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ORBITAL FIXED POINT CONDITIONS IN GEODESIC SPACES

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Abstract. Many metric fixed point results can be formulated in an abstract 'convexity structure' setting. This discussion contains a review of some of these, as well as a discussion of other results which seem to require a bit more structure on the space. A metric space (X, d) is said to be Γ-uniquely geodesic if Γ is a family of geodesic segments in X and for each $x, y \in X$ there is a unique geodesic $[x, y] \in \Gamma$ with endpoints x and y. Let X be Γ-uniquely geodesic and let $\mathfrak{C}(X)$ denote the family of all bounded closed convex (relative to Γ) subsets of X. Assume that the family $\mathfrak{C}(X)$ is compact in the sense that every descending chain of nonempty subsets of $\mathfrak{C}(X)$ has a nonempty intersection. This is a brief discussion of what additional conditions on a mapping $T : K \to K$ for $K \in \mathfrak{C}(X)$ always assure that has at least one fixed point. In particular it is shown that if the balls in X are Γ-convex and if the closure of a Γ-convex set in X is again Γ-convex then a mapping $T : K \to K$ always has a fixed point if it is nonexpansive with respect to orbits in the sense of Amini-Harandi, et al., and if for each $x \in K$ with $x \neq T(x)$,

$$\inf_{m\in\mathbb{N}} \left\{ \limsup_{n\to\infty} d\left(T^m(x), T^n(x)\right) \right\} < diam\left(O(x)\right).$$

Mappings of the above type include those which are pointwise contractions in the sense that for each $x \in K$ there exists $\alpha(x) \in (0, 1)$ such that

$$d\left(T(x), T(y)\right) \leq \alpha(x)\, d(x, y) \text{ for all } y \in K.$$

The results discussed here extend known results if K is a weakly compact (convex) subset of a Banach space. A number of open questions are raised in connection with characterizations of normal structure in certain geodesic spaces.

Key Words and Phrases: Normal structure, compact convexity structures, nonexpansive mappings, fixed points, dimihishing orbital diameters, pointwise contractions, mappings nonexpansive with respect to orbits, strictly contractive mappings, geodesic spaces.
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1. Introduction

The observations discussed here have evolved, over time, from the study of fixed point theory for nonexpansive and related classes of mappings in a Banach space setting. The origin of this study dates back to 1965, when Felix Browder [8], Dietrich Göhde [11], and the first author simultaneously (and independently) published almost identical fixed point theorems for the class of nonexpansive mappings. We begin with a brief description of the original setting. For a historical development of the early theory see [19] and, for a more recent survey, [28].

Let $(X, \|\cdot\|)$ be a Banach space. The *diameter* of a subset D of X is denoted $diam\,(D)$; thus

$$diam\,(D) = \sup\{\|u - v\| : u, v \in D\}.$$

The concept of normal structure in a Banach space originates with Brodskii and Milman [7]. A closed convex subset K of X is said to have *normal structure* [resp., *weak normal structure*] if given any bounded convex [resp., weakly compact convex] subset H of K for which $diam\,(H) > 0$ there exists $u \in H$ such that

$$\sup\{\|u - x\| : x \in H\} < diam\,(H).$$

A mapping $T : K \to K$ is said to be *nonexpansive* if $\|T(u) - T(v)\| \leq \|u - v\|$ for each $u, v \in K$.

The following is the original result of Kirk [14]. Browder and Göhde obtained the same result under the stronger assumption that the set K is bounded closed and convex and the space X is uniformly convex.

Theorem 1.1. (*Kirk* [14]) *Let K be a bounded closed convex subset of a reflexive Banach space X, and suppose K has normal structure. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

The above result as well as those of Browder and Göhde were motivated by an attempt to extend the classical fixed point theorem of Banach from the class of contraction mappings, that is, mappings for which $\|T(u) - T(v)\| \leq \lambda\|u - v\|$ for each $u, v \in K$ for some $\lambda \in (0, 1)$ to the wider class of mappings for which the Lipschitz constant λ may be equal to 1. Of course Banach's theorem holds in any complete metric space, thus assuring the existence of a fixed point if K is any closed subset of a Banach space. This fact immediately suggests that several assumptions in Theorem 1.1 might be relaxed. For example reflexivity is only utilized in the original proof to assure that any descending chain of nonempty closed convex subsets of K has a nonempty intersection which is also closed and convex. This observation motivated a 'convexity structure' approach introduced by Penot in [27]. This approach is the main focus of the present paper.

2. Convexity structures: a summary of some known facts

All the results discussed below are formulated in a metric setting. The underlying framework is that of an abstract convexity structure in a metric space X, and we use the following notation: For $D \subseteq X$,

$$\begin{aligned} diam\,(D) &= \sup\{d\,(u,v) : u, v \in D\}\,; \\ r_u\,(D) &= \sup\{d\,(u,v) : v \in D\} \quad (v \in X)\,. \end{aligned}$$

Definition 2.1. *Let (X, d) be a metric space. A family $\mathfrak{C}(X)$ of subsets of X is said to be a* convexity structure *if X and $\emptyset$ are in $\mathfrak{C}(X)$ and if $\mathfrak{C}(X)$ is closed under arbitrary intersections. $\mathfrak{C}(X)$ is said to be* compact *[resp.,* sequentially compact*] if the intersection of every descending chain [resp., sequence] of nonempty sets in $\mathfrak{C}(X)$ is nonempty.*

Definition 2.2. *A member K of a convexity structure $\mathfrak{C}(X)$ in (X, d) is said to be*

(a) normal *if given any set $D \in \mathfrak{C}(X)$ with $D \subseteq K$,*

$$diam\,(D) > 0 \Rightarrow \text{there exists } x \in D \text{ such that } r_x\,(D) < diam\,(D)\,;$$

(b) uniformly normal *if there exists* $c \in (0, 1)$ such that *given any set $D \in \mathfrak{C}(X)$ with $D \subseteq K$,*

$$diam\,(D) > 0 \Rightarrow \text{there exists } x \in D \text{ such that } r_x\,(D) \leq c\,diam\,(D)\,;$$

(c) relatively normal *if given any $D \in \mathfrak{C}(X)$ with $D \subseteq K$,*

$$diam\,(D) > 0 \Rightarrow \text{there exists } x \in K \text{ such that } r_x\,(D) < diam\,(D)\,;$$

