Fixed Point Theory, 21(2020), No. 1, 259-270
DOI: 10.24193/fpt-ro.2020.1.18
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SEVERAL FIXED POINT THEOREMS ON PARTIALLY ORDERED BANACH SPACES AND THEIR APPLICATIONS TO INTEGRAL EQUATIONS

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Abstract. In this paper, we prove several fixed point theorems on partially ordered Banach spaces, in which the considered mappings satisfy some order monotone conditions. In addition, if the considered mappings satisfy some continuity conditions, then some iterated schemes can be constructed to approximate the fixed points. Then we apply these theorems to prove an existence theorem of solutions to some Hammerstein integral equations.
Key Words and Phrases: Partially ordered Banach space, regular partially ordered Banach space, fixed point, C-continuity of set-valued mappings, Hammerstein integral equations.
2010 Mathematics Subject Classification: 06F30, 06F30, 45G10, 47H10.

1. Introduction

In 1955, Tarski [22] first proved a fixed point theorem on complete lattices for single-valued mappings. It was extended to chain-complete partially ordered sets by Abian and Brown [1] in 1961. In 1984, Fujimoto [9] extended the Tarski's fixed point theorem from single-valued mappings to set-valued mappings on complete lattices. In 2014, Li [13] extended the Abian-Brown fixed point theorem from single-valued mappings to set-valued mappings on chain-complete partially ordered sets, which is also an extension of Fujimoto fixed point theorem from complete lattices to chain-complete partially ordered sets. In 2015, Li [17] proved the inductive properties of fixed points of some mappings on chain-complete partially ordered sets. Since then, the fixed point theorems of set-valued mappings on chain-complete partially ordered sets have been applied to equilibrium problems with incomplete preferences on partially ordered topological vector spaces; solving ordered variational inequalities in partially ordered Banach spaces; solving nonlinear Hammerstein integral equations (see [13-18]).

In all theorems proved in the above-mentioned papers, in the underlying spaces, the ordering structures are only considered (they may be equipped with neither algebraic structures nor topology structures), the considered mappings are only required to satisfy some order monotone conditions. It is clear that this is a new aspect in fixed point theory, in which some fixed point theorems are proved without any continuity conditions for the considered mappings.

On the other hand, since any non-singleton pointed closed and convex cone in a Banach space can induce a partial order on it and then, this Banach space becomes a partially ordered Banach space. In any partially ordered Banach space, in addition to the algebraic and topology structures, it is naturally (from a non-singleton pointed closed and convex cone) equipped with an ordering structure, which can play an important role in the analysis in Banach spaces. For example, the ordering structures in partially ordered Banach spaces have been used in fixed point theory (see [2-6], [21], [23]). In the fixed point theorems on partially ordered Banach spaces, the considered mappings may be required to satisfy both of order monotone conditions and some continuity conditions. These theorems have been widely applied to: solving integral equations (see [8-9], [10-12], [18]); solving differential equations and nonlinear fractional evolution equations (see [19-20], [24]); solving equilibrium problems (see [13], [25]).

In this paper, we first prove some fixed point theorems on partially ordered Banach spaces for single-valued mappings. We will take account the order monotone properties and some continuity of the considered mappings to show the existence of a fixed point. By the continuity, we construct some iteration schemes to approximate some fixed points of some mappings. Then, we introduce C-continuity of set-valued mappings. Combining the C-continuity and order monotony, we prove the existence of fixed points for some set-valued mappings on partially ordered Banach spaces. Finally, we use these theorems to prove an existence theorem of solutions to some Hammerstein integral equations, in which the conditions are relatively weaker and are simpler than that used by some publications.

2. Preliminaries

2.1. **Partially ordered Banach spaces.** Let X be a Banach space and K a non-singleton pointed closed and convex cone in X satisfying $K \neq \{0\}$. An ordering relation $\succcurlyeq$ is defined on X by K as follows:

$$y \succcurlyeq x \text{ if and only if } y - x \in K, \text{ for } x, y \in X. \tag{2.1}$$

Then $\succcurlyeq$ is a partial order on X and X, equipped with this partial order $\succcurlyeq$, is called a partially ordered Banach space induced by K. It is denoted by $(X, \succcurlyeq)$. The partial order $\succcurlyeq$ on X has the following properties:

(O_1) $x \succcurlyeq y$ implies $x + z \succcurlyeq y + z$, for $x, y, z \in X$.

