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DISCUSSION ON THE EXISTENCE OF BEST PROXIMITY POINTS IN METRIC SPACES

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Abstract. In this paper, we modify the definition of some generalized proximal contractions and enumerate a list of equivalent conditions for various versions of generalized proximal contractions of non-self set-valued mappings on (ordered) metric spaces. By using the fixed point means, we establish the existence of best proximity points for mappings involving such contractions which extend and improve many existing related results, as well as, reveal that most of existing best proximity point theorems on metric spaces are in fact equivalent and immediate consequences of well-known fixed point theorems. Finally, some examples are given to support our results.
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1. Introduction

Let (X, d) be a metric space and $A, B \subset X$ be nonempty. A set-valued mapping $T : A \to 2^B$ (or a single-valued mapping $T : A \to B$) is called non-self if $A \not\subset B$. In this situation, T may have no fixed point since $T(A) \cap A$ may be empty. Thus, to investigate the existence of best proximity points, that is, to establish a point x such that $d(x, Tx) = d(A, B) = \inf\{d(x, y) : x \in A \text{ and } y \in B\}$, has received much attention in the last decades with a rapidly increasing number of related results on various contractive or nonexpansive mappings. For instance, see [1, 18] and many others. To survey the idea of the references, authors often follows the line of constructing fixed point theorems to establish the existence of best proximity points since a best proximity point is essentially a fixed point if the underlying mapping is a selfmapping. However, in [2] the author showed that some best proximity point theorems can be obtained from related fixed point theorems if the pair (A, B) has the P-property; in [27] the authors observed that the most best proximity point results on a metric space

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endowed with a partial order (under the P-property) can be deduced from existing fixed point theorems in the literature; in [25] the authors noticed that some existing fixed point results and recently announced best proximity point results are equivalent. In this paper, motivated by the above mentioned work, we will reveal the universality of the method in [2, 27, 25]. To wit, without the P-property, we can convert the existence of best proximity points for contractive mappings into the counterpart of fixed points for other mappings and then the desired results can be deduced from the corresponding fixed point theorems. In other words, most of existing best proximity point theorems are in fact immediate consequences of well-known fixed point theorems in the existing literature.

On the other hand, a number of authors generalize Banach's [7] and Nadler's [30] result and introduce the new concepts of (set-valued) contractions of Banach or Nadler type. Moreover, they study the problem concerning the existence of best proximity points for so-called proximal contractions. The existing various versions of generalized proximal contractions for non-self mappings seem to be based on the global sense, that is, the contractive conditions are required to hold for all $u_1, u_2, x_1, x_2 \in A$ with $d(u_1, Tx_1) = d(A, B)$ and $d(u_2, Tx_2) = d(A, B)$. In this paper, we weaken such conditions, i.e., we want contractive conditions to hold for partial points in A for which satisfy the above equations. This is where the main novelty of the present work lies, since such contraction condition is easier to satisfy and more convenient to be applied than those in the above mentioned literature. The importance of the present work also consists in establishing the existence of best proximity points for mappings involving such contractions, which extend and improve many existing relative results and our argumentation process is also simpler and clearer by using the fixed point means. As well as, we enumerate a list of equivalent conditions for various versions of such contractions for non-self set-valued mappings on (ordered) metric spaces, which further reveal that most of existing best proximity point theorems on metric spaces are equivalent indeed.

2. Preliminaries

For the metric space (X, d) and nonempty subsets $A, B \subset X$, we adopt the following notations:

$$\begin{aligned}
&d(x, B) = \inf\{d(x, b) : b \in B\},\ d(A, B) = \inf\{d(a, b) : a \in A, b \in B\},\\
&H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\},\\
&A_0 = \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\} \text{ and}\\
&B_0 = \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.
\end{aligned}$$

Definition 2.1. Let X be a metric space. A subset $C \subset X$ is said to be approximative if the set-valued mapping

$$\mathcal{P}_C(x) = \{y \in C : d(x, y) = d(x, C)\}, \forall x \in X$$

has nonempty values.

