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A NOTE ON A RATIONAL FORM CONTRACTIONS WITH DISCONTINUITIES AT FIXED POINTS

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Abstract. In this paper, we investigate one of the classical problems of the metric fixed point theory: Whether there is a contraction condition which does not force the mapping to be continuous at the fixed point. We propose a contraction conditions in rational form that has a unique fixed point but not necessarily continuous at the given fixed point.
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1. Introduction

In the metric fixed point theory literature, the contraction conditions in the fixed point theorems preponderantly concern the continuous mapping. Only in a few of published papers, the discontinuous operators were investigated whether they posses a fixed point. In 1969, Kannan [8] proved the first metric fixed point theorem that is not necessarily continuous. Following this initial result, a number of authors have proposed some contraction conditions which do not force the mapping to be continuous at the fixed point, see e.g. [15, 13, 2]. For the sake of completeness, we recollect some fundamental results.

Throughout this paper, we shall denote the set of positive numbers and the set of real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Moreover, we set $\mathbb{R}_0^+ = [0, \infty)$.

In 1999, Pant [13] proved the following fixed point theorem in which the continuity of mapping at the fixed point is not necessary.

Theorem 1.1. ([13]) *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

- (*i*) $d(Tx, Ty) \leq \phi(\max\{d(x, Tx), d(y, Ty)\})$, *where ϕ is a self-mapping on $\mathbb{R}_0^+$ such that $\phi(t) < t$ for each $t > 0$;*
- (*ii*) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < \max\{d(x, Tx), d(y, Ty)\} < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon$$

then T has a unique fixed point, say z. Moreover, T is continuous at z if and only if

$$\lim_{x\to z}\max\{d(x,Tx),d(z,Tz)\}=0. \tag{1.1}$$

Very recently, the result of Pant [13] was extended by Bisht and Pant [2] in the following way:

Theorem 1.2. ([2]) *If a self-mapping T of a complete metric space (X,d) satisfies the conditions;*

(i) *T^2 is continuous,*
(ii) *$d(Tx,Ty)\le\phi(M(x,y))$, where ϕ is a self-mapping on $\mathbb{R}_0^+$ such that $\phi(t)<t$ for each $t>0$;*
(iii) *for a given $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that*

$$\varepsilon<M(x,y)<\varepsilon+\delta(\varepsilon) \text{ implies } d(Tx,Ty)\le\varepsilon,$$

where

$$M(x,y)=\left\{d(x,y),d(x,Tx),d(Ty,y),\frac{d(x,Ty)+d(Tx,y)}{2}\right\}$$

then T has a unique fixed point, say z, and and $T^n x\to z$ for each $x\in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x\to z}M(x,z)\neq 0.$$

In what follows, we recall two interesting contraction types that involve rational expression (see also e.g. [1, 4, 9, 12]).

Theorem 1.3. *Let (X,d) be a complete metric space and $T:X\to X$ be a continuous mapping. If there exist $\alpha,\beta\in[0,1)$, with $\alpha+\beta<1$ such that*

$$d(Tx,Ty)\le\alpha\cdot\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+\beta d(x,y), \tag{1.2}$$

for all distinct $x,y\in X$, then, T posses a unique fixed point in X.

Theorem 1.4. ([3]) *Let (X,d) be a complete metric space and $T:X\to X$ be a self-mapping. If there exist $\alpha,\beta\in[0,1)$, with $\alpha+\beta<1$ such that*

$$d(Tx,Ty)\le\alpha\cdot d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}+\beta d(x,y) \tag{1.3}$$

for all $x,y\in X$, then, T has a unique fixed point $u\in X$. Moreover, the sequence $\{T^n x\}$ converges to the fixed point u for all $x\in X$.

In this paper, we provide answers for the question whether there is a contraction condition which does not force the mapping to be continuous at the fixed point. In particular, we propose a contraction conditions in rational forms that possess a fixed point but not need to be continuous at the given fixed point.

2. Main results

The following is the first main results of this paper.