(O_2) $x \succcurlyeq y$ implies $\alpha x \succcurlyeq \alpha y$, for $x, y \in X$ and $\alpha \succcurlyeq 0$.

(O_3) There are distinct points $x, y \in X$ satisfying $x \succcurlyeq y$.

(O_4) For any $u, w \in X$, with $u \preccurlyeq w$, the $\succcurlyeq$-intervals $[u)$, $(w]$ and $[u, w]$ are closed, where

$$[u) = \{x \in X : x \succcurlyeq u\},\ (w] = \{x \in X : x \preccurlyeq w\}$$

and

$$[u, w] = [u) \cap (w] = \{x \in X : u \preccurlyeq x \preccurlyeq w\}.$$

The first two properties $(O_1) - (O_2)$ are called the order-linearity of the partial order $\succcurlyeq$. The cone K is the $\succcurlyeq$-positive cone (simply written as positive cone) of $(X, \succcurlyeq)$, which is rewritten as

$$K = [0) = \{x \in X : x \succcurlyeq 0\}.$$

It is well known that if a Banach space X is equipped with a partial order $\succcurlyeq$ which satisfies the conditions $(O_1) - (O_4)$, then there is a non-singleton pointed closed and convex cone $K \subset X$ such that $\succcurlyeq$ is induced by K as defined in (2.1).

Hence, consequently, for every Banach space X there is a one-to-one correspondence between the family of partial orders satisfying conditions $(O_1) - (O_4)$ and the family of non-singleton pointed closed and convex cones in X.

Let $(X, \succcurlyeq)$, $(Y, \succcurlyeq^Y)$ be partially ordered Banach spaces. Let D and C be nonempty subsets of X and Y, respectively. A single-valued mapping f from $(D, \succcurlyeq)$ to $(C, \succcurlyeq^Y)$ is said to be order-increasing whenever $x \preccurlyeq y$ implies $f(x) \preccurlyeq^Y f(y)$. f is said to be strictly order-increasing whenever $x \prec y$ implies $f(x) \prec^T f(y)$.

Let $F : D \to 2^C \setminus \{\emptyset\}$ be a set-valued mapping. F is said to be isotonic, or to be order-increasing upward, if $x \preccurlyeq y$ in D implies that, for any $z \in Fx$, there is a $w \in Fy$ such that $z \succcurlyeq^Y w$. F is said to be order-increasing downward, if $x \preccurlyeq y$ in D implies, for any $z \in Fx$, there is a $w \in Fy$ such that $z \preccurlyeq^Y w$. F is said to be order-increasing downward, if $x \preccurlyeq y$ in D implies, for any $w \in Fy$, there is a $z \in Fx$ such that $z \succcurlyeq^Y w$. If F is both order-increasing upward and order-increasing downward, then F is said to be order-increasing.

2.2. **Fixed point theorems on partially ordered Banach spaces with continuity conditions.** As discussed in the previous section, in the literature of fixed point theory on partially ordered Banach spaces, many authors proved fixed point theorems for some mappings that, in addition to the continuity condition, have some order increasing properties. For example, Amann proved the following fixed point theorem on partially ordered Banach spaces for single-valued mappings.

Theorem 1.0. [3]. *Let $(X, \succcurlyeq)$ be a partially ordered Banach space induced by a pointed closed and convex cone K in X and let D be an order convex subset of X. Suppose that $f : D \to X$ is an $\succcurlyeq$-increasing map which is compact on every order interval in D.*

If there exist $\overline{y}, \widetilde{y} \in D$ with $\overline{y} \preccurlyeq \widetilde{y}$ such that $\overline{y} \preccurlyeq f(\overline{y})$ and $f(\widetilde{y}) \preccurlyeq \widetilde{y}$, then f has a minimal fixed point $\overline{x}$ in $(\overline{y} + K) \cap D$. Moreover, $\overline{x} \preccurlyeq \widetilde{y}$ and $\overline{x} = \lim_{n \to \infty} f^n(\overline{y})$, that is, the minimal fixed point $\overline{x}$ can be computed iteratively by means of the iteration scheme

$$x_0 = \overline{y}$$
$$x_{n+1} = f(x_n),\ n = 0, 1, 2, \ldots$$

Moreover, the sequence $\{x_n\}$ is $\succcurlyeq$-increasing.

In the above fixed point theorem, the underlying space D is supposed to be an order convex subset of the partially ordered Banach space E. That is,

$$x, y \in D \text{ implies } [x, y] \subseteq D. \tag{2.2}$$

The considered mapping, in addition to the order increasing condition, f is required to be a compact operator that is a very strong continuity condition. So, in this fixed point theorem, the considered mapping is required to satisfy both of an order increasing condition and a strong continuity condition.

