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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS ON BIPOLAR METRIC SPACES

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Abstract. In this article, we introduce concepts of Pompeiu-Hausdorff bipolar metric, multivalued covariant and contravariant contraction mappings in bipolar metric spaces. In addition to these, we express two main fixed point theorems, which are supported with four important corollaries, related to these multivalued mappings. Finally we give an example which presents the applicability of our obtained results.

Key Words and Phrases: Bipolar metric space, multivalued contraction, fixed point.
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1. Introduction

In 2016, Mutlu and Gürdal [15] introduced the notion of bipolar metric space as a type of partial distance. Moreover, they stated the link between metric spaces and bipolar metric spaces, especially in the context of completeness, and gave some extensions of known fixed point theorems as Banach's and Kannan's. After that, Mutlu, Özkan and Gürdal [16] gave some generalizations of coupled fixed point theorems in these metric spaces.

There exist various generalizations of the Banach contraction principle. One of them belongs to Nadler. In 1969, Nadler [17] introduced the concept of multivalued contraction mapping and he gave some important results, examples and many elementary fixed point theorems for these contraction mappings. After that, many authors examined multivalued contraction mappings in various metric spaces and they expressed Nadler's and some well known fixed point theorems for these contraction mappings [4, 9, 10, 12, 14]. Recently, some authors continue to study on this area [1, 2, 3, 5, 6, 7, 8, 11, 13, 18, 19, 20, 21, 22].

In this paper, we introduce the concepts of Pompeiu-Hausdorff bipolar metric, multivalued covariant and contravariant contraction mappings in bipolar metric spaces.

In addition to these, we express the two main fixed point theorems, which are supported with four important corollaries, related to these multivalued mappings. Finally we give an example which presents the applicability of our obtained results.

2. Bipolar metric spaces

In this section, we give some definitions and notions related to bipolar metric spaces.

Definition 2.1. [15] A bipolar metric space is a triple (X, Y, d) such that $X, Y \neq \emptyset$ and $d : X \times Y \to \mathbb{R}^+$ is a function which satisfies the properties

(B0) if $d(x, y) = 0$, then $x = y$,
(B1) if $x = y$, then $d(x, y) = 0$,
(B2) if $x, y \in X \cap Y$, then $d(x, y) = d(y, x)$,
(B3) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$,

for all $(x, y), (x_1, y_1), (x_2, y_2) \in X \times Y$, where $\mathbb{R}^+$ denotes the set of all non-negative real numbers. Then d is called a bipolar metric on the pair (X, Y).

Definition 2.2. [15] Let (X_1, Y_1) and (X_2, Y_2) be pairs of sets and given a function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$. If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$, we call f a covariant map from (X_1, Y_1) to (X_2, Y_2) and denote this with $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$. If $f(X_1) \subseteq Y_2$ and $f(Y_1) \subseteq X_2$, then we call f a contravariant map from (X_1, Y_1) to (X_2, Y_2) and write $f : (X_1, Y_1) \nearrow\!\!\!\!\swarrow (X_2, Y_2)$. In particular, if d_1 and d_2 are bipolar metrics on (X_1, Y_1) and (X_2, Y_2), respectively, we sometimes use the notations $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $f : (X_1, Y_1, d_1) \nearrow\!\!\!\!\swarrow (X_2, Y_2, d_2)$.

Definition 2.3. [15] Let (X, Y, d) be a bipolar metric space. A point $u \in X \cup Y$ is called a left point if $u \in X$, a right point if $u \in Y$ and a central point if it is both left and right point. Similarly a sequence (x_n) on the set X is called a left sequence and a sequence (y_n) on Y is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence. A sequence (u_n) is said to be convergent to a point u, iff (u_n) is a left sequence, u is a right point and $\lim_{n\to\infty} d(u_n, u) = 0$; or (u_n) is a right sequence, u is a left point and $\lim_{n\to\infty} d(u, u_n) = 0$. A bisequence (x_n, y_n) on (X, Y, d) is a sequence on the set $X \times Y$. If the sequences (x_n) and (y_n) are convergent, then the bisequence (x_n, y_n) is said to be convergent, and if (x_n) and (y_n) converge to a common point, then (x_n, y_n) is called biconvergent. (x_n, y_n) is a Cauchy bisequence, if $\lim_{n,m\to\infty} d(x_n, y_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent. A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 2.4. [15] Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces.