Theorem 2.1. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

(i) *T^2 is continuous,*
(ii) *$d(Tx, Ty) \leq \phi(R(x, y))$, where ϕ is a self-mapping on $\mathbb{R}_0^+$ such that $\phi(t) < t$ for each $t > 0$;*
(iii) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < R(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon,$$

where

$$R(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}, x \neq y,$$

then T has a unique fixed point, say z, and and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x \to z} R(x, z) \neq 0.$$

Remark 2.2. The last conclusion of Theorem 1.1 can be written as, T is continuous at z if and only if

$$\lim_{x \to z} d(x, Tx) = 0, \tag{2.1}$$

since $d(z, Tz) = 0$. The same remark is also valid for Theorem 2.1. Thus, the second conclusion of Theorem 2.1 could be represented as (2.1).

Proof. Let $x_0 \in X$. We built an iterative sequence $\{x_n\}$ in X by letting

$$x_n = T^n x_0 = Tx_{n-1}, \text{ for } n \in \mathbb{N}.$$

In case of $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}_0$, we conclude that $x^* = x_{n_0}$ forms a fixed point for T which completes the proof. Consequently, throughout the proof, we assume that

$$x_n \neq x_{n+1} \Leftrightarrow d(x_n, x_{n+1}) > 0 \text{ for all } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{2.2}$$

Furthermore, we shall assume that

$$x_n \neq x_{n+k} \Leftrightarrow d(x_n, x_{n+k}) > 0 \text{ for all } n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \text{ and } k \in \mathbb{N}. \tag{2.3}$$

Indeed, if $x_n = x_{n+k}$, then $T^k x_n = x_n$, that is, x_n is a fixed point of T^k. Now, we shall show that , x_n is also a fixed point of T. To show this, it is sufficient to use (ii) with the method of *Reductio ad Absurdum.*

Suppose, on the contrary that $d(Tx_n, x_n) > 0$.

$$\begin{aligned}
0 < d(Tx_n, x_n) &= d(T^{k+1}x_n, T^k x_n) = d(T^k(Tx_n), T^k x_n) \\
&\le \phi(R(T^{k-1}(Tx_n), T^{k-1}x_n)) = \phi(R(T^k x_n, T^{k-1}x_n)) \\
&< R(T^k x_n, T^{k-1}x_n) \\
&= \max\left\{\frac{d(T^k x_n, TT^k x_n)d(T^{k-1}x_n, TT^{k-1}x_n)}{d(T^k x_n, T^{k-1}x_n)}, d(T^k x_n, T^{k-1}x_n)\right\} \\
&= \max\left\{\frac{d(T^k x_n, T^{k+1}x_n)d(T^{k-1}x_n, T^k x_n)}{d(T^k x_n, T^{k-1}x_n)}, d(T^k x_n, T^{k-1}x_n)\right\} \\
&= d(T^k x_n, T^{k-1}x_n). \qquad (2.4)
\end{aligned}$$

Notice that this is the only possible case, since, the case

$$R(T^k x_n, T^{k-1}x_n) = d(T^k x_n, T^{k+1}x_n)$$

brings a contradiction. Recursively, after $k-1$ steps, we find that

$$\begin{aligned}
0 < d(Tx_n, x_n) \quad &= d(T^k(Tx_n), T^k x_n) \le \phi(R(T^{k-1}(Tx_n), T^{k-1}x_n)) \\
&< R(T^{k-1}(Tx_n), T^{k-1}x_n) = d(T^{k-1}(Tx_n), T^{k-1}x_n) \\
&\le \cdots \\
&\le \phi(d(Tx_n, x_n)) < d(Tx_n, x_n),
\end{aligned} \qquad (2.5)$$

a contradiction. Hence, we deduce the validity of (2.3).
By taking $x = x_n$ and $y = x_{n+1}$ in the inequality in (ii) together with (2.2), we derive that

$$\begin{aligned}
d(x_n, x_{n+1}) \quad &= d(Tx_{n-1}, T_n) \le \phi(R(x, y)) \\
&= \phi\left(\max\left\{\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\}\right) \\
&= \phi\left(\max\left\{\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\right\}\right) \\
&< \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}.
\end{aligned} \qquad (2.6)$$

Since the case

$$d(x_n, x_{n+1}) \ge d(x_{n-1}, x_n)$$

yields a contradiction, we conclude that the inequality (2.6) turns into

$$d(x_n, x_{n+1}) \le \phi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n). \qquad (2.7)$$

Thus, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below by 0. Accordingly, the sequence $\{d(x_n, x_{n+1})\}$ is convergent to some $L \ge 0$. We shall show that $L = 0$.

