [46] for more details). Thus, if $A = 0$ and $T = \partial\phi$ in Theorems 3.3, 3.5, 3.6 and 3.8, we obtain convergence results for approximating solutions of CMP (4.1).

4.2. **Application to mixed variational inequality problem.** Here, we consider the following mix variational inequality problem:

$$\text{Find } x \in \operatorname{intdom} f \text{ such that } \exists z \in Ax : [\langle z, y - x\rangle \geq \varphi(x) - \varphi(y)\ \forall y \in \operatorname{dom} f], \tag{4.2}$$

where $\varphi : E \to \mathbb{R}$ is a proper convex and lower semi-continuous function and $A : E \to E^*$ be an operator which satisfy the condition:

$$\emptyset \neq \operatorname{dom} A \cap \operatorname{intdom} f \text{ and } \operatorname{ran}(\nabla f - A) \subseteq \operatorname{intdom} f^*. \tag{4.3}$$

Lemma 4.1. ([14]) *Suppose $\varphi : E \to \mathbb{R}$ is a proper convex and lower semi-continuous function. For any $z \in \operatorname{intdom} f^*$ there exists a unique global minimizer, denoted $Prox^f_\varphi(z)$ of the function $\varphi(.) + V_f(., z)$. The vector $Prox^f_\varphi(z)$ is contained in $\operatorname{dom}\partial\varphi \cap \operatorname{intdom} f$ and we have $Prox^f_\varphi(z) = (\partial\varphi + \nabla f)^{-1}(z)$.*

Observe that since the proximal mapping $prox^f_\varphi = (\partial\varphi + \nabla f)^{-1} \circ \nabla f$ (see [4]), then $prox^f_\varphi = Prox^f_\varphi \circ \nabla f$.

Lemma 4.2. ([14]) *Suppose $\varphi : E \to \mathbb{R}$ is a proper convex and lower semi-continuous function and $z \in \operatorname{intdom} f^*$. If $\widehat{x} \in \operatorname{dom}\partial\varphi \cap \operatorname{intdom} f$ then the following conditions are equivalent:*
(a) $\widehat{x} = Prox^f_\varphi(z)$;
(b) $\widehat{x}$ is a solution of the variational inequality

$$\langle z - \nabla f(x), y - x\rangle \leq \varphi(y) - \varphi(x),\ \forall y \in \operatorname{dom}\varphi \cap \operatorname{dom} f;$$

(c) $\widehat{x}$ *is a solution of the variational inequality*

$$V_f(z,x) + V_f(\nabla f(x), y) - V_f(z,y) \leq \varphi(y) - \varphi(x), \ \forall y \in \text{dom}\varphi \cap \text{dom}f.$$

Theorem 4.3. *Let* $\varphi : E \to \mathbb{R}$ *be a proper convex and lower semi-continuous function with* $\bar{x} \in \text{dom}\partial\varphi \cap \text{intdom}f$ *and suppose* $A : E \to E^*$ *is a Bregman inverse strongly monotone operator. The* $\widehat{x}$ *is a solution of the variational inequality* (4.2) *if and only if* $0 \in (\partial\varphi + A)\bar{x}$.

Proof. Clearly, $\widehat{x}$ is a solution of (4.2) if and only if there exists $z \in E^*$ such that $z = A\widehat{x}$ and

$$\langle (\nabla f(\widehat{x}) - z) - \nabla f(\widehat{x}), y - \bar{x} \rangle \leq \varphi(y) - \varphi(\widehat{x}), \ \forall y \in \text{dom}\varphi \cap \text{dom}f. \tag{4.4}$$

Thus from Lemma 4.2, we have that

$$\begin{aligned} \widehat{x} &= Prox^f_\varphi(\nabla f(\widehat{x}) - z) \\ &= Prox^f_\varphi(\nabla f(\widehat{x}) - A\widehat{x}) \\ &= prox^f_\varphi \circ A^f(\widehat{x}) \\ &= Res^f_{\partial\varphi} \circ A^f(\widehat{x}). \end{aligned} \tag{4.5}$$

That is $\widehat{x} \in F(Res^f_{\partial\varphi} \circ A^f)$. Therefore it follows from the maximal monotonicity of $\partial\varphi$ and Proposition 3.1 (a) that $0 \in (\partial\varphi + A)\widehat{x}$. □

Theorem 4.3 shows that we can apply our results in Theorems 3.3, 3.5, 3.6 and 3.8 to approximate solutions of (4.2).

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