A set-valued mapping $T : C \to 2^C$ is said to have approximative values on C if Tx is approximative for each $x \in C$

Definition 2.2. Let $T : A \to 2^B$ be a set-valued mapping. An element $x \in A$ is called a generalized best proximity point of T if $d(gx, Tx) = d(A, B)$, where, $g : A \to A$ is an isometry, i.e., $d(gx, gy) = d(x, y)$ for all $x, y \in A$. In particular, x is called a best proximity point if $g = I$, where, I defined by $Ix = x$ for $x \in A$ is an identity.

$x \in A$ is called a Picard iterated (generalized) best proximity point if x is a (generalized) best proximity point and, for any $x_0 \in A_0$, there exists an iterated sequence $\{x_n\} \subset A$ such that $d(gx_n, Tx) = d(A, B))$ and $\lim_{n\to\infty} x_n = x$.

To harmonize relations between some contractions, we need the following functional families:

- Φ consists of all nondecreasing functions ϕ from $\mathbb{R}_+$, the set of all nonnegative reals, into itself and $\phi(t) = 0$ iff $t = 0$;
- $\widehat{\Phi}$ consists of all continuous functions $\phi \in \Phi$;
- $\widetilde{\Phi}$ consists of all functions $\phi \in \widehat{\Phi}$ with $\lim_{t\to\infty} \phi(t) = \infty$;
- Ψ consists of all lower semicontinuous functions ψ from $\mathbb{R}_+$ into itself and $\psi(t) = 0$ iff $t = 0$;
- $\widehat{\Psi}$ consists of all functions $\psi \in \Psi$ with $\liminf_{t\to\infty} \psi(t) > 0$;
- Γ consists of all functions γ from $\mathbb{R}_+$ into itself such that $\gamma(0) = 0$ and $\gamma(t_n) \to 0$ implies $t_n \to 0$ for any positive sequence $\{t_n\} \subset \mathbb{R}$;
- $\widehat{\Gamma}$ consists of all functions $\widehat{\gamma}$ from $\mathbb{R}_+$ into $[0, 1)$ such that $\widehat{\gamma}(t_n) \to 1$ implies $t_n \to 0$ for any bounded positive sequence $\{t_n\} \subset \mathbb{R}$;
- Ω consists of all right continuous and nondecreasing functions ω from $\mathbb{R}_+$ into itself such that $\omega(t) < t$ for $t > 0$;
- $\widehat{\Omega}$ consists of all continuous and nondecreasing functions $\widehat{\omega}$ from $\mathbb{R}_+$ into itself such that $\widehat{\omega}(t) < t$ for $t > 0$.

By virtue of Lemma 1 in [24], we enumerate a series of contractions which are each other equivalent.

Lemma 2.3. *Let (X, d) be a metric space, $T : X \to 2^X$ a set-valued mapping. Then the following definitions of generalized contractions are equivalent:*

(g1) *T is called g1-contractive if there exist $\widehat{\phi}, \overline{\phi} \in \widehat{\Phi}$ such that, for any $x, y \in X$,*

$$\widehat{\phi}(H(Tx, Ty)) \le \widehat{\phi}(M(x, y)) - \overline{\phi}(M(x, y)), \tag{1}$$

where, $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$;

(g2) *T is called g2-contractive if there exist $\widehat{\phi} \in \widehat{\Phi}$ and $\overline{\phi} \in \widehat{\Psi}$ such that (1) holds for any $x, y \in X$;*

(g3) *T is called g3-contractive if there exist $\widehat{\phi} \in \widehat{\Phi}$ and $\overline{\phi} \in \Phi$ such that (1) holds for any $x, y \in X$;*

(g4) *T is called g4-contractive if there exist $\widehat{\phi} \in \widehat{\Phi}$ and $\overline{\phi} \in \Gamma$ such that (1) holds for any $x, y \in X$;*

(g5) *T is called g5-contractive if there exist $\widehat{\phi} \in \widetilde{\Phi}$ and $\overline{\phi} \in \Psi$ such that (1) holds for any $x, y \in X$;*