Suppose, on the contrary, that $L > 0$. For this $L = \varepsilon$, there exists a positive integer $k_0 \in \mathbb{N}$ such that

$$L < d(x_n, x_{n+1}) < L + \delta(L), \text{ for all } n \geq k_0. \tag{2.8}$$

Employing (iii) together with (2.7), we conclude that $d(x_n, x_{n+1}) < L$, a contradiction. Accordingly, we conclude that $L = 0$.

As a next step, we shall show that the recursive sequence $\{x_n\}$ is Cauchy. Fix an $\varepsilon > 0$. Without loss of generality, we may assume that $\delta = \delta(\varepsilon) < \varepsilon$. On account of the fact that $\{d(x_n, x_{n+1})\}$ converges to 0, there positive integer $k \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{2}\delta, \text{ for all } n \geq k. \tag{2.9}$$

By using the method of the induction, we shall show that, for any $n \in \mathbb{N}$,

$$d(x_n, x_{n+k}) \leq \varepsilon + \frac{1}{2}\delta. \tag{2.10}$$

It is clear that the inequality (2.10) holds for $n = 1$. Now, we suppose that the inequality (2.10) is satisfied for n. To show our assertion, we shall prove it for $n+1$. Taking the triangle inequality into account, we have

$$d(x_k, x_{n+k+1}) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{n+k+1}) \tag{2.11}$$

To conclude our claim, it is enough to indicate that

$$d(x_{k+1}, x_{n+k+1}) \leq \varepsilon. \tag{2.12}$$

Indeed, when we prove that $R(x_k, x_{n+k}) \leq \varepsilon + \delta(\varepsilon)$, by (iii) together with (2.3), we deduce (2.12), where

$$R(x_k, x_{n+k}) = \max\left\{\frac{d(x_k, Tx_k)d(x_{n+k}, Tx_{n+k})}{d(x_k, x_{n+k})}, d(x_k, x_{n+k})\right\}. \tag{2.13}$$

Regarding the induction assumptions (2.9) and (2.10), we derive that

$$R(x_k, x_{n+k}) < \varepsilon + \delta. \tag{2.14}$$

On account of (iii), we get that $d(x_{k+1}, x_{n+k+1}) < \varepsilon$ that completes the induction. Moreover, (2.10) yields that the constructed sequence $\{x_n\}$ is Cauchy.

Regarding that X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n = x_{n+1} \to z$ and $T^2x_n \to z$. By continuity of T^2, we have $T^2x_n \to T^2z$. This implies $T^2z = z$.

As a final step, we assert that $Tz = z$. Suppose, on the contrary, that $z \neq Tz$. So, we have

$$\begin{aligned} d(z, Tz) = &\ d(T^2z, Tz) \leq \phi(R(z, Tz)) < R(z, Tz) \\ = &\ \max\left\{\frac{d(z, Tz)d(Tz, T^2z)}{d(z, Tz)}, d(z, Tz)\right\} = d(z, Tz), \end{aligned} \tag{2.15}$$

which is a contradiction. Consequently, z is a fixed point of T. It is easy to see that the uniqueness of the fixed point follows from (ii).

Example 2.3. Let d be the standard metric on $X := \mathbb{R}_0^+$. Define a self-mapping T on X by

$$T(x) = 1 \text{ if } x \leq 1, T(x) = 0 \text{ if } x > 1.$$

It is easy to see that T fulfills the axioms of Theorem 2.1 and has a unique fixed point $x = 1$. The mapping T fulfills the contractive condition (ii) with $\phi(t) = 1$ for $t > 1$ and $\phi(t) = \frac{t}{3}$ for $t \leq 1$. Also, T fulfills the condition (iii) with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$. Notice also that

$$\lim_{x \to 1} R(x, 1) \neq 0$$

and T is discontinuous at the fixed point $x = 1$. Furthermore, $\phi(t)$ is not upper semi-continuous at $t = 1$ and $\delta(\varepsilon)$ is not lower semi-continuous at ε =1. On the other hand, T^2 is continuous, since $T^2(x) = 1$ for all $x \in X$.