2.3. **Fixed point theorems on partially ordered Banach spaces without continuity conditions.** Numerous fixed point theorems on partially ordered Banach spaces have been proved, in which the considered mappings are not required to satisfy any continuity conditions and are only required to be order monotone. Meanwhile, the underlying spaces are not required to be compact that is replaced by chain-complete condition. These theorems have been applied to solving some equilibrium problems with incomplete preferences and to solving ordered variational inequalities, etc. (see [15], [16]). Since the chain-complete condition is broader than the compactness (see Examples 2.1 and 2.2 below), from both aspects of the conditions for the considered mappings and the underlying spaces, the conditions for the fixed point theorems proved in [13], [14], [17] are weaker than the conditions in the fixed point theorems proved in [3] which are required conditions of continuity and compactness.

Before we list some fixed point theorems on partially ordered Banach spaces without continuity conditions, we recall some concepts and some results about chain-complete and universally inductive for easy reference.

Let $(X, \succcurlyeq)$ be a partially ordered Banach space and D a subset of X. $(D, \succcurlyeq)$ is said to be chain-complete if every chain in D has the least $\succcurlyeq$-upper bound that is contained in D. A partially ordered Banach space $(X, \succcurlyeq)$ is said to have *the chain-complete property* (or it is simply said to be *chain-complete*) if, for every $w \in X$, the $\succcurlyeq$-interval $(w]$ of $(X, \succcurlyeq)$ is chain-complete.

We provide the following examples to demonstrate that the chain-complete condition is broader than the compactness.

Example 2.1. Let $(R, \geq)$ be the totally ordered Banach space of real numbers with the ordinary order $\geq$. Let $D = [0, 1) \cup [2, 3]$. Then $(D, \geq)$ is $\geq$-chain-complete, which is considered as a partially ordered set. But D is not a compact subset in R.

Example 2.2. Every nonempty bounded closed and convex subset of a partially ordered reflexive Banach space is chain-complete, which may not be compact.

Now we recall some examples from [10], [12] about chain-complete subsets in partially ordered Banach spaces, which have been used in the proofs of some fixed point theorems in some publications and will be used in this paper.

Lemma 2.1. *Every non-empty compact subset in a partially ordered Banach space is chain-complete.*

Lemma 2.2. *Every non-empty norm-bounded closed and convex subset of a partially ordered reflexive Banach space is chain-complete.*

Lemma 2.3. *Let $(X, \succcurlyeq)$ be a normal partially ordered reflexive Banach space. Then very order interval $[u, v]$, for $u, v \in X$ with $u \preccurlyeq v$, is chain-complete.*

Lemma 2.4. *Every regular partially ordered Banach space has the chain-complete property.*

We found that when we prove some fixed point theorems on partially ordered Banach spaces, the concept of universally inductive subsets plays important role (it is also true even on posets, which are broader than partially ordered Banach space. In this paper, we concentrate to partially ordered Banach spaces).

A nonempty subset A of a partially ordered Banach space $(X, \succcurlyeq)$ is said to be universally inductive in X, whenever for any given chain $\{x_a\} \subseteq X$, if it satisfies that every element $x_\beta in \{x_\alpha\}$ has an $\succcurlyeq$-upper cover in A, then the chain $\{x_\alpha\}$ has an $\succcurlyeq$-upper bound in A.

Some useful universally inductive subsets in partially ordered Banach spaces (and in posets) are provided in [11], which shows that the class of universally inductive subsets in partially ordered Banach spaces are relatively broad that includes many useful cases.

Lemma 2.5. *Every nonempty compact subset of a partially ordered Banach space is universally inductive.*

Lemma 2.6. *Every nonempty bounded closed and convex subset of a partially ordered reflexive Banach space is universally inductive.*

Using the concept of universally inductive subsets, the following fixed point theorem on partially ordered Banach spaces with set-valued mappings has been proved (we only list the case that the underlying spaces are partially ordered Banach spaces. For the case of posets, which are more general, the readers are referred to [17]).

As usual, we denote by $\mathcal{F}(F)$ the set of fixed points of a set-valued mapping F.

Theorem 2.1. *Let $(X, \succcurlyeq)$ be a partially ordered Banach space and C a chain-complete subset in X and let $F : C \to 2^C \setminus \{\emptyset\}$ be a set-valued mapping satisfying the following three conditions:*

A_1. F is order-increasing upward.