(g6) *T is called g6-contractive if there exists $\omega \in \widehat{\Omega}$ such that for any $x, y \in X$,*

$$H(Tx, Ty) \leq \omega(M(x, y));$$

(g7) *T is called g7-contractive if there exist $\phi \in \widehat{\Phi}$ and $\omega \in \Omega$ such that for any $x, y \in X$,*

$$\phi(H(Tx, Ty)) \leq \omega(\phi(M(x, y)));$$

(g8) *T is called g8-contractive if there exists $\gamma \in \widehat{\Gamma}$ such that for any $x, y \in X$,*

$$H(Tx, Ty) \leq \gamma(M(x, y))M(x, y).$$

Proof. It follows directly from Lemma 1 in [24] applied to the set

$$D := \{(M(x, y), H(Tx, Ty)) : x, y \in X\}.$$

Recently, in order to establish the existence of best proximity points, the proximal contractions are employed in the non-self mappings. For instance, in [14] authors assumed that T satisfies the following contractive condition:

$$d(u_1, u_2) \leq \alpha d(x_1, x_2) + \beta d(x_1, u_1) + \gamma d(x_2, u_2) + \delta[d(x_1, u_2) + d(x_2, u_1)]$$

with $\alpha + \beta + \gamma + 2\delta < 1$, $d(u_1, Tx_1) = d(A, B)$ and $d(u_2, Tx_2) = d(A, B)$ for all $u_1, u_2, x_1, x_2 \in A$. This is the globality requirement. We introduce the following equivalent proximal contractions which relax the above "globality". To this end, we define the set-valued mapping $T^\sharp : A_0 \to 2^{A_0}$ as follows

$$T^\sharp(x) = \{u \in A : d(gu, Tx) = d(A, B)\}$$

for T and isometry g given as in Definition 2.2.

Definition 2.4. The set-valued mapping $T : A \to 2^B$ is said to have proximal approximative values if $T^\sharp$ has approximative values.

For the sake of convenience, let us define the set-valued mapping $\mathfrak{P} : A_0 \to 2^{A_0}$ by

$$\mathfrak{P}(x) = \mathcal{P}_{T^\sharp x}(x) = \{u \in T^\sharp x : d(x, u) = d(x, T^\sharp x)\} \text{ for } x \in A_0.$$

Lemma 2.5. *Let (X, d) be a metric space and $T : A \to 2^B$ have proximal approximative values. Then, for all $x \in A$, $\mathfrak{P}(x)$ is nonempty and $\mathfrak{P}(x) \subset \mathcal{P}_{\mathfrak{P}(x)}(x)$ which shows that $\mathfrak{P}$ has approximative values.*

Lemma 2.6. *Let (X, d) be a metric space and $T : A \to 2^B$ have proximal approximative values. Then the following definitions of generalized proximal contractions are equivalent:*

(gp1) *T is called gp1-contractive if there exist $\widehat{\phi}, \overline{\phi} \in \widehat{\Phi}$ such that, for any $x, y \in A$ and $u \in \mathfrak{P}(x)$, one possesses at least an element $v \in \mathfrak{P}(y)$ satisfying*

$$\widehat{\phi}(d(u,v)) \leq \widehat{\phi}(N(x,y,u,v)) - \overline{\phi}(N(x,y,u,v)), \tag{2}$$

where, $N(x,y,u,v) = \max\left\{d(x,y), d(x,u), d(y,v), \frac{d(x,v)+d(y,u)}{2}\right\}$.