(1) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at a point $x_0 \in X_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x_0, y) < \delta$ implies $d_2(f(x_0), f(y)) < \varepsilon$ all $y \in Y_1$.
(2) A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right-continuous at a point $y_0 \in Y_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x, y_0) < \delta$ implies $d_2(f(x), f(y_0)) < \varepsilon$ for all $x \in X_1$.

(3) A map f is called continuous, if it is left-continuous at each point $x \in X_1$ and right-continuous at each point $y \in Y_1$.
(4) A contravariant map $f : (X_1, Y_1, d_1) \nearrow\!\!\!\!\searrow (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (Y_2, X_2, d_2)$

It can be seen from the definition that a covariant or a contravariant map f from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) is continuous if and only if $(u_n) \to v$ on (X_1, Y_1, d_1) implies $(f(u_n)) \to f(v)$ on (X_2, Y_2, d_2).

3. Main results

Definition 3.1. Let (X, Y, d) be a bipolar metric space. A set $A \subseteq X \cup Y$ is called closed if every limit of convergent sequence in A belong to A.

Definition 3.2. Let (X, Y, d) be a bipolar metric space.

(1) A set $A \subseteq X$ is called bounded if $\delta(A) = \sup\{d(a, y) : a \in A\} < \infty$ for all $y \in Y$.
(2) A set $B \subseteq Y$ is called bounded if $\delta(B) = \sup\{d(x, b) : b \in B\} < \infty$ for all $x \in X$.

Definition 3.3. Let (X, Y, d) be a bipolar metric space. We denote

$$\begin{aligned} CB(X) &= \{A : A \text{ is a nonempty closed and bounded subset of } X\} \\ CB(Y) &= \{B : B \text{ is a nonempty closed and bounded subset of } Y\} \\ D(a, B) &= \inf\{d(a, b) : b \in B \subset Y\},\ a \in X \\ D(A, b) &= \inf\{d(a, b) : a \in A \subset X\},\ b \in Y \\ H(A, B) &= \max\{\sup\{D(a, B) : a \in A\}, \sup\{D(A, b) : b \in B\}\} \end{aligned}$$

for all $A \in CB(X)$ and $B \in CB(Y)$. Then H is a bipolar metric on $(CB(X), CB(Y))$, called the Pompeiu-Hausdorff bipolar metric induced by bipolar metric d.

Definition 3.4. Let (X, Y, d) be a bipolar metric space.

(1) A covariant mapping $T : (X, Y) \rightrightarrows (CB(X), CB(Y))$ is said to be a multivalued covariant contraction mapping if there exists a fixed real number $\lambda \in (0, 1)$ such that
$$H(Tx, Ty) \leq \lambda d(x, y) \tag{3.1}$$
for all $x \in X$ and $y \in Y$.
(2) A contravariant mapping $T : (X, Y) \nearrow\!\!\!\!\searrow (CB(X), CB(Y))$ is said to be a multivalued contravariant contraction mapping if there exists a fixed real number $\lambda \in (0, 1)$ such that
$$H(Ty, Tx) \leq \lambda d(x, y) \tag{3.2}$$
for all $x \in X$ and $y \in Y$.
(3) A point $u \in X \cup Y$ is called a fixed point of a multivalued covariant or contravariant map T, if $u \in Tu$.

Lemma 3.5. *Let (X,Y,d) be a bipolar metric space, $A \in CB(X)$, $B \in CB(Y)$ and $\varepsilon > 0$. Then there exists a $b = b(a) \in B$ for any $a \in A$ (or there exists an $a = a(b) \in A$ for any $b \in B$) such that*

$$d(a,b) \le H(A,B) + \varepsilon. \tag{3.3}$$

Proof. As $\varepsilon > 0$, from definition of $D(a,B)$, $D(A,b)$ and $H(A,B)$, it is clear that there exists a $b \in B$ for any $a \in A$ such that