A_2. $(F(x), \succcurlyeq)$ is universally inductive, for every $x \in C$.

A_3. There is an element x_0 in C and $v_0 \in F(x_0)$ with $x_0 \preccurlyeq v_0$.

Then

(i) $(\mathcal{F}(F), \succcurlyeq)$ is a nonempty inductive poset;

(ii) $(\mathcal{F}(F) \cap [x_0), \succcurlyeq)$ is a nonempty inductive poset.

Consequently, we have

(iii) F has an $\succcurlyeq$-maximal fixed point;

(iv) F has an $\succcurlyeq$-maximal fixed point x^* with $x^* \succcurlyeq x_0$.

In Theorem 2.1, the underlying set C is just required to be chain-complete. From Examples 2.1, it is a weaker condition than the compactness. The considered mapping F is only needed to have the order monotone property and with universally inductive ranges (if F is a single valued mapping, condition A_2 is automatically satisfied). In contrast with Theorem 1.0, the conditions in Theorem 2.1 are much weaker. The results in Theorem 2.1 are stranger than that in Theorem 1.0 [3], except the convergent properties of the iteratively calculated iteration scheme in which the limit is a fixed point of the considered mapping. It is because that the considered mapping in Theorem 2.1 is not required to be continuous.

Theorem 2.1 has been applied in solving ordered or vector variational inequalities, equilibrium problems with incomplete preferences, etc.

In the following sections of this paper, we will prove more fixed point theorems on partially ordered Banach spaces, in which the conditions are weaker than that in Theorem 1.0 and stranger than that in Theorem 2.1. For the case of single valued mappings, to iteratively construct an iteration scheme like Theorem 1.0 to approach a fixed point of the considered mapping, the considered mappings will be required to be continuous.

3. Fixed point theorems for mappings with chain-complete ranges

3.1. **Fixed point theorems on posets for set-valued mappings.** Notice that in the fixed point theorems proved in [1], [10], [13-18], and [22], or the theorems provided in the previous sections 1 and 2, the underlying spaces are required to be chain-complete (the regularity and inductivity of partially ordered Banach spaces imply the chain-completeness). In the applications, chain-completeness of a given partial order may not be satisfied. On the other hand, the ranges of some useful mappings (operators) may be chain-complete (for example, the compactness implies the chain-completeness). So, in this section, we prove some fixed point theorems in which the underlying spaces may not be chain-complete and the ranges of the considered mappings are chain-complete.

Theorem 3.1. *Let $(C, \succcurlyeq)$ be a poset. Let $F : C \to 2^C \setminus \{0\}$ be a set-valued mapping. Suppose that, in addition to conditions $A_1 - A_3$ in Theorem 2.1, F satisfies the following condition*

A_0. the range $R(F) = \cup\{F(x) : x \in C\}$ is $\succcurlyeq$-chain-complete.

Then F and $\mathcal{F}(F)$ have the properties (i)-(iv) stated in Theorem 2.1.

Proof. Let $F_C = F|_{R(F)}$ be the restriction of F on $R(F)$. Then F_C is a set-valued mapping. F_C and its domain $R(F)$ satisfy all conditions $A_1 - A_2$ on the chain-complete set $(R(F), \succcurlyeq)$. Next we show that F_C satisfies condition A_3. To this end, from condition A_3 for the mapping F, there is an element x_0 in C and $v_0 \in F(x_0)$ with $x_0 \preccurlyeq v_0$. It implies that $v_0 \in R(F)$. From the increasing condition on F, for $x_0 \preccurlyeq v_0$ and $v_0 \in F(x_0)$, there is $w_0 \in F(v_0)$ with $v_0 \preccurlyeq w_0$. So $w_0 \in R(F)$ and F_C satisfies condition A_3 on $R(F)$. Hence F_C and $\mathcal{F}(F_C)$ have the properties (i)-(iv) listed in Theorem 2.1. It is clear to see that $\mathcal{F}(F_C) = \mathcal{F}(F)$. It completes the proof of this theorem. □

3.2. **Fixed point theorems on partially ordered Banach spaces for set-valued mappings.** We immediately obtain the following corollaries as consequences of Theorem 3.1, as it is applied to partially ordered Banach spaces.