(gp2) *T is called gp2-contractive if (2) holds for $\widehat{\phi} \in \widehat{\Phi}$ and $\overline{\phi} \in \widehat{\Psi}$ instead of $\widehat{\phi}, \overline{\phi} \in \widehat{\Phi}$.*

(gp3) *T is called gp3-contractive if (2) holds for $\widehat{\phi} \in \widehat{\Phi}$ and $\overline{\phi} \in \Phi$ instead of $\widehat{\phi}, \overline{\phi} \in \widehat{\Phi}$.*

(gp4) *T is called gp4-contractive if (2) holds for $\widehat{\phi} \in \widehat{\Phi}$ and $\overline{\phi} \in \Gamma$ instead of $\widehat{\phi}, \overline{\phi} \in \widehat{\Phi}$.*

(gp5) *T is called gp5-contractive if (2) holds for $\widehat{\phi} \in \widetilde{\Phi}$ and $\overline{\phi} \in \Psi$ instead of $\widehat{\phi}, \overline{\phi} \in \widehat{\Phi}$.*

(gp6) *T is called gp6-contractive if there exists $\omega \in \widehat{\Omega}$ such that, for any $x, y \in A$ and $u \in \mathfrak{P}(x)$, one possesses at least an element $v \in \mathfrak{P}(y)$ satisfying*

$$d(u,v) \leq \omega(N(x,y,u,v)).$$

(gp7) *T is called gp7-contractive if there exist $\phi \in \widehat{\Phi}$ and $\omega \in \Omega$ such that, for any $x, y \in A$ and $u \in \mathfrak{P}(x)$, one possesses at least an element $v \in \mathfrak{P}(y)$ satisfying*

$$\phi(d(u,v)) \leq \omega(\phi(N(x,y,u,v))).$$

(gp8) *T is called gp8-contractive if there exists $\gamma \in \widehat{\Gamma}$ such that, for any $x, y \in A$ and $u \in \mathfrak{P}(x)$, one possesses at least an element $v \in \mathfrak{P}(y)$ satisfying*

$$H(Tx,Ty) \leq \gamma(N(x,y,u,v))N(x,y,u,v).$$

Proof. It follows directly from Lemma 1 in [24] applied to the set

$$D := \{(M(x,y,u,v), d(u,v)) : x, y \in A, u \in \mathfrak{P}(x), v \in \mathfrak{P}(y)\}.$$

Remark 2.7. Under hypotheses of Lemma 2.3, if the inequality (2) holds for all $u, v \in A$ with $d(gu, Ty) = d(A,B)$ and $d(gv, Ty) = d(A,B)$, then T is called proximal g1-contractive. Similarly, we can define the proximal gi-contraction for $i = 2, 3, \ldots, 8$. Clearly, each proximal gi-contraction implies gpi-contraction for $i = 1, 2, \ldots, 8$ but the converse is not true. Moreover, the proximal gi-contractions reduces the common corresponding proximal contractions when $g = I$.

In addition, if T is a single-valued mapping and set pair (A,B) satisfies P-property, namely, for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$, $d(x_1,y_1) = d(A,B) = d(x_2,y_2)$ implies that $d(x_1,x_2) = d(y_1,y_2)$ (and it is called weak P-property if $d(x_1,x_2) \leq d(y_1,y_2)$), then each gi-contraction implies proximal gi-contraction for $i = 1, 2, \ldots, 8$, as well as, each proximal gi-contraction of S implies gi-contraction for $i = 1, 2, \ldots, 8$ if $T(A_0) \subset B_0$, where, S is the restriction on A_0 of T.

It is clear that one can convert seeking the best proximity point of T into establishing the fixed point of $T^\sharp$, that is,

Lemma 2.8. *An element $x \in A$ is a generalized best proximity point of T if and only if it is a fixed point of $T^\sharp$. Moreover, a fixed point of $\mathfrak{P}$ is the fixed point of $T^\sharp$.*

The set-valued mapping $T^\sharp$ possesses the following properties.