$$d(a,b) \le D(a,B) + \varepsilon \le H(A,B) + \varepsilon$$

and there exists an $a \in A$ for any $b \in B$ such that

$$d(a,b) \le D(A,b) + \varepsilon \le H(A,B) + \varepsilon. \qquad \square$$

Lemma 3.6. *Let (X,Y,d) be a bipolar metric space, $A \in CB(X)$, $B \in CB(Y)$ and $h \in (0,1)$. Then there exists $b = b(a) \in B$ for any $a \in A$ (or there exists $a = a(b) \in A$ for any $b \in B$) such that*

$$hd(a,b) \le H(A,B). \tag{3.4}$$

Proof. If $H(A,B) = 0$, then $a \in B$ and for $b = a$ the inequality (3.4) is satisfied. If $H(A,B) > 0$, then, since $h \in (0,1)$, we can take

$$\varepsilon = (h^{-1} - 1)H(A,B) > 0. \tag{3.5}$$

From Lemma 3.5 and equality (3.5), we obtain

$$hd(a,b) \le H(A,B). \qquad \square$$

Theorem 3.7. *Let (X,Y,d) be a complete bipolar metric space. If*

$$T : \ (X,Y) \rightrightarrows (CB(X), CB(Y))$$

is a multivalued covariant contraction mapping, then
(i) T has at least one fixed point;
(ii) for each $(x_0,y_0) \in X \times Y$ there exists a sequence $(x_n,y_n) \in X \times Y$ such that $x_{n+1} \in T(x_n)$, $y_{n+1} \in T(y_n)$ and $x_n \to u$, $y_n \to u$ as $n \to \infty$, with $u \in T(u)$.

Proof.
(i) Let $x_0 \in X$, $y_0 \in Y$ and $h = \sqrt{\lambda}$. Denote $x_1 \in Tx_0$. From Lemma 3.6, we can choose

$$\begin{array}{rcl} \exists y_1 \in Ty_0 & ; & hd(x_1,y_1) \le H(Tx_0,Ty_0), \\ \exists x_2 \in Tx_1 & ; & hd(x_2,y_1) \le H(Tx_1,Ty_0), \\ \exists y_2 \in Ty_1 & ; & hd(x_2,y_2) \le H(Tx_1,Ty_1), \\ & & \vdots \\ \exists y_n \in Ty_{n-1} & ; & hd(x_n,y_n) \le H(Tx_{n-1},Ty_{n-1}), \\ \exists x_{n+1} \in Tx_n & ; & hd(x_{n+1},y_n) \le H(Tx_n,Ty_{n-1}), \end{array}$$

for $n \ge 1$. So, we obtain a bisequence $(x_n,y_n) \in X \times Y$. Thus, from (3.1) we get

$$\begin{array}{rcl} hd(x_n,y_n) & \le & \lambda d(x_{n-1},y_{n-1}) = h^2 d(x_{n-1},y_{n-1}) \\ \Rightarrow d(x_n,y_n) & \le & hd(x_{n-1},y_{n-1}) \end{array}$$

$$\begin{array}{rcl} hd(x_{n+1},y_n) & \le & \lambda d(x_n,y_{n-1}) = h^2 d(x_n,y_{n-1}) \\ \Rightarrow d(x_{n+1},y_n) & \le & hd(x_n,y_{n-1}) \end{array}$$

Repeating this process $n-$times, we get

$$\begin{array}{rcl} d(x_n, y_n) & \leq & h^n d(x_0, y_0) \\ d(x_{n+1}, y_n) & \leq & h^n d(x_1, y_0). \end{array}$$

For any $n, m \in N$ with $n \leq m$, we obtain

$$\begin{array}{rcl} d(x_n, y_m) & \leq & d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) + d(x_{n+2}, y_{n+1}) \\ & & + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m), \\ & \leq & h^n d(x_0, y_0) + h^n d(x_1, y_0) + h^{n+1} d(x_0, y_0) + h^{n+1} d(x_1, y_0) \\ & & + \cdots + h^{m-1} d(x_1, y_0) + h^m d(x_0, y_0), \\ & = & (h^n + h^{n+1} + \cdots + h^m) d(x_0, y_0) \\ & & + (h^n + h^{n+1} + \cdots + h^m) d(x_1, y_0), \\ & = & (\frac{h^n}{1-h}) d(x_0, y_0) + (\frac{h^n}{1-h}) d(x_1, y_0). \end{array}$$