Corollary 3.1. *Let $(X, \succcurlyeq)$ be a partially ordered Banach space and C a subset in X. Let $F : C \to 2^C \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that, in addition to conditions $A_1 - A_3$ in Theorem 2.1, the following condition is satisfied:*

A_0. the range $R(F) = \cup\{F(x) : x \in C\}$ is chain-complete.

Then F and $\mathcal{F}(F)$ have the properties (i)-(iv) stated in Theorem 2.1.

Corollary 3.2. *Let $(X, \succcurlyeq)$ be a partially ordered Banach space and C a subset in X. Let $F : C \to 2^C \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that, in addition to conditions $A_1 - A_3$ in Theorem 2.1, the following condition is satisfied*

A_0. the range $R(F) = \cup\{F(x) : x \in C\}$ is weakly compact.

Then F and $\mathcal{F}(F)$ have the properties (i)-(iv) stated in Theorem 2.1.

Corollary 3.3. *Let $(X, \succcurlyeq)$ be a partially ordered reflexive Banach space and C a subset in X. Let $F : C \to 2^C \setminus \{\emptyset\}$ be a set-valued mapping. Suppose that, in addition to conditions $A_1 - A_3$ in Theorem 2.1, the following condition is satisfied:*

A_0. the range $R(F) = \cup\{F(x) : x \in C\}$ is norm bounded closed and convex. Then F and $\mathcal{F}(F)$ has the properties (i)-(iv) stated in Theorem 2.1.

4. Fixed point theorems on regular partially ordered Banach spaces for single-valued mappings

We recall some concepts about the normality and regularity of the partial orders that are induced by respected cones on partially ordered Banach spaces. These properties have been applied to solve some equations which includes integral equations, ordinary differential equations and partial differential equations. For details, the readers are referred to [3-4] and [10-11].

Let $(X, \succcurlyeq)$ be a partially ordered Banach space induced by a non-singleton pointed closed and convex cone K in X. If there is a constant $\lambda > 0$ such that

$$0 \preccurlyeq x \preccurlyeq y \text{ implies that } \|x\| \leq \lambda\|y\|,$$

then $\succcurlyeq$ (or the cone K) is said to be normal and $(X, \succcurlyeq)$ is called a normal partially ordered Banach space. If every $\succcurlyeq$-upper bounded and $\succcurlyeq$-increasing sequence $\{x_n\}$ of X is a $\|\cdot\|$-convergent sequence, that is,

$$x_1 \preccurlyeq x_2 \preccurlyeq \ldots \preccurlyeq y, \text{ for some } y \in X$$

$$\Rightarrow \text{there is } x \in X \text{ such that } \|x_n - x\| \to 0 \text{ as } n \to \infty,$$

then $\succcurlyeq$ is said to be regular and $(X, \succcurlyeq)$ is called a regular partially ordered Banach space. It is well known that the regularity implies the normality and the following statements are equivalent:
(i) $\succcurlyeq$ is normal;
(ii) the norm $\|\cdot\|$ has an equivalent norm $\|\cdot\|_1$ such that $0 \preccurlyeq x \preccurlyeq y$ implies $\|x\|_1 \leq \|y\|_1$;
(iii) every $\succcurlyeq$-interval $[x, y] = \{z \in X : x \preccurlyeq z \preccurlyeq y\}$ is $\|\cdot\|$-bounded.

Some fixed point theorems on regular partially ordered Banach spaces for set valued mappings have been proved in [17]. We list one of them below.

Theorem 3.9 in [17]. *Let $(X, \succcurlyeq)$ be a regular partially ordered Banach space. Let C be a closed inductive subset of X. Let $F : C \to 2^C \setminus \{\emptyset\}$ be a set-valued isotonic mapping with universally inductive values. Suppose that there are points $x_0 \in C$, $x_1 \in Fx_0$ satisfying $x_0 \preccurlyeq x_1$. Then*

(i) $(\mathcal{F}(F), \succcurlyeq)$ is a nonempty inductive subset of C;

(ii) $(\mathcal{F}(F) \cap [x_0), \succcurlyeq)$ is a nonempty inductive subset of C.

By the above theorem, we prove a fixed point theorem for single-valued mappings.

Theorem 4.1. *Let $(X, \succcurlyeq)$ be a regular partially ordered Banach space and C a closed inductive subset of X. Let $f : C \to C$ be a single-valued $\succcurlyeq$-increasing mapping. Suppose that there is a point $x_0 \in C$ satisfying $x_0 \preccurlyeq f(x_0)$. Then*

(i) $(\mathcal{F}(f), \succcurlyeq)$ is a nonempty inductive poset;

(ii) $(\mathcal{F}(f) \cap [x_0), \succcurlyeq)$ is a nonempty inductive poset;

(iii) f has an $\succcurlyeq$-maximal fixed point;

(iv) f has an $\succcurlyeq$-maximal fixed point x^* with $x^* \succcurlyeq x_0$.