(d) uniformly relatively normal *if there exists $c \in (0, 1)$ such that given any set $D \in \mathfrak{C}(X)$ with $D \subseteq K$,*

$$diam\,(D) > 0 \Rightarrow \text{there exists } x \in K \text{ such that } r_x\,(D) \leq c\,diam\,(D)\,;$$

(e) quasi-normal *if given any set $D \in \mathfrak{C}(X)$ with $D \subseteq K$,*

$$diam\,(D) > 0 \Rightarrow \text{there exists } x \in D \text{ such that } d\,(x, y) < diam\,(D) \;\; \forall\, y \in D.$$

We now summarize some known facts, many of which were originally formulated in a Banach space setting. One of our objectives is to extend results of this type to related classes of mappings in certain geodesic spaces. See, for example, [13, Chapter 5] for the historical origins of these results.

Theorem 2.1. ([18]) *Let (X, d) be a bounded metric space possessing a convexity structure $\mathfrak{C}(X)$ which is countably compact and contains the closed balls in X, and suppose $K \in \mathfrak{C}(X)$ is normal. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

Theorem 2.2. ([9]*, also cf.* [30]) *Let (X, d) be a bounded metric space possessing a convexity structure $\mathfrak{C}(X)$ which is compact and contains the closed balls in X, and suppose $K \in \mathfrak{C}(X)$ is uniformly relatively normal. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

In the next theorem we assume that $T : K \to K$ is *Kannan nonexpansive*. This means that

$$d(T(x), T(y)) \leq \frac{1}{2}[d(x, T(x)) + d(y, T(y))] \text{ for each } x, y \in K.$$

Theorem 2.3. ([25]) *Let (X, d) be a bounded metric space possessing a convexity structure $\mathfrak{C}(X)$ which is compact and contains the closed balls in X, and suppose $K \in \mathfrak{C}(X)$ is quasi-normal. Then every Kannan nonexpansive mapping $T : K \to K$ has a fixed point.*

Remark 2.1. There are two interesting facts related to Theorem 2.3 which have been known for some time. Chi-Song Wong has shown in [32] that a weakly compact convex subset of a Banach space is quasi-normal if and only if every Kannan map $T : K \to K$ has a fixed point. Wong also proved *(*see [31]*)* that a weakly compact convex subset of a Banach space is quasi-normal if either X strictly convex or K is separable.

3. Diminishing orbital diameters and pointwise contractions

Since there are rather easily defined classes of mappings which lie between the class of contraction mappings and the nonexpansive mappings, it is natural to investigate conditions under which they also have fixed points. Two mappings of this type seem rather natural. We formulate both conditions in the setting of an arbitrary metric space (M, d).

Definition 3.1. *A mapping $T : M \to M$ is called a* pointwise contraction *if for each $x \in M$ there exists $\alpha(x) \in (0, 1)$ such that for each $y \in M$,*

$$d(T(x), T(y)) \leq \alpha(x)\, d(x, y). \tag{3.1}$$

The following class of mappings lies between the pointwise contractions and the nonexpansive mappings.

Definition 3.2. *A mapping $T : M \to M$ is called a* strict contraction *(*or strictly contractive*) if for each $x, y \in M$* with $x \neq y$,

$$d(T(x), T(y)) < d(x, y). \tag{3.2}$$

Obviously the following implications hold for a mapping $T : M \to M$:

$$\begin{aligned} T \text{ is a contraction } &\Rightarrow T \text{ is a pointwise contraction} \\ &\Rightarrow T \text{ is strictly contractive} \Rightarrow T \text{ is nonexpansive} \end{aligned}$$

Theorem 1.1 subsequently motivated a number of related results involving weakening the normal structure assumption or replacing that assumption with conditions on the behavior of the orbits of the mapping. The following extension of normal structure was introduced in [15] (also see [30]), and an example is given in that paper which shows that it is properly weaker than normal structure. This weaker assumption yields a fixed point theorem for a class of mappings which lie strictly between the class of nonexpansive mappings and the Banach contractions.

Definition 3.3. *A convex subset K of a Banach space X is said to have normal structure relative to K if given any bounded convex subset H of K for which $diam\,(H) > 0$ there exists $x \in K$ and $r < diam\,(H)$ such that $H \subseteq U\,(x; r)$,* where $U\,(x; r) = \{y \in X : ||x - y|| < r\}$.

The following is Corollary 1 of [15].

Theorem 3.1. *Let K be a weakly compact convex subset of a Banach space X, suppose $T : K \to K$ is strictly contractive, and suppose K has normal structure relative to K. Then T has a (unique) fixed point.*

Let $A \subseteq M$, and $T : A \to A$. For $x \in A$, let $O\,(x) = \left\{x, T\,(x), T^2\,(x), \cdots\right\}$, and for $n \geq 0$, let $O\,(T^n\,(x)) = \bigcup_{i=n}^{\infty} \left\{T^i\,(x)\right\}$. The set $O\,(x)$ is called the *orbit* of T at x. The sequence $\{diam\,(O\,(T^n\,(x)))\}$ is nondecreasing and has limit $r\,(x)$, called the *limiting orbital diameter* of T at x. (In general, this limit may be infinite, but typically the domain of T is a bounded set.)

Definition 3.4. ([4]) *Let T be a mapping of a metric space M into itself. If for each $x \in M$ it is the case that $diam\,(O\,(x)) < \infty$ and $r\,(x) < diam\,(O\,(x))$ when $diam\,(O\,(x)) > 0$, then T is said to have* diminishing orbital diameters *on M.*

It was shown in [4] that if K is a weakly compact convex subset of a Banach space then every nonexpansive mapping $T : K \to K$ which has diminishing orbital diameters has a fixed point. Subsequently Kirk observed in [17] that the convexity assumption on the domain of K in the may be dropped.

There is another class of mapping which are relevant to our discussion. These are the asymptotic pointwise contractions introduced by Kirk and Xu in [21]. Once again the underlying setting is a metric space (M, d).