Thus, $d(x_n, y_m) \to 0$ as $n, m \to \infty$. Similarly, for any $n, m \in \mathbb{N}$ with $m < n$ we obtain

$$\begin{array}{rcl} d(x_n, y_m) & \leq & d(x_n, y_{n-1}) + d(x_{n-1}, y_{n-1}) + d(x_{n-1}, y_{n-2}) + d(x_{n-1}, y_{n-2}) \\ & & + \cdots + d(x_{m+1}, y_{m+1}) + d(x_{m+1}, y_m), \\ & \leq & h^{n-1} d(x_1, y_0) + h^{n-1} d(x_0, y_0) + h^{n-2} d(x_1, y_0) + h^{n-2} d(x_0, y_0) \\ & & + \cdots + h^{m+1} d(x_0, y_0) + h^m d(x_1, y_0), \\ & = & (h^{n-1} + h^{n-2} + \cdots + h^m) d(x_1, y_0) \\ & & + (h^{n-1} + h^{n-2} + \cdots + h^{m+1}) d(x_0, y_0), \\ & = & (\frac{h^{n-1}}{1-h}) d(x_1, y_0) + (\frac{h^{n-1}}{1-h}) d(x_0, y_0). \end{array}$$

Then $d(x_n, y_m) \to 0$ as $n, m \to \infty$. We conclude that (x_n, y_n) is a Cauchy bisequence. Since (X, Y, d) is complete, (x_n, y_n) converges (in particular biconverges) to some $u \in X \cap Y$. So,

$$\lim_{n\to\infty} x_n = u \text{ and } \lim_{n\to\infty} y_n = u.$$

From hypothesis and property (B3) we have that

$$\begin{array}{rcl} D(u, Tu) & \leq & d(u, y_{n+1}) + d(x_{n+1}, y_{n+1}) + D(x_{n+1}, Tu), \\ & \leq & d(u, y_{n+1}) + d(x_{n+1}, y_{n+1}) + H(Tx_n, Tu), \\ & \leq & d(u, y_{n+1}) + d(x_{n+1}, y_{n+1}) + \lambda(Tx_n, Tu) \to 0 \end{array}$$

as $n \to \infty$. Thus, $D(u, Tu) = 0$. Since Tu is closed, we get $u \in Tu$. Hence u is a fixed point of T.
(ii) The sequences (x_n) and (y_n) that are defined in the proof of *(i)*, satisfies the conditions in *(ii)*. □

Theorem 3.8. *Let (X, Y, d) be a complete bipolar metric space. If*

$$T : (X, Y) \rightleftarrows (CB(X), CB(Y))$$

is a multivalued contravariant mapping such that

$$H(Ty, Tx) \leq \alpha d(x, y) + \beta[D(x, Tx) + D(Ty, y)] \tag{3.6}$$

for all $x \in X$ and $y \in Y$ where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$. Then T has a fixed point.

Proof. Let $x_0 \in X$, $y_0 \in Tx_0$ and $r = \frac{\alpha+\beta}{1-\beta}$. We choose an $x_1 \in Ty_0$. Then it follows from Lemma 3.6 that

$$\begin{array}{rcl} \exists y_1 \in Tx_1 & ; & d(x_1, y_1) \leq H(Ty_0, Tx_1) + r, \\ \exists x_2 \in Ty_1 & ; & d(x_2, y_1) \leq H(Ty_1, Tx_1) + r^2, \\ \exists y_2 \in Tx_2 & ; & d(x_2, y_2) \leq H(Ty_1, Tx_2) + r^3, \\ & & \vdots \\ \exists y_n \in Tx_n & ; & d(x_n, y_n) \leq H(Ty_{n-1}, Tx_n) + r^{2n-1}, \\ \exists x_{n+1} \in Ty_n & ; & d(x_{n+1}, y_n) \leq H(Ty_n, Tx_n) + r^{2n}, \end{array}$$

for $n \geq 1$. From (3.6) and Lemma 3.5, we get

$$\begin{array}{rcl} d(x_n, y_n) & \leq & \alpha d(x_n, y_{n-1}) + \beta[D(x_n, Tx_n) + D(Ty_{n-1}, y_{n-1})] + r^{2n-1} \\ & \leq & \alpha d(x_n, y_{n-1}) + \beta[d(x_n, y_n) + d(x_n, y_{n-1})] + r^{2n-1} \end{array}$$