We compute iteratively the following iteration scheme

$$x_{n+1} = f(x_n), \ n = 0, 1, 2, \ldots$$

Then

(a) the sequence $\{x_n\}$ *is* $\succcurlyeq$*-increasing;*

(b) $\vee\{x_n\}$ *exists.*

Furthermore, in addition, if f *is continuous, then*

$$x_n \to \vee\{x_n\}, \text{ as } n \to \infty \text{ and } \vee\{x_n\} \in \mathcal{F}(f). \tag{4.1}$$

Proof. Parts (i)-(iii) follow immediately from Theorem 3.9 in [17]. Part (a) is a consequence of the $\succcurlyeq$-increasing property of f. In [17], it is proved that every closed and inductive subset in a regular partially ordered Banach space is chain-complete. It implies (b). So, we only need to prove part (2). Since $\{x_n\}$ is $\succcurlyeq$-increasing and C is a closed and inductive subset of X, from the regularity of X, there is $x \in X$ such that

$$x_n \to x, \text{ as } n \to \infty. \tag{4.2}$$

For every n, since $[x_n)$ is closed, from (4.2), it implies that $x \in [x_n)$. That is $x_n \preccurlyeq x$, for $n = 1, 2, \ldots$ So, x is an $\succcurlyeq$-upper bound of $\{x_n\}$. It follows that

$$\vee\{x_n\} \preccurlyeq x. \tag{4.3}$$

Since the regularity implies the normality, from (4.3), we have

$$\|x - \vee\{x_n\}\| \leq \lambda\|x - x_m\|, \text{ for } m = 1, 2, \ldots, \tag{4.4}$$

where λ is the normality constant. Combining (4.2) and (4.4), we obtain

$$\vee\{x_n\} = x. \tag{4.5}$$

It proves (4.1). From (4.2), (4.5), the iterated construction of $\{x_n\}$ and the continuity of f, we have

$$f(\vee\{x_n\}) = \vee\{x_n\}.$$

□

Next, we extend Theorem 4.1 to set-valued mappings. We first need the following concept.

Definition 4.1. Let $(X, \succcurlyeq)$ be a partially ordered Banach space and C a nonempty subset of X. Let $F : C \to 2^C \setminus \{\emptyset\}$ be a set-valued mapping. F is said to be C-continuous if, for any convergent sequence $\{x_n\} \subseteq C$ with

$$x_n \to x, \text{ as } n \to \infty, \text{ for some } x \in C,$$

and, for any convergent sequence $\{y_n\} \subseteq C$ satisfying

$$y_n \in F(x_n), \text{ for } n = 1, 2, \ldots, \text{ with } y_n \to y, \text{ as } n \to \infty, \text{ for some } y \in C,$$

we have $y \in F(x)$.

Theorem 4.2. *Let* $(X, \succcurlyeq)$ *be a regular partially ordered Banach space and* C *a closed inductive subset of* X*. Let* $F : C \to 2^C \setminus \{\emptyset\}$ *be a set-valued mapping satisfying the conditions* A_1*,* A_2 *and* A_3 *listed in Theorem 2.1 in Section 2. In addition, suppose that* F *is* C*-continuous. Then,* $\mathcal{F}(F)$ *has the properties (i)-(iii) given in Theorem 2.1. Furthermore, we can iteratively choose an iteration scheme*

$$x_{n+1} \in F(x_n) \text{ such that } x_n \preccurlyeq x_{n+1}, \text{ for } n = 0, 1, 2, \ldots \tag{4.6}$$

It satisfies that $\vee\{x_n\} \in \mathcal{F}(f)$ *and*

$$x_n \to \vee\{x_n\}, \text{ as } n \to \infty.$$

Proof. Let $x_0 \in C$, $x_1 \in F(x_0)$ be the points given in condition A_3, which satisfy $x_0 \preccurlyeq x_1$. By condition A_1, there is $x_2 \in F(x_1)$ satisfying $x_1 \preccurlyeq x_2$. Repeating condition A_1, we can iterated choose a sequence $\{x_n\}$ satisfying (4.6). By using the C-continuity of F, rest of the proof is like the proof of Theorem 4.1. □

5. Applications to Hammerstein integral equations

Let (Σ, τ, μ) be a τ-compact topological space with a σ-finite measure μ satisfying $0 < \mu(\Sigma) \leq 1$. For given nonlinear real functions T and g defined on $\Sigma \times \Sigma$ and $\Sigma \times R$, respectively, we consider a nonlinear integral equation of Hammerstein type on Σ of the form

$$x(t) = \int_{\Sigma} T(t,s)g(s,x(s))d\mu(s), \tag{5.1}$$

where $T(t,s)$ is called the kernel of this Hammerstein integral equation.