Lemma 2.9. *Let (X,d) be a metric space, $A_0, B_0 \subset X$ be nonempty subsets. Suppose that $g : A \to A$ is an isometry with $A_0 \subset g(A_0)$ and $T : A \to 2^B$ satisfies $T(A_0) \subset B_0$. We obtain*

- (a1) *For each $x \in A_0$, $T^\sharp x$ is nonempty. Moreover, $T^\sharp$ has closed values when A_0 is closed.*
- (a2) *T is gpi-contractive with $N(x,y,u,v) = d(x,y)$ if and only if $\mathfrak{P}$ is gi-contractive with $M(x,y) = d(x,y)$ for $i = 1,2,\dots,8$.*
- (a3) *If T is proximal gi-contractive with $N(x,y,u,v) = d(x,y)$ for $i = 1,2,\dots,8$, then $T^\sharp$ is a single-valued mapping.*

Proof. (a1). It is clear that $Tx \subset B_0$ for any $x \in A_0$. Hence, for any $w \in Tx$, there exists $\overline{u} \in A$ such that $d(\overline{u}, w) = d(A,B)$ and this evidently implies $\overline{u} \in A_0$. Note that $d(\overline{u}, w) \geq d(\overline{u}, Tx)$ by $w \in Tx$, we have $d(A,B) \geq d(\overline{u}, Tx)$ which yields that $d(\overline{u}, Tx) = d(A,B)$. By virtue of $A_0 \subset g(A_0)$, there exists $u \in A_0$ such that $\overline{u} = gu$ and hence $d(gu, Tx) = d(A,B)$. This shows that $u \in T^\sharp x$, i.e., $T^\sharp x$ is nonempty.

To prove that $T^\sharp x$ is closed, it suffices to show the implication of $u \in T^\sharp x$ for the limit point u of the sequence $\{u_n\}$ with $u_n \in T^\sharp x$. We observe that $u \in A_0$ since $u_n \in A_0$ and A_0 is closed. Moreover, by the continuity of g and the distance function, we have

$$d(gu, Tx) = \lim_{n\to\infty} d(gu_n, Tx) = d(A,B).$$

Therefore, $u \in T^\sharp x$.

(a2). By Lemma 2.3 and Lemma 2.4 it is sufficient to check that T is gp6-contractive with $N(x,y,u,v) = d(x,y)$ if and only if $\mathfrak{P}$ is g6-contractive with $M(x,y) = d(x,y)$. If $\mathfrak{P}$ is g6-contractive with $M(x,y) = d(x,y)$, there exists $\omega \in \widehat{\Omega}$ such that $H(\mathfrak{P}x, \mathfrak{P}y) \leq \omega(d(x,y))$. For any $u \in \mathfrak{P}x$, there exists $v \in \mathfrak{P}y$ such that $d(u,v) = d(u, \mathfrak{P}y)$ and hence

$$d(u,v) = d(u, \mathfrak{P}y) \leq \sup_{u^* \in \mathfrak{P}x} d(u^*, \mathfrak{P}y) \leq H(\mathfrak{P}x, \mathfrak{P}y) \leq \omega(d(x,y)).$$

This yields that T is gp6-contractive. For the proof of "only if" it will be able to refer to the following Lemma 3.1.

(a3). Let $x \in X$ and $u_1, u_2 \in T^\sharp x$, then $d(gu_1, Tx) = d(A,B) = d(gu_2, Tx)$. Without the lost of generality, suppose that T is a proximal g1-contraction with $N(x,y,u,v) = d(x,y)$. There exist $\widehat{\phi}, \overline{\phi} \in \widehat{\Phi}$ such that

$$\widehat{\phi}(d(u_1,u_2)) \leq \widehat{\phi}(d(x,x)) - \overline{\phi}(d(x,x)).$$

This implies $u_1 = u_2$ and hence $T^\sharp$ is single-valued.

Example 2.10. Let $X = \mathbb{R}^2$ be a metric space with the usual distance. Consider subsets

$$A = \left\{ (x,y) : x - y = 1,\ x \in \left\{\frac{1}{2}\right\} \cup \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \right\}$$

and

$$B = \{(x, y) : x - y = -1,\ -1 \leq x \leq 0\}$$

and the set-valued mapping $T : A \to 2^B$ defined as

$$T(x,y) = \begin{cases} (-x, -y), & (x,y) \in A \text{ and } x \neq \frac{1}{2}, \\ ([-\frac{2}{3}, -\frac{1}{3}] \times [\frac{1}{3}, \frac{2}{3}]) \cap B, & (x,y) = (\frac{1}{2}, -\frac{1}{2}). \end{cases}$$

Then T has proximal approximative values and is a gp6-contraction but not a proximal g6-contraction.