Theorem 2.4. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

(*i*) T^2 *is continuous,*
(*ii*) $d(Tx, Ty) \leq \phi(Q(x, y))$, *where ϕ is a self-mapping on $\mathbb{R}_0^+$ such that $\phi(t) < t$ for each $t > 0$;*
(*iii*) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < Q(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon,$$

where

$$Q(x, y) = \max \left\{ d(y, Ty) \frac{1 + d(x, Tx)}{1 + d(x, y)}, d(x, y) \right\}$$

then T has a unique fixed point, say z, and and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x \to z} Q(x, z) \neq 0.$$

Proof. The proof is the mimic of the proof of Theorem 2.1. As in the proof of Theorem 2.1, we construct an recursive sequence $\{x_n\}$ in X by letting $x_n = T^n x_0 = T x_{n-1}$, for $n \in \mathbb{N}$. Analogously, we derive that (2.2).

On account of (ii), we derive that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, T_n) \leq \phi(Q(x, y)) \\ &= \phi\left(\max \left\{ d(x_n, Tx_n) \frac{1 + d(x_{n-1}, Tx_{n-1})}{1 + d(x_{n-1}, x_n)}, d(x_{n-1}, x_n) \right\} \right) \\ &= \phi\left(\max \left\{ d(x_n, x_{n+1}) \frac{1 + d(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_n)}, d(x_{n-1}, x_n) \right\} \right) \\ &< \max \left\{ d(x_n, x_{n+1}), d(x_{n-1}, x_n) \right\}. \end{aligned} \tag{2.16}$$

Note that the case

$$d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$$

bring a contradiction. Accordingly, we deduce that the inequality (2.16) turns into

$$d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n). \tag{2.17}$$

Consequently, we observe that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below by 0. Thus, the sequence $\{d(x_n, x_{n+1})\}$ is convergent to some $L \geq 0$. We shall show that $L = 0$. Suppose, on the contrary, that $L > 0$. For this $L = \varepsilon$, there exists a positive integer $k_0 \in \mathbb{N}$ such that

$$L < d(x_n, x_{n+1}) < L + \delta(L), \text{ for all } n \geq k_0. \tag{2.18}$$

Taking (iii) into account together with (2.17), we find that $d(x_n, x_{n+1}) < L$, a contradiction. Hence, we derive that $L = 0$.

To show that the sequence $\{x_n\}$ is Cauchy, we shall use the induction again, as in the proof of Theorem 2.1. First, we fix an $\varepsilon > 0$ and suppose that $\delta = \delta(\varepsilon) < \varepsilon$, without loss of generality. Since $\{d(x_n, x_{n+1})\}$ converges to 0, there positive integer $k \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) \leq \frac{1}{2}\delta, \text{ for all } n \geq k. \tag{2.19}$$

Our goal is to get the inequality below

$$d(x_n, x_{n+k}) \leq \varepsilon + \frac{1}{2}\delta, \text{for any } n \in \mathbb{N}, \tag{2.20}$$

by using the induction steps. The inequality (2.20) trivially holds for $n = 1$. Now, we assume that the inequality (2.20) is fulfilled for n, and show it holds for $n+1$. By the triangle inequality, we observe that

$$d(x_k, x_{n+k+1}) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{n+k+1}) \tag{2.21}$$

To conclude our claim, it is enough to indicate that

$$d(x_{k+1}, x_{n+k+1}) \leq \varepsilon. \tag{2.22}$$

Indeed, when we prove that $Q(x_k, x_{n+k}) \leq \varepsilon + \delta(\varepsilon)$, by $(iii$, we get (2.22), where

$$Q(x_k, x_{n+k}) = \max\left\{d(x_{n+k}, Tx_{n+k})\frac{1 + d(x_k, Tx_k)}{1 + d(x_k, x_{n+k})}, d(x_k, x_{n+k})\right\}. \tag{2.23}$$