Definition 3.5. *A mapping $T : M \to M$ is called an* asymptotic pointwise contraction *if for all $n \in \mathbb{N}$ there exists functions $\alpha_n : M \to \mathbb{R}^+$ such that for all $x, y \in M$,*

$$d\,(T^n\,(x), T^n\,(y)) \leq \alpha_n\,(x)\,d\,(x, y), \tag{3.3}$$

where $\alpha_n \to \alpha : M \to [0, 1)$ pointwise on M as $n \to \infty$.

A special case of the central result of Belluce and Kirk in [4] is the fact that every pointwise contraction $T : K \to K$ of a weakly compact convex set has a unique fixed point, and that the Picard iterates of T at each point $x \in K$ converge to this fixed point. The following asymptotic version of this fact was subsequently obtained by Kirk and Xu in [21]. (We show below that this is actually a special case of more general results.)

Theorem 3.2. *Let K be a weakly compact convex subset of a Banach space X and let $T : K \to K$ be an asymptotic pointwise contraction. Then T has a unique fixed point z, and for each $x \in K$ the Picard sequence $\{T^n\,(x)\}$ converges to z.*

4. Γ-convex geodesic spaces

Many of the Banach space results described above extend in a fairly natural way to certain geodesic spaces. Here we follow the terminology of [26]. Let (X,d) be a metric space and $x,y\in X$. A geodesic path from x to y is a mapping $\gamma:[0,\ell]\to X$ with $\gamma(0)=x$, $\gamma(\ell)=y$, and $d(\gamma(t),\gamma(t'))=|t-t'|$ for all $t,t'\in[0,\ell]$. The image of $\gamma[0,\ell]$ in X is a *geodesic (or metric) segment* which joins x and y. When no confusion arises, and in particular when this segment is unique, we shall denote it $[x,y]$.

Definition 4.1. *A metric space (X,d) is said to be (*uniquely*)* geodesic *if every two distinct points $x,y\in X$ are joined by a (unique) geodesic.*

Definition 4.2. *Let (X,d) be a metric space and $\Gamma\subseteq\Lambda$ a family of geodesic segments. The space (X,d) is said to be* Γ-uniquely geodesic *if for each $x,y\in X$ there is a unique geodesic in Γ which joins x and y. (*In this context we use $[x,y]$ to denote the unique geodesic in Γ which joins x and y.*) A subset K of X is said to be* Γ-convex *if $[x,y]\subseteq K$ whenever $x,y\in K$. Also for $x,y\in X$ and $t\in[0,1]$, we use the symbol $(1-t)x\oplus ty$ to denote the unique point $u\in[x,y]$ for which*

$$d(x,u)=td(x,y).$$

Proposition 4.1. ([3]) *Let (X,d) be a Γ-uniquely geodesic space. The following assertions are equivalent.*

(i) *For any $x\in X$ and $r>0$ the closed ball $B(x;r)$ is convex.*

(ii) *For any $x\in X$ and $r>0$ the open ball $U(x;r)$ is convex.*

(iii) *For any $x,y,z\in X$, the function $t\mapsto d(x,(1-t)y\oplus tz)$ is quasi-convex, i.e.,*

$$d(x,(1-t)y\oplus tz)\le\max\{d(x,y),d(x,z)\}\qquad \text{for all } t\in[0,1].$$

Definition 4.3. *A Γ-uniquely geodesic space (X,d) is said to have* property (P) *if*

$$\lim_{\varepsilon\searrow 0}\sup\{d((1-t)x\oplus ty,(1-t)x\oplus tz):t\in[0,1],\ x,y,z\in X,\ d(y,z)\le\varepsilon\}=0.$$

Example 4.1. ([3]) A metric space (X,d) is Γ-hyperbolic space in the sense of Reich-Shafrir [29] if X is Γ-uniquely geodesic and the following inequality holds:

$$d\left(\frac{1}{2}x\oplus\frac{1}{2}y,\frac{1}{2}x\oplus\frac{1}{2}z\right)\le\frac{1}{2}d(y,z)\ \text{ for all } x,y,z\in X. \tag{4.1}$$

It is shown in [3] that such spaces have property (P) but also that there are Γ-uniquely geodesics space, for example certain CAT(κ) spaces, which have property (P) but which fail to be hyperbolic.

Definition 4.4. *A nonempty subset C of a Γ-uniquely geodesic space is said to be* Γ-convex *if $[x,y]\in C$ whenever $x,y\in C$.*

Remark 4.1. It is immediate from the definitions that in a Γ-uniquely geodesic space which satisfies property (P) the closure of a Γ-convex set is again Γ-convex. [*To see this,* let K be a Γ-convex set and let $x,y\in\overline{K}$, and suppose $x_n,y_n\in K$ with $x_n\to x$

and $y_n \to y$ as $n \to \infty$. We need to show that $[x, y] \subset \overline{K}$. Fix $m \in \mathbb{N}$ and $t \in [0,1]$. By property (P)

$$(1-t)\, x_m \oplus t y_n \to (1-t)\, x_m \oplus t y \text{ as } n \to \infty.$$

Thus $(1-t)\, x_m \oplus ty \subset \overline{K}$ for each m.
Letting $m \to \infty$ we conclude that $(1-t)\, x \oplus ty \in \overline{K}$.]

The fact that an analog of Schauder's Theorem holds in Γ-uniquely geodesic spaces which have property (P) highlights the significance of this class of spaces. The following is the central result of Ariza, et al. in [3].

Theorem 4.1. *Let (X, d) be a Γ-uniquely geodesic space such that it satisfies property (P) and all balls are Γ-convex. Let K be a nonempty closed Γ-convex subset of X. Then every continuous mapping $T : K \to K$ for which $\overline{T(K)}$ is compact has a fixed point.*

5. A condition on orbits

We now turn a concept introduced by Nicolae [24] in 2010 and studied further in a Banach space setting by Amini-Harandi, et al. in [2]. The following is the metric space formulation.

Definition 5.1. (cf. [24], [2]) *Let C be a subset of a metric space X. A mapping $T : C \to C$ is said to be* nonexpansive wrt orbits *if for all $x, y \in C$,*

$$d\,(T\,(x)\,, T\,(y)) \leq r_x\,(O\,(y))$$

where

$$r_x\,(O\,(y)) = \sup\,\{d\,(x, u) : u \in O\,(y)\}\,.$$

If $x \neq y \Rightarrow d\,(T\,(x)\,, T\,(y)) < r_x\,(O\,(y))$ then T is said to be strictly contractive wrt orbits.