Proof. It is easy to see that $A = A_0$, $B = B_0$, $T(A_0) \subset B_0$ and $d(A, B) = \sqrt{2}$. Let $g = I$. Next, for any $(x, y) \in A$, either $(x, y) = \left(\frac{1}{2}, -\frac{1}{2}\right)$, we have

$$T^\sharp\left(\frac{1}{2}, -\frac{1}{2}\right) = \left\{\left(\frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{2}{3}, -\frac{1}{3}\right)\right\};$$

or $(x, y) \neq (\frac{1}{2}, -\frac{1}{2})$, we have $T^\sharp(x, y) = (1 - x, -x)$. It reduces that T has proximal approximative values. In addition,

$$\mathfrak{P}\left(\left(\frac{1}{2}, -\frac{1}{2}\right)\right) = \left\{\left(\frac{1}{2}, -\frac{1}{2}\right)\right\}$$

and $\mathfrak{P}((x, y)) = (1 - x, -x)$ with $(x, y) \neq (\frac{1}{2}, -\frac{1}{2})$.

In order to check that T is a gp6-contraction, we put $\omega(t) = \frac{2}{3}t$ for $t \geq 0$. To avoid elementary computation, we omit the verification process.

To check that T is not a proximal g6-contraction, we take

$$(x_1, x_2) = \left(\frac{1}{2}, -\frac{1}{2}\right),\ (x_2, y_2) = \left(\frac{2}{3}, -\frac{1}{3}\right) = (u_1, v_1)$$

and

$$(u_2, v_2) = \left(\frac{1}{3}, -\frac{2}{3}\right).$$

Then $d((u_1, v_1), (u_2, v_2)) = \frac{\sqrt{2}}{3}$ and $N = \frac{\sqrt{2}}{6}$. Hence, $d((u_1, v_1), (u_2, v_2)) > \omega(N)$.

3. Results for contractive mappings

In the sequel, we always assume that (X, d) is a metric space and $A_0, B_0 \subset X$ are nonempty subsets unless otherwise specified. We first need the following auxiliary results.

Lemma 3.1. *Suppose that $g : A \to A$ is an isometry and $T : A \to 2^B$ has proximal approximative values.*

Then $\mathfrak{P}$ is a gi-contraction if T is gpi-contractive for $i = 1, 2, \ldots, 8$.

Proof. Let $x, y \in A_0$ be any given. For any $\varepsilon > 0$, the definition of supremum guarantees that there exists $u \in \mathfrak{P}(x)$ such that

$$d(u, \mathfrak{P}y) \geq \sup_{u^* \in \mathfrak{P}x} d(u^*, \mathfrak{P}y) - \varepsilon. \tag{3}$$

In virtue of Lemma 2.4 it is sufficient to prove that $\mathfrak{P}$ is g6-contractive under the hypothesis that T is gp6-contractive. Thus there exists $\omega \in \widehat{\Omega}$ such that one possesses an element $v \in \mathfrak{P}(y)$ with

$$d(u,v) \leq \omega(N(x,y,u,v)). \tag{4}$$

Therefore, by (3) and the monotonicity of ω we have

$$\sup_{u^* \in \mathfrak{P}x} d(u^*, \mathfrak{P}y) \leq d(u, \mathfrak{P}y) + \varepsilon \leq d(u,v) + \varepsilon \leq \omega(N(x,y,u,v)) + \varepsilon. \tag{5}$$