Regarding the induction assumptions (2.19) and (2.20), we derive that

$$R(x_k, x_{n+k}) < \varepsilon + \delta. \tag{2.24}$$

On account of (iii), we get that $d(x_{k+1}, x_{n+k+1}) < \varepsilon$ that completes the induction. Moreover, (2.10) yields that the constructed sequence $\{x_n\}$ is Cauchy.
Regarding that X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n = x_{n+1} \to z$ and $T^2x_n \to z$. By continuity of T^2, we have $T^2x_n \to T^2z$. This implies $T^2z = z$.
As a final step, we assert that $Tz = z$. Suppose, on the contrary, that $z \neq Tz$. So, we have

$$\begin{aligned} d(z, Tz) = \;& d(T^2z, Tz) \leq \phi(Q(z, Tz)) < Q(z, Tz) \\ &= \max\left\{d(Tz, T^2z)\frac{1 + d(z, Tz)}{1 + d(z, Tz)}, d(z, Tz)\right\} = d(z, Tz), \end{aligned} \tag{2.25}$$

a contradiction. As a result, we derive that z is a fixed point of T. The uniqueness of the fixed point follows from (ii).

Example 2.5. Let d be the standard metric on $X := [0, 2]$. Define a self-mapping T on X by

$$T(x) = \frac{1}{2} \text{ if } x \leq \frac{1}{2}, \text{ and } T(x) = \frac{x}{8} \text{ if } x > \frac{1}{2}.$$

It is easy to see that T fulfills the axioms of Theorem 2.4 and has a unique fixed point $x = \frac{1}{2}$.

Corollary 2.6. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

(*i*) *T^2 is continuous,*
(*ii*) *$d(Tx, Ty) < R(x, y)$,*
(*iii*) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < R(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon,$$

where

$$R(x, y) = \max\left\{\frac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}$$

then T has a unique fixed point, say z, and and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x \to z} R(x, z) \neq 0.$$

Corollary 2.7. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

(*i*) *T^2 is continuous,*
(*ii*) *$d(Tx, Ty) < Q(x, y)$,*
(*iii*) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < R(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon,$$

where

$$Q(x, y) = \max\left\{d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}, d(x, y)\right\}$$

then T has a unique fixed point, say z, and and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x \to z} Q(x, z) \neq 0.$$

Corollary 2.8. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

(*i*) *T^2 is continuous,*
(*ii*) *$d(Tx, Ty) \leq \phi(d(x, y))$, where ϕ is a self-mapping on $\mathbb{R}_0^+$ such that $\phi(t) < t$ for each $t > 0$;*
(*iii*) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < R(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon,$$

then T has a unique fixed point, say z, and and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x \to z} d(x, z) \neq 0.$$

Corollary 2.9. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

(i) *T^2 is continuous,*

(ii) *$d(Tx, Ty) \leq \phi(r_i(x, y))$ holds for $i = 1$ or $i = 2$ where ϕ is a self-mapping on $\mathbb{R}_0^+$ such that $\phi(t) < t$ for each $t > 0$;*

(iii) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < r_i(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon, \text{ for } i = 1 \text{ or } i = 2$$

where

$$r_1(x, y) = \frac{d(y, Ty)d(x, Tx)}{d(x, y)} \text{ and } r_2(x, y) = d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}.$$

Then T has a unique fixed point, say z, and and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x \to z} r_i(x, z) \neq 0, \text{ for } i = 1 \text{ or } i = 2.$$

Corollary 2.10. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

(i) *T^2 is continuous,*

(ii) *$d(Tx, Ty) < r_i(x, y)$ holds for $i = 1$ or $i = 2$ where ϕ is a self-mapping on $\mathbb{R}_0^+$ such that $\phi(t) < t$ for each $t > 0$;*

(iii) *for a given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that*

$$\varepsilon < r_i(x, y) < \varepsilon + \delta(\varepsilon) \text{ implies } d(Tx, Ty) \leq \varepsilon, \text{ for } i = 1 \text{ or } i = 2$$

where

$$r_1(x, y) = \frac{d(y, Ty)d(x, Tx)}{d(x, y)} \text{ and } r_2(x, y) = d(y, Ty)\frac{1 + d(x, Tx)}{1 + d(x, y)}.$$

Then T has a unique fixed point, say z, and and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z if and only if

$$\lim_{x \to z} r_i(x, z) \neq 0, \text{ for } i = 1 \text{ or } i = 2.$$

Remark 2.11. Note that the discussion in Remark 2.2 is also valid for Theorem 2.4 and the consequences of both Theorem 2.1 and Theorem 2.4.

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