If T is nonexpansive then $d\,(T\,(x)\,, T\,(y)) \leq d\,(x, y) \leq r_x\,(O\,(y))$ so it follows that nonexpansive mappings are always nonexpansive wrt orbits.

The significance of mappings which are nonexpansive wrt orbits is that (unlike the nonexpansive mappings) they provide a characterization of (weak) normal structure. Also pointwise contractions (see Section 3) are nonexpansive wrt orbits in a very strong sense. The following characterization of normal structure is the central result of [2].

Theorem 5.1. *A Banach space X has weak normal structure if and only if every weakly compact convex subset of X has the fixed point property for mappings which are nonexpansive wrt orbits.*

Pointwise contractions have also been recently studied in the context of geodesic spaces, most notably by Nicolae [24] in the class of CAT(0) spaces. See [24] (also [6]) for the relevant notation and definitions. The following two observations are among several obtained in [24].

Theorem 5.2. *Let X be a bounded metric space for which the family $\mathfrak{A}(X)$ of all admissible sets in X (i.e., the intersections of closed balls in X) forms a compact convexity structure. Suppose $T : X \to X$ is a mapping for which there exists $\alpha : X \to [0,1)$ such that for each $x, y \in X$,*

$$d(T(x), T(y)) \leq \alpha(x) r_x(O(y)).$$

Then T has a unique fixed point $z \in X$ and every sequence $\{T^n(x)\}$ for $x \in X$ converges to z.

Theorem 5.3. *Let X be a bounded complete $CAT(0)$ space and suppose $T : X \to X$ is nonexpansive wrt orbits. Then the fixed point set of T is nonempty, closed and convex.*

In what follows we say that a family $\mathfrak{C}(X)$ of subsets of a metric space X is compact if the intersection of every descending chain of nonempty members of $\mathfrak{C}(X)$ is a nonempty member of $\mathfrak{C}(X)$. If $\mathfrak{C}(X)$ is closed under nonempty intersections this means that $\mathfrak{C}(X)$ is a compact convexity structure in the sense of Penot [27] which we discussed in Section 2. In particular, this assumption is properly weaker than the assumption that each set $C \in \mathfrak{C}(X)$ is compact in the metric sense.

We now prove an apparently new result which has the following simple corollary. In the following we use $\mathfrak{C}(X)$ to denote the family of all bounded closed Γ-convex subsets of a given Γ-uniquely geodesic space (X, d).

Theorem 5.4. *Let (X, d) be a Γ-uniquely geodesic space which has property (P) and for which all balls are Γ-convex, suppose the family $\mathfrak{C}(X)$ is compact, and suppose $D \in \mathfrak{C}(X)$. Then every pointwise contraction $f : D \to D$ has a fixed point.*

Theorem 5.4 is a consequence of the following much more general (although surely less natural) result. Here we use the fact that in the context of Γ-uniquely geodesic spaces the intersection of two convex sets in Γ is always in Γ. Since property (P) assures that the closure of a Γ-convex set is again Γ-convex, it is possible to take the closed convex hull, $\overline{conv}(K)$, of a subset K of such a space X to be the intersection of all members of $\mathfrak{C}(X)$ which contain K. In this case,

$$\overline{conv}(K) = cl\left(\bigcup_{n=0}^{\infty} K_n\right)$$

where $K_0 = K$, K_n is the set of all points which lie on a Γ-geodesic with endpoints in K_{n-1}, and cl is the closure in the usual sense. This observation is crucial to the proof.
The proof of the following result follows the one of Theorem 5.2 of Kirk and Shahzad [20].

Theorem 5.5. *Let (X, d) be a Γ-uniquely geodesic space which has property (P), for which all balls are Γ-convex, and for which the family $\mathfrak{C}(X)$ is compact. Let $K \in \mathfrak{C}(X)$ and suppose $T : K \to K$ is nonexpansive wrt orbits. Suppose also that T satisfies the following condition: For each $x \in K$ with $x \neq T(x)$,*

$$\inf_{m \in \mathbb{N}} \left\{ \limsup_{n \to \infty} d(T^m(x), T^n(x)) \right\} < diam(O(x)). \tag{5.1}$$

Then T has a fixed point.

Proof. Let $K_1 \in \mathfrak{C}(X)$ be a subset of K which is minimal with respect to being nonempty and invariant under T. We suppose $diam\,(K_1) > 0$ and show that this leads to a contradiction.

Let $x \in K_1$ and let $d_x = diam\,(O(x))$. If $T(x) \neq x$ then by assumption

$$\exists\ m, n \in \mathbb{N}, \text{ and } r \in (0, d_x) \text{ such that } O(T^n(x)) \subseteq B(T^m(x); r).$$

Let $\varepsilon > 0$ and let

$$S_\varepsilon = \left\{ y \in X : T^i(x) \in B(y; r+\varepsilon) \text{ for almost all } i \right\}.$$

Let

$$S = \bigcap_{\varepsilon > 0} S_\varepsilon.$$

Since $T^m(x) \in S$, $S \neq \emptyset$. Now suppose $\{z_n\} \subset S$ with $\lim_{n\to\infty} z_n = z$, and let $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that $d(z, z_n) < \varepsilon/2$. Since $z_n \in S_{\varepsilon/2}$ there exists i_0 such that $i \geq i_0 \Rightarrow d\left(z_n, T^i(x)\right) \leq r + \varepsilon/2$. Thus $i \geq i_0 \Rightarrow$

$$d\left(z, T^i(x)\right) \leq d(z, z_n) + d\left(z_n, T^i(x)\right) \leq r + \varepsilon.$$

Thus $z \in S_\varepsilon$ for each $\varepsilon > 0$, so $z \in S$; hence S is closed. Also if $y_1, y_2 \in S$ and $\varepsilon > 0$, then for almost all i:

$$d\left(y_1, T^i(x)\right) \leq r + \varepsilon \text{ and } d\left(y_2, T^i(x)\right) \leq r + \varepsilon.$$