implies

$$\begin{array}{rcl} d(x_n, y_n) & \leq & \frac{\alpha+\beta}{1-\beta} d(x_n, y_{n-1}) + \frac{r^{2n-1}}{1-\beta} \\ & = & r d(x_n, y_{n-1}) + \frac{r^{2n-1}}{1-\beta} \end{array}$$

and

$$\begin{array}{rcl} d(x_n, y_{n-1}) & \leq & \alpha d(x_{n-1}, y_{n-1}) + \beta[D(x_{n-1}, Tx_{n-1}) + D(Ty_{n-1}, y_{n-1})] + r^{2n-2} \\ & \leq & \alpha d(x_{n-1}, y_{n-1}) + \beta[d(x_{n-1}, y_{n-1}) + d(x_n, y_{n-1})] + r^{2n-2} \end{array}$$

implies

$$\begin{array}{rcl} d(x_n, y_{n-1}) & \leq & \frac{\alpha+\beta}{1-\beta} d(x_{n-1}, y_{n-1}) + \frac{r^{2n-2}}{1-\beta} \\ & = & r d(x_{n-1}, y_{n-1}) + \frac{r^{2n-2}}{1-\beta}. \end{array}$$

Then we conclude that

$$\begin{array}{rcl} d(x_n, y_n) & \leq & r^2 d(x_{n-1}, y_{n-1}) + \frac{r^{2n-2}}{1-\beta} + \frac{r^{2n-1}}{1-\beta} \\ & \vdots & \\ & \leq & r^{2n} d(x_0, y_0) + \sum_{i=1}^{2n} r^i \frac{1}{1-\beta} \\ & \leq & r^{2n} d(x_0, y_0) + \frac{(2n) r^{2n}}{1-\beta}. \end{array}$$

On the other hand, similarly we get

$$\begin{array}{rcl} d(x_{n+1}, y_n) & \leq & \alpha d(x_n, y_n) + \beta[D(x_n, Tx_n) + D(Ty_n, y_n)] + r^{2n} \\ & \leq & \alpha d(x_n, y_n) + \beta[d(x_n, y_n) + d(x_{n+1}, y_n)] + r^{2n} \end{array}$$

implies

$$\begin{array}{rcl} d(x_{n+1}, y_n) & \leq & \frac{\alpha+\beta}{1-\beta} d(x_n, y_n) + \frac{r^{2n}}{1-\beta} \\ & = & r d(x_n, y_n) + \frac{r^{2n}}{1-\beta} \\ & \vdots & \\ & \leq & r^{2n+1} d(x_0, y_0) + \sum_{i=1}^{2n+1} r^i \frac{1}{1-\beta} \\ & \leq & r^{2n+1} d(x_0, y_0) + \frac{(2n+1) r^{2n+1}}{1-\beta}. \end{array}$$

For any $n, m \in N$ with $n \leq m$, we have

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_{n+1}) \\ &\quad + \cdots + d(x_m, y_{m-1}) + d(x_m, y_m) \\ &\leq r^{2n} d(x_0, y_0) + \tfrac{2nr^{2n}}{1-\beta} + r^{2n+1} d(x_0, y_0) + \tfrac{(2n+1)r^{2n+1}}{1-\beta} \\ &\quad + r^{2n+2} d(x_0, y_0) + \tfrac{(2n+2)r^{2n+2}}{1-\beta} + \cdots + r^{2m-1} d(x_0, y_0) \\ &\quad + \tfrac{(2m-1)r^{2m-1}}{1-\beta} + r^{2m} d(x_0, y_0) + \tfrac{(2m)r^{2m}}{1-\beta} \\ &\vdots \\ &\leq (r^{2n} + r^{2n+1} + \cdots + r^{2m}) d(x_0, y_0) \\ &\quad + \tfrac{(2n)r^{2n} + (2n+1)r^{2n+1} + \cdots + (2m)r^{2m}}{1-\beta}. \end{aligned}$$