Nonlinear integral equations of Hammerstein type on a measure space Σ have been studied by many authors and have a lot of applications, such as: to differential equations, to the theory of feedback of control systems, etc. For example, see [3-4], [8] and [11]. In [8], the existence of solutions to some Hammerstein integral equations has been deeply studied. The techniques used in [8] are as follows: Hammerstein integral equations can be converted to operator problems such that the solutions of the corresponding Hammerstein integral equations are the fixed points of the constructed operators. Meanwhile, some useful iterated schemes are provided for the approximations of the solutions.

In [3-4] and [11], the proofs of the existence of solutions to some Hammerstein integral equations are based on fixed point theorems on partially ordered Banach spaces. In book [11], for a given class of integral equations of Hammerstein type, a pointed closed convex cone is constructed which induces a partial order on the considered Banach space. Then the existence problems become fixed point problems on partial ordered Banach spaces, in which, in addition to the continuity, the order increasing properties of the corresponding operators play important roles.

In contrast to [3-4], [11], in this section, we apply the fixed point theorems proved in the previous section to prove the existence of nonnegative solutions to some Hammerstein integral equations by seeking some weaker conditions.

Let $C(\Sigma)$ denote the set of continuous real functions on Σ with the usual maximum norm $\|\cdot\|$. Then $(C(\Sigma), \|\cdot\|)$ is a Banach space. Let $C_+(\Sigma)$ denote the positive cone in $C(\Sigma)$, that is,

$$C_+(\Sigma) = \{x \in C(\Sigma) : x(t) \geq 0, \text{ for every } t \in \Sigma\}. \tag{5.2}$$

Then $C_+(\Sigma)$ is a pointed closed and convex cone in $C(\Sigma)$ and it induces a partial order on $C(\Sigma)$, denoted by $\succcurlyeq$. It follows that, for any $x, y \in C(\Sigma)$,

$$y \succcurlyeq x \text{ if and only if } y(t) \geq x(t), \text{ for every } t \in \Sigma. \tag{5.3}$$

For a given continuous kernel T defined on $\Sigma \times \Sigma$ and a continuous real function g defined on $\Sigma \times R$ of a Hammerstein integral equation, define a mapping

$$F : C(\Sigma) \to C(\Sigma)$$

by

$$(Fx)(t) = \int_{\Sigma} T(t,s)g(s,x(s))d\mu(s), \text{ for all } x \in C(\Sigma). \tag{5.4}$$

One can see that this mapping $F : C(\Sigma) \to C(\Sigma)$ is well-defined. It will be used in the sequel.

Now we apply Theorem 3.1 to prove a theorem for the existence of continuous solutions of nonlinear Hammerstein integral equations like [18]. In here, the proof is much simpler than that in [18].

Theorem 5.1. *Let T and g be continuous real functions respectively defined on $\Sigma \times \Sigma$ and on $\Sigma \times R$. Suppose that T and g satisfy the following conditions:*

(E_1) *$T(t,s) \geq 0$, for all $(t,s) \in \Sigma \times \Sigma$;*

(E_2) *for every $s \in \Sigma$, $g(x,\cdot)$ is increasing and $g(s,0) \geq 0$;*

(E_3) *there is $M > 0$, such that $\int_{\Sigma} T(t,s)g(s,M)d\mu(s) \leq M$, for every $t \in \Sigma$;*

(E_4) *$\int_{\Sigma} T(t,s)g(s,0)d\mu(s) > 0$, for some $t \in \Sigma$.*

Then the Hammerstein integral equation (5.1) has a positive continuous solution $x^* \in C_+(\Sigma)$ satisfying

$$0 < \|x^*\| \leq M. \tag{5.5}$$

Furthermore, for any given $z \in C_+(\Sigma)$ with $\|z\| \leq M$ satusfying $z \preccurlyeq F(z)$, we construct an iterative scheme as below

$$z, F(z), F^2(z), F^3(z), \ldots$$

Then

(a) $F^n(z) \preccurlyeq F^{n+1}(z)$, for $n = 1, 2, \ldots$;

(b) $F^n(z) \to z^*$, as $n \to \infty$, for some $z^* \in C_+(\Sigma)$ satisfying

$$\vee\{z, F(z), F^2(z), F^3(z), \ldots\} = z^*;$$

(c) z^* is a solution of the Hammerstein integral equation (5.1).