Since closed balls in X are Γ-convex,

$$\begin{aligned} d\left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2, T^i(x)\right) &\leq \max\left\{d\left(y_1, T^i(x)\right), d\left(y_2, T^i(x)\right)\right\} \\ &\leq r + \varepsilon \end{aligned}$$

for almost all i and it follows that $\frac{1}{2}y_1 \oplus \frac{1}{2}y_2 \in S_\varepsilon$ for each $\varepsilon > 0$; hence $\frac{1}{2}y_1 \oplus \frac{1}{2}y_2 \in S$ and it follows from this that S is convex. Therefore $S \in \mathfrak{C}(X)$; hence $C = S \cap K_1 \in \mathfrak{C}(X)$. Also C is nonempty since $T^n(x) \in C$. Now suppose $d\left(y, T^i(x)\right) \leq r + \varepsilon$ for almost all i, say for $i \geq N$. Then, since T is nonexpansive wrt orbits,

$$d\left(T(y), T^{i+1}(x)\right) \leq r_y\left(O\left(T^i(x)\right)\right) = \sup\left\{d(y, u) : u \in O\left(T^i(x)\right)\right\} \leq r + \varepsilon$$

for $i \geq N$. Thus $d\left(T(y), T^i(x)\right) \leq r + \varepsilon$ for almost all i and therefore $T : C \to C$. It now follows from minimality of K_1 that $C = K_1$.

Now let

$$W = \left(\bigcap_{x \in K_1} B(x; r)\right) \cap K_1 = \{y \in K_1 : K_1 \subseteq B(y; r)\}.$$

Therefore W is nonempty and clearly $W \in \mathfrak{C}(X)$. Also, if $y \in W$ and $x \in K_1$ then $d(y, x) \leq r$, i.e., $K_1 \subseteq B(y; r)$ for each $y \in W$. Also if $x \in K_1$ then $O(x) \subseteq K_1$. It follows that $r_y(O(x)) \leq r$ and since T is nonexpansive wrt orbits,

$$d(T(y), T(x)) \leq r_y(O(x)) \leq r.$$

Therefore $T(K_1) \subseteq B(T(y); r)$ for each $y \in W$. But this implies

$$\overline{conv}\,(T(K_1)) \subseteq B(T(y); r)$$

where $\overline{conv}\,(T\,(K_1))$ denotes the closure of the Γ-convex hull of $T\,(K_1)$. Since property (P) implies $\overline{conv}\,(T\,(K_1)) \in \mathfrak{C}\,(X)$ and since

$$T\left(\overline{conv}\,(T\,(K_1))\right) \subseteq \overline{conv}\,(T\,(K_1)),$$

the minimality of K_1 implies $\overline{conv}\,(P) = K_1$. It follows that $T\,(y) \in W$.
Hence $T : W \to W$. Again by minimality of K_1, it follows that $W = K_1$.
But $diam\,(W) \leq r < diam\,(K_1)$ – a contradiction. □

The above theorem raises the question of whether a more abstract version of Theorem 3.1 holds. The following theorem shows that this is indeed the case. It also shows that the pointwise contractive assumption in Theorem 5.4 can be weakened if the domain K of the mapping is relatively normal (as defined in Section 2).

Lemma 5.1. *Let (X, d) be a hyperbolic space for which the family $\mathfrak{C}\,(X)$ is compact, and let H and K be two nonempty disjoint members of $\mathfrak{C}\,(X)$. Then there exist $u \in H, v \in K$ such that*

$$d\,(u, v) = \inf\left\{d\,(w, z) : w \in H \text{ and } z \in K\right\}.$$

Proof. As pointed out in [3] (also see [29, p. 104]) inequality (4.1) is equivalent to the following inequality:

$$d\,((1-t)\,x \oplus ty, (1-t)\,w \oplus tz) \leq (1-t)\,d\,(x, w) + td\,(y, z) \tag{5.2}$$

for all $t \in [0, 1]$ and all $x, y, z, w \in W$. Let $\varepsilon > 0$ and set

$$d := \inf\left\{d\,(w, z) : w \in H \text{ and } z \in K\right\}.$$

By assumption the set

$$H_\varepsilon := \{u \in H : \exists\; v \in K \text{ such that } d\,(u, v) \leq d + \varepsilon\} \neq \emptyset$$

and using inequality (5.2) it is easy to see that H_ε is convex. Moreover since X has property (P) the closure $\overline{H}_\varepsilon$ of H_ε is in $\mathfrak{C}\,(X)$. Therefore $\bigcap_{\varepsilon>0} \overline{H}_\varepsilon$ is a nonempty subset of H.

Now let $u \in \bigcap_{\varepsilon>0} \overline{H}_\varepsilon$ and let $K_\varepsilon = \{v \in K : d\,(u, v) \leq d + \varepsilon\}$. Then $\bigcap_{\varepsilon>0} \overline{K}_\varepsilon \neq \emptyset$ and the conclusion follows upon taking $v \in \bigcap_{\varepsilon>0} \overline{K}_\varepsilon$. □

Theorem 5.6. *Let (X, d) be a Γ-hyperbolic space for which all balls are Γ-convex and for which the family $\mathfrak{C}\,(X)$ is compact. Suppose $K \in \mathfrak{C}\,(X)$ is relatively normal. Then every mapping $T : K \to K$ which is strictly contractive has a fixed point.*

Proof. (This follows the proof of Theorem 4.2 of [15].) As above, let $K_1 \in \mathfrak{C}\,(X)$ be a subset of K which is minimal with respect to being nonempty and invariant under T, and assume $diam\,(K_1) > 0$. By assumption there exist $x \in K$ and $r < diam\,(K_1)$ such that $K_1 \subseteq B\,(x; r)$. Let

$$W = \{x \in K : K_1 \subseteq B\,(x; r)\}.$$

Then $W = \left(\bigcap_{x \in K_1} B(x;r) \right) \cap K$; hence $W \in \mathfrak{C}(X)$. Also $\overline{conv}(T(K_1)) \in \mathfrak{C}(X)$ (because hyperbolic spaces satisfy property (P)) so the minimality of K_1 implies $\overline{conv}(T(K_1)) = K_1$. Let $x \in W$. Since T is strictly contractive (hence nonexpansive) it follows that $T(K_1) \subseteq B(T(x);r)$; hence $K_1 = \overline{conv}(T(K_1)) \subseteq B(T(x);r)$. Therefore $T : W \to W$. However if $W \cap K_1 \neq \emptyset$ then $T : W \cap K_1 \to W \cap K_1$. This contradicts the minimality of K_1 because

$$diam(W \cap K_1) \leq r < diam(K_1)$$

so $W \cap K_1$ is a proper subset of K_1. On the other hand, if $W \cap K_1 = \emptyset$ then by Lemma 5.1 there exist points $u \in K_1$ and $v \in W$ such that

$$d(u,v) = \inf\{d(w,z) : w \in K_1 \text{ and } z \in W\}.$$

However this is not possible because in this case $u \neq v$; thus $d(T(u), T(v)) < d(u,v)$ with $T(u) \in K_1$ and $T(v) \in W$. We conclude therefore that K_1 consists of a single point which is fixed under T. □