Since $r < 1$, $d(x_n, y_m) \to 0$ as $n, m \to \infty$. Similarly, if $m < n$, it is clear that $d(x_n, y_m) \to 0$ for $n, m \to \infty$. Then (x_n, y_m) is a Cauchy bisequence. Since (X, Y, d) is complete bipolar metric space, (x_n, y_n) converges (in particular biconverges) to a point $u \in X \cap Y$. Then

$$\lim_{n\to\infty} x_n = u \text{ and } \lim_{n\to\infty} y_n = u.$$

From hypothesis and property (B3) we have that

$$\begin{aligned} D(u, Tu) &\leq d(u, y_{n+1}) + d(x_{n+1}, y_{n+1}) + D(x_{n+1}, Tu) \\ &\leq d(u, y_{n+1}) + d(x_{n+1}, y_{n+1}) + H(Ty_n, Tu) \\ &\leq d(u, y_{n+1}) + d(x_{n+1}, y_{n+1}) + \alpha d(u, y_n) + \beta[D(Ty_n, y_n) + D(u, Tu)] \\ &\leq d(u, y_{n+1}) + d(x_{n+1}, y_{n+1}) + \alpha d(u, y_n) + \beta[d(x_{n+1}, y_n) + D(u, Tu)] \end{aligned}$$

for all $n \in \mathbb{N}$. As $n \to \infty$, we get

$$D(u, Tu) \leq \beta D(u, Tu).$$

Since $b < 1$, $D(u, Tu) = 0$. This implies $u \in Tu$. Then u is a fixed point of T. □

From Theorem 3.8, we obtain the following corollaries.

Corollary 3.9. *Let (X, Y, d) be a complete bipolar metric space and*

$$T : (X, Y) \rightleftarrows (X, Y)$$

be a contravariant mapping such that

$$d(Ty, Tx) \leq \alpha d(x, y) + \beta[d(x, Tx) + d(Ty, y)] \tag{3.7}$$

for all $x \in X$ and $y \in Y$, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$. Then T has a fixed point.

Corollary 3.10. *Let (X, Y, d) be a complete bipolar metric space and*

$$T : (X, Y) \rightleftarrows (CB(X), CB(Y))$$

be a multivalued contravariant mapping such that

$$H(Ty, Tx) \leq a_1 d(x, y) + a_2 D(x, Tx) + a_3 D(Ty, y) \tag{3.8}$$

for all $x \in X$ and $y \in Y$ where $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 < 1$. Then T has a fixed point.

Corollary 3.11. *Let (X,Y,d) be a complete bipolar metric space. If*

$$T:(X,Y) \rightleftarrows (CB(X),CB(Y))$$

is a multivalued contravariant mapping, then T has a fixed point.

Corollary 3.12. *Let (X,Y,d) be a complete bipolar metric space and*

$$T:(X,Y) \rightleftarrows (CB(X),CB(Y))$$

be a multivalued contravariant mapping such that

$$H(Ty,Tx) \leq \beta[D(x,Tx)+D(Ty,y)] \tag{3.9}$$

for all $x \in X$ and $y \in Y$ where $\beta \in \left[0,\frac{1}{2}\right)$. Then T has a fixed point.

Example 3.13. Let $X=\{0,1,2\}$ and $Y=\{1,2,3\}$. Define $d: X\times Y \to \mathbb{R}^+$ such that

$$\begin{aligned}
&d(1,1)=d(2,2)=0,\\
&d(1,2)=d(2,1)=5,\\
&d(0,1)=13,\ d(0,2)=12,\\
&d(0,3)=15,\ d(1,3)=14,\ d(2,3)=9.
\end{aligned}$$

Then (X,Y,d) is a complete bipolar metric space. A covariant mapping

$$T:\ (X,Y) \rightrightarrows (CB(X),CB(Y))$$

be defined by

$$T(0)=\{1\},\ T(1)=T(2)=\{2\},\ T(3)=\{1,2\}.$$

Note that, Tx and Ty are closed and bounded for all $x \in X$ and $y \in Y$ with respect to the bipolar metric space (X,Y,d). Then we conclude that the condition

$$H(Tx,Ty) \leq kd(x,y)$$

for all $x \in X$ and $y \in Y$ is satisfied for the constant $k=\frac{5}{9}$. From Theorem 3.1, we say that T has a fixed point. It is $2 \in X \cap Y$.

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