Proof. From the definition of the partial order $\succcurlyeq$ on $C(\Sigma)$ by (5.2) and (5.3), $(C(\Sigma), \succcurlyeq)$ is a partially ordered Banach space. From the given positive number M in this theorem, we write 0 and M for the constant functions on Σ with values 0 and M, respectively. Then we define the $\succcurlyeq$-interval $[0, M] \subseteq C_+(\Sigma)$ by

$$C_M(\Sigma) = [0, M] = \{x \in C(\Sigma) : 0 \preccurlyeq x \preccurlyeq M\}.$$

(It is equivalently rewritten as

$$C_M(\Sigma) = [0, M] = \{x \in C(\Sigma) : 0 \leq x(t) \leq M, \text{ for every } t \in \Sigma\}$$
$$= \{x \in C_+(\Sigma) : \|x\| \leq M\}).$$

Let $F : C(\Sigma) \to C(\Sigma)$ be the mapping defined in (5.4). From conditions $E_1 - E_3$ in this theorem, we have that $F(C_M(\Sigma)) \subseteq C_M(\Sigma)$. Condition E_2 implies that $F : C_M(\Sigma) \to C_M(\Sigma)$ is $\succcurlyeq$-increasing. From condition E_4, the zero function satisfies $0 \preccurlyeq F(0)$. So F satisfies the conditions $A_1 - A_3$ in Theorem 2.1 with respect to the set $C_M(\Sigma)$ (since F here is a single-valued mapping, condition A_2 in Theorem 2.1 is automatically satisfied).

On the other hand, since T and g are continuous real functions respectively defined on $\Sigma \times \Sigma$ and on $\Sigma \times R$ and Σ is compact, so T and g are uniformly continuous on $\Sigma \times \Sigma$, $\Sigma \times [0, M]$, respectively. Then from the Ascoli-Arzela Theorem, we can show that $F(C_M(\Sigma))$ is a compact subset in $C_M(\Sigma)$. From Lemma 2.1 listed in section 2, $(F(C_M(\Sigma)), \succcurlyeq)$ is an $\succcurlyeq$-chain-complete subset in $(C_M(\Sigma), \succcurlyeq)$. Hence, from Theorem 3.1, F has a fixed point $x^* \in C_M(\Sigma)$. By (5.4), x^* is a solution to the Hammerstein integral equation (5.1). It is clear to see that $F(0) \neq 0$ (it is the zero function). So, x^* satisfies (5.5).

For any given $z \in C_+(\Sigma)$ with $\|z\| \leq M$ satisfying $z \preccurlyeq F(z)$, since $F : C_M(\Sigma) \to C_M(\Sigma)$ is an $\succcurlyeq$-increasing mapping, it yields that $\{F^n(z)\}$ is $\succcurlyeq$-increasing, which proves (a). From the fact that $F(C_M(\Sigma))$ is a compact subset in $C_M(\Sigma)$, $\{F^n(z)\}$ is a Cauchy sequence in $C_M(\Sigma)$, by (a), $\{F^n(z)\}$ is a convergent sequence. So there is $z^* \in C_M(\Sigma) \subseteq C_+(\Sigma)$ such that $F^n(z) \to z^*$, as $n \to \infty$. By the $\succcurlyeq$-increasing property of $\{F^n(z)\}$ again, one can show

$$\vee\{z, F(z), F^2(z), F^3(z), \ldots\} = z^*.$$

From (5.4) and the fact that T and g are uniformly continuous on $\Sigma \times \Sigma$, $\Sigma \times [0, M]$, respectively, $F : C_M(\Sigma) \to C_M(\Sigma)$ is a continuous mapping. By $F^n(z) \to z^*$, it follows that $F(z^*) \to z^*$. So z^* is a fixed point of F. It implies that z^* is a solution of (5.1). □

Acknowledgements. The author is very grateful to the National Natural Science Foundation of China (11771194) for partially support about this research.

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Received: October 11, 2018; Accepted: January 17, 2019.