Remark 5.1. Obviously it is possible to weaken the assumptions of Theorem 5.6 in various ways without significantly altering the proof. For example it suffices to assume that T is strictly contractive on K or, more generally, that nonexpansive of K and eventually strictly contractive in the sense that if $x, y \in K$ and $x \neq y$, then $d(T^n(y), T^n(y)) < d(x,y)$ for some $n \in \mathbb{N}$.

We now turn to an asymptotic extension of Theorem 5.4. As we show below this is actually a special case of Theorem 5.5 but we include the proof because it is much simpler. This theorem is essentially an alternate version of Theorem 4.2 of Hussain and Khamsi [12]. (We remark that the method of defining the set C below goes back to a 1969 paper of Kirk [16].)

Theorem 5.7. *Let (X,d) be a Γ-uniquely geodesic space which has property (P) and for which all balls are Γ-convex, suppose the family $\mathfrak{C}(X)$ is compact, and suppose $K \in \mathfrak{C}(X)$. Then every asymptotic pointwise contraction $T : K \to K$ has a unique fixed point, and the sequence of Picard iterates $\{T^n(x)\}$ at each $x \in K$ converges to this fixed point.*

Proof. Since T is an asymptotic pointwise contraction, for all $x \in M$ and $n \in \mathbb{N}$ there exist $\alpha_n(x) \in \mathbb{R}^+$ such that for each $y \in M$,

$$d(T^n(x), T^n(y)) \leq \alpha_n(x)\, d(x,y), \tag{5.3}$$

where $\alpha_n \to \alpha : M \to [0,1)$ pointwise on M. Fix $x \in K$ and define $r : K \to \mathbb{R}$ by setting

$$r(u) = \limsup_{n\to\infty} d(T^n(x), u)$$

and let $r_0 = \inf\{r(u) : u \in K\}$. Then given $\varepsilon > 0$ each of the sets

$$\bigcap_{i=k}^{\infty} B\left(T^i(x); r + \varepsilon\right) \neq \emptyset$$

is nonempty closed and Γ-convex for k sufficiently large. It follows that

$$C_\varepsilon = \bigcup_{k=1}^{\infty} \left(\bigcap_{i=k}^{\infty} B\left(T^i(x); r+\varepsilon\right) \neq \emptyset \right)$$

is the union of an ascending sequence of closed Γ-convex sets. Hence by property (P) the closure $\overline{C}_\varepsilon$ of C_ε is nonempty closed and Γ-convex. Thus

$$C = \bigcap_{\varepsilon>0} \overline{C}_\varepsilon$$

is nonempty closed and Γ-convex. We can now follow the proof of Theorem 3.1 of [21]. Notice that for each $u \in K$,

$$r\left(T^m(u)\right) \leq \alpha_m(u).$$

Now take $u \in C$. Since $T^m(u) \in K$ we have, for $m \geq 1$,

$$r(u) \leq r\left(T^m(u)\right) \leq \alpha_m(u)\, r(u).$$

Since $\alpha_m(u) \to \alpha(u) < 1$ as $m \to \infty$ we conclude that $r(u) = 0$ and hence, from the above inequality, $r\left(T^m(u)\right) = 0$ for all $m \geq 1$. In particular, $r(T(u)) = 0$. It follows that $T^n(x) \to u$ and $T^n(x) \to T(u)$ as $n \to \infty$ so $T(u) = u$ and $\{T^n(x)\}$ converges to u. The uniqueness of u is immediate. □

We now show that asymptotic pointwise contractions satisfy the conditions of Theorem 5.5. Specifically, we show that $(B) \Rightarrow (A)$ where:

(A) Mapping $T : M \to M$ satisfies the assumptions of Theorem 5.5 (i.e., T is nonexpansive wrt orbits and for each $x \in K$ with $x \neq T(x)$,

$$\inf_{m\in\mathbb{N}} \left\{ \limsup_{n\to\infty} d\left(T^m(x), T^n(x)\right) \right\} < diam\left(O(x)\right).) \tag{5.4}$$

(B) $T : M \to M$ is an asymptotic pointwise contraction.

To see that $(B) \Rightarrow (5.4)$ take $n = m$ in (A). Fix $x \in K$ with $x \neq Tx$. Then for each $y \in M$,

$$d\left(T^m(x), T^m(y)\right) \leq \alpha_m(x)\, d(x,y).$$

Now take $n \geq m$ and take $y = T^{n-m}(x)$ in (B). Then we have

$$d\left(T^m(x), T^n(x)\right) \leq \alpha_m(x)\, d\left(x, T^{n-m}(x)\right) \leq \alpha_m(x)\, diam\left(O(x)\right)$$

and thus

$$\limsup_{n\to\infty} d\left(T^m(x), T^n(x)\right) \leq \alpha_m(x)\, diam\left(O(x)\right).$$

Since $\alpha_m(x) \to \alpha(x) < 1$ as $m \to \infty$,

$$\inf_{m\in\mathbb{N}} \left\{ \limsup_{n\to\infty} d\left(T^m(x), T^n(x)\right) \right\} < diam\left(O(x)\right).$$

Thus $(B) \Rightarrow (5.4)$. Since asymptotic pointwise contractions are nonexpansive the conclusion follows.

The preceding results may be summarized as follows. (Part (I) of this theorem is a special case.)

Theorem 5.8. *Let* (X,d) *be a* Γ*-uniquely geodesic space for which all balls are* Γ*-convex. Let* $\mathfrak{C}(X)$ *denote the family of all closed* Γ*-convex subsets of* X*, and suppose* $\mathfrak{C}(X)$ *is compact.*

(I) Suppose X *has property* (P) *and suppose* $K \in \mathfrak{C}(X)$. *Then every mapping* $T : K \to K$ *which is nonexpansive wrt orbits and which has diminishing orbital diameters has a fixed point.*

(II) Suppose X *has property* (P) *and suppose* $K \in \mathfrak{C}(X)$. *Then every asymptotic pointwise contraction* $T : K \to K$ *has a unique fixed point.*

(III) Suppose X *is* Γ*-hyperbolic and suppose* $K \in \mathfrak{C}(X)$ *is relatively normal. Then every strictly contractive mapping* $T : K \to K$ *has a fixed point.*

Remark 5.2. Some assumptions in the above theorem may be further weakened without affecting the proofs involved. For example property (P) is used only to assure that the closure of a Γ-convex set in X is again Γ-convex. Also the Γ-hyperbolic assumption in *(*III*)* is needed only to assure the validity of Lemma 5.1.

Remark 5.3. The proofs of Theorems 5.5 and 5.6 given above make strong use of Zorn's Lemma. However, as in the case of Theorems 2.1 and 2.2, more constructive proofs likely exist.

6. Orbitally nonexpansive mappings - some questions

We begin with an obvious question.

Question 6.1. Can the nonexpansive assumptions on the mappings in Theorems 2.1 and 2.2 be replaced by the assumption that the mappings are nonexpansive wrt orbits?

We next turn to a concept recently introduced (in a Banach space setting) by E. Llorens-Fuster [23]. This result gives rise to a number of related questions.

Definition 6.1. *A mapping* T *of a metric space* (X,d) *into itself is said to be* orbitally nonexpansive *if for every nonempty closed convex* T*-invariant subset* D *of* X *there exists* $x_0 \in D$ *such that for every* $x \in D$,

$$\limsup_{n\to\infty} d\left(T\left(x\right), T^n\left(x_0\right)\right) \leq \limsup_{n\to\infty} d\left(x, T^n\left(x_0\right)\right).$$

Obviously every nonexpansive mapping is orbitally nonexpansive. Other examples are given in [23]. The following is the central result of [23].

Theorem 6.1. *Let* K *be a weakly compact convex subset of a Banach space and suppose* K *has normal structure. Then every orbitally nonexpansive mapping* $T : K \to K$ *has a fixed point.*

A key component in Llorens-Fuster's proof of the above Theorem is the following characterization of normal structure due to Bogin [5].

Theorem 6.2. ([5], Corollary 1) *A convex subset K of a Banach space X has normal structure if and only if it possesses the following property:*
For each non-constant bounded sequence $\{x_n\}$ in K, the function

$$g(x) := \lim\sup_{n\to\infty} \|x_n - x\|$$

is not constant on conv$(\{x_n\})$.

Question 6.2. Is there a Γ-geodesic space analog of the above result? Specifically, does the above result remain true if $K \in \mathfrak{C}(X)$ is normal where (X, d) is a Γ-uniquely geodesic space which satisfies property (P) and for which all balls are Γ-convex $\mathfrak{C}(X)$ is the family of all closed Γ-convex subsets of X is compact?

The above question essentially reduces to the following.

Question 6.3. Does the Banach space proof of Theorem 6.1 carry over to the geodesic space setting? In particular, is the real valued function $g : D \to [0, \infty)$ is non-constant where $K \in \mathfrak{C}(X)$ and D is a closed convex subset of K with $diam(D) > 0$?

This opens up a new avenue of questions which evolve from Brodskii and Milman's original concept of normal structure. A non-constant bounded sequence $\{x_n\}$ of points in a Banach space X is said to be a *diametral sequence* if

$$\lim_{n\to\infty} dist(x_{n+1}, conv(\{x_1, \cdots, x_n\})) = diam(\{x_1, x_2, \cdots\}).$$

The following is a fundamental characterization of normal structure which appears in Brodskii and Milman's original paper [7]. It is this fundamental characterization that provides the basis for Bogin's characterization of [5].

Theorem 6.3. ([7]) *A bounded convex subset K of a Banach space has normal structure if and only if it does not contain a diametral sequence.*

Question 6.4. Does this characterization carry over to the geodesic setting of Section 4? Specifically*:* Suppose X is a Γ-uniquely geodesic space for which the balls are Γ-convex, and suppose X has property (P). Suppose $K \in \mathfrak{C}(X)$ fails to be normal. Then does K contain a diametral sequence? Of course it is important here to be precise about the definition of $conv(\{x_1, \cdots, x_n\})$. The typical approach is to take $F_0 = \{x_1, \cdots, x_n\}$ and define

$$conv(\{x_1, \cdots, x_n\}) = \bigcup_{k=1}^{\infty} F_k$$

where F_k is the collection of all points which lie on a Γ-geodesic with endpoints in F_{k-1}.

Another curious question is whether a Banach space characterization of normal structure due to T. C. Lim carries over to the geodesic setting considered here. Again, suppose X is a Γ-uniquely geodesic space for which the balls are Γ-convex, and suppose

X has property (P). For a nonempty subset W of X and a decreasing net $\{W_\alpha\}_{\alpha\in A}$ of nonempty bounded subsets of W (A is a directed set), let

$$\begin{aligned} r_\alpha(x) &= \sup\{d(x,y) : y \in W_\alpha\} \text{ for each } x \in W; \\ r(x) &= \inf\{r_\alpha(x) : \alpha \in A\}; \\ r &= \inf\{r(x) : x \in W\}; \\ \mathfrak{C}(\{W_\alpha\}, W) &= \{x \in W : r(x) = r\}. \end{aligned}$$

The number r is called the *asymptotic radius* of $\{W_\alpha\}$ in W and the set $\mathfrak{C}(\{W_\alpha\}, W)$ is called the *asymptotic center* of $\{W_\alpha\}$ in W.

Definition 6.2. *With X as above, we say that a set $K \in \mathfrak{C}(X)$ of a Banach space is asymptotically normal if, given any bounded Γ-convex subset W of K which contains more than one point and given any decreasing decreasing net $\{W_\alpha\}_{\alpha\in A}$ of nonempty subsets of W, $\mathfrak{C}(\{W_\alpha\}, W)$ is a proper subset of W.*

The following characterization of normal structure due to T. C. Lim [22] is a very important application of the Brodskii-Milman characterization.

Theorem 6.4. *A bounded convex subset K of a Banach space has normal structure if and only if it has asymptotic normal structure.*

This characterization motivates yet another question.

Question 6.5. Suppose X is a Γ-uniquely geodesic space for which the balls a Γ-convex, and suppose X has property (P). Is it the case that a set $K \in \mathfrak{C}(X)$ is normal if and only if it is asymptotically normal?

The answer to the above question is likely affirmative, but because of the intricate structure of convex combinations in geodesic spaces (see, e.g., [1]) the details would appear to be difficult.

7. Normal structure and countable compactness

The following is Lemma 1 of [18]. It is based on an earlier result of Gillespie and Williams [10]. One consequence of this lemma is that if in addition $\mathfrak{C}(X)$ is countably compact then T always has a fixed point for mappings that are nonexpansive wrt orbits. (It is also interesting to note that the proof of this theorem is constructive in the sense that it does not require the full axiom of choice.)

Lemma 7.1. *Let (X,d) be a non-trivial bounded metric space that possesses a convexity structure $\mathfrak{C}(X)$ which is normal and contains the closed balls of X. Suppose $T : X \to X$ is nonexpansive wrt orbits. Then there exists $M \in \mathfrak{C}(X)$ such that $T : M \to M$ and for which diam$(M) <$ diam(X).*

Proof. After obvious notational changes, the proof is identical with the one given in [18] with one exception. The step in the proof where it is asserted that

$$d(T(x), T(y) \leq d(x,y) \leq \rho$$

for each $y \in L$ should be replaced with the following assertion: Since $O(y) \subset L$ for each $y \in L$ and since T is nonexpansive wrt orbits,

$$d(T(x), T(y)) \leq r_x(O(y)) \leq r_x(L) \leq \rho. \qquad \square$$

The proof of Theorem 1 in [18] (Theorem 2.1 above) now carries over to yield the following result.

Theorem 7.1. *Let (X, d) be a bounded metric space which possesses a countably compact normal convexity structure $\mathfrak{C}(X)$ that contains the closed balls of X. Then any mapping $T : X \to X$ which is nonexpansive wrt orbits has a fixed point.*

In view of Theorem 5.1 it is natural to ask whether the above theorem in some sense characterizes normality of the convexity structure. Specifically:

Question 7.1. Under the assumptions of Theorem 7.1 suppose some set $D \in \mathfrak{C}(X)$ fails to be normal. Then does there exist a mapping $T : D \to D$ which is nonexpansive wrt orbits yet fails to have a fixed point?

8. Appendix

For convenience of the careful reader we provide details of the proof of Lemma 7.1.

Proof of Lemma 7.1. Since $\mathfrak{C}(X)$ is normal there exists $x \in X$ such that

$$\rho := r_x(X) < diam(X).$$

Thus $C := \{z \in X : X \subset B(z; \rho)\} \neq \emptyset$. Set

$$\mathfrak{F} = \{D \in \mathfrak{C}(X) : C \subset D \text{ and } T : D \to D\}$$

and let $L = \bigcap \mathfrak{F}$. Then $X \in \mathfrak{F}$ so $\mathfrak{F} \neq \emptyset$. Also $L \in \mathfrak{F}$.

Now let $A = C \cup T(L)$. Then, since $T(L) \subset L$ (thus $C \cup T(L) \subset C \cup L = L$) it follows that $A \subset L$, and hence

$$cov(A) := \bigcap \{D \in \mathfrak{C}(X) : A \subset D\} \subset L.$$

Therefore $T(cov(A)) \subset T(L) \subset A \subset cov(A)$. Since $cov(A) \in \mathfrak{C}(X)$ it must be the case that $cov(A) \subset \mathfrak{F}$; hence $cov(A) = L$. Let

$$M = \{x \in L : L \subset B(x; \rho)\}.$$

Then $x \in C \Rightarrow z \in L$, and since $X \subset B(z; \rho)$ it follows that $L \subset B(z; \rho)$. Therefore $C \subset M$ so $M \neq \emptyset$. Also, $x \in M \Rightarrow T(x) \in L$ (because $L \in \mathfrak{F}$). Since T is nonexpansive wrt orbits and $O(y) \subset L$ for each $y \in L$ we have

$$d(T(x), T(y)) \leq r_x(O(y)) \leq \rho.$$

Further if $z \in C$ then $d(T(z), x) \leq \rho$ (because $X \subset B(z; \rho)$).
This proves $A \subset B(T(x); \rho)$. This in turn implies that $L = cov(A) \subset B(T(x); \rho)$. Therefore $T : M \to M$. Finally,

$$M = \left\{ \bigcap_{u \in L} B(u; \rho) \right\} \cap L.$$

Thus M is the intersection of sets in $\mathfrak{C}(X)$, so $M \in \mathfrak{C}(X)$. Since $x, y \in M \Rightarrow d(x,y) \leq \rho$ we have $diam(M) < diam(X)$ completing the proof. □

Finally, to facilitate comparison with our Theorem 5.7, we now describe Theorem 4.2 of Hussain-Khamsi [12].

Definition 8.1. *Let X be a metric space and $\mathfrak{C}(X)$ a convexity structure on X. A function $\Phi : X \to [0,\infty)$ is $\mathfrak{C}$-convex if $\{X : \Phi(x) \leq r\} \in \mathfrak{C}(X)$ for any $r \geq 0$. Also a type on X is a function of the form*

$$\Phi(u) = \limsup_{n\to\infty} d(x_n, u)$$

where $\{x_n\}$ is a bounded sequence in X. A convexity structure is said to be T-stable if all types are $\mathfrak{C}$-convex.

Theorem 8.1. *Let X be a bounded metric space. Assume that there exists a convexity structure $\mathfrak{C}(X)$ on X which is compact and T-stable. Let $T : X \to X$ be an asymptotic pointwise contraction. Then T has a unique fixed point x_0. Moreover the orbit $\{T^n(x)\}$ converges to x_0 for each $x \in X$.*

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