Note that $v \in T^\sharp y$ and $d(y,v) = d(y, T^\sharp y)$, we have $d(x,v) \leq d(x,y) + d(y,T^\sharp y)$. Similarly, $d(y,u) \leq d(x,y) + d(x,T^\sharp x)$. This yields that $N(x,y,u,v) \leq M(x,y)$. By means of this, together with (5) and letting $\varepsilon \to 0$, we have

$$\sup_{u^* \in \mathfrak{P}x} d(u^*, \mathfrak{P}y) \leq \omega(N(x,y,u,v)) \leq \omega(M(x,y)).$$

We can similarly infer

$$\sup_{v^* \in \mathfrak{P}y} d(v^*, \mathfrak{P}x) \leq \omega(M(x,y)).$$

Hence

$$H(\mathfrak{P}x, \mathfrak{P}y) \leq \omega(M(x,y)).$$

This yields that $\mathfrak{P}$ is g6-contractive.
The following fixed point theorem for the proof is analogue to Lemma 3.1 of [23].

Lemma 3.2. *Let X be complete and $C \subset X$ be a closed subset. If the set-valued mapping $S : C \to 2^C$ is a g6-contraction and has approximative values on C, then S has a Picard iterated fixed point x, namely, x is a fixed point of S and, for any $x_0 \in C$, there exists an iterated sequence $\{x_n\}$ with $x_n \in Sx_{n-1}$ such that $\lim_{n\to\infty} x_n = x$.*

Theorem 3.3. *Let X be complete, A_0 be closed and g be an isometry with $A_0 \subset g(A_0)$. Let $T : A \to 2^B$ satisfy the following conditions:*

(a) $T(A_0) = \bigcup_{x \in A_0} Tx \subset B_0$.

(b) *T is one of gpi-contractions for $i = 1, 2, \ldots, 8$.*

(c) *Either A_0 is compact or T has proximal approximative values.*

Then T has a Picard iterated generalized best proximity point $x \in A$.

Proof. By means of Lemma 2.8, it suffices to prove the existence of fixed points of $\mathfrak{P}$. By virtue of Lemma 2.4 it is sufficient to assume that T is gp6-contractive. Lemma 3.1 guarantees that $\mathfrak{P}$ is a g6-contraction. Thus (c) and Lemma 3.2 show that $\mathfrak{P}$ has at least a Picard iterated fixed point $x \in A_0$. This completes the proof.

Theorem 3.3 subsumes the main results in [25]. In the case that T is a single-valued mapping and the pair (A, B) satisfies P-property, by Remark 2.7 the g8-contraction of T implies its proximal g8-contraction and hence Theorem 3.3 generalizes the main results in [5, 16, 29, 39].

Corollary 3.4. *Let X be complete, A_0 be closed and g be an isometry with $A_0 \subset g(A_0)$. Let $T : A \to 2^B$ be proximal gi-contractive for $i = 1, 2, \ldots, 8$ and the conditions (a), (c) be valid. Then T has a Picard iterated generalized best proximity point $x \in A$.*

We remark that the "P-property" is unnecessary in Corollary 3.4. Therefore, it essentially extends and improves the main results of [3, 19, 36] in the sense that they deal with the proximal contractions of the first kind. Moreover, it shows the above mentioned results are equivalent.

Corollary 3.5. *Let X be complete, A_0 closed and g an isometry with $A_0 \subset g(A_0)$. Suppose that $T : A \to 2^B$ has proximal approximative values such that $T(A_0) \subset B_0$ and one of the following conditions holds*

(1) *there exist non-negative numbers $\alpha, \beta, \gamma, \delta$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that $u \in \mathfrak{P}x$ and $v \in \mathfrak{P}y$ for any $x, y \in A_0$ imply*
$$d(u,v) \le \alpha d(x,y) + \beta d(x,u) + \gamma d(y,v) + \delta[d(x,v) + d(y,u)].$$
(2) *For any $x, y, u, v \in A$ with $d(u, Tx) = d(A,B) = d(y, Ty)$ one has*
$$d(u,v) \le d(x,y) - \phi(d(x,y))$$
for $\phi \in \tilde{\Phi}$.
(3) *there exists a mapping $\gamma \in \widehat{\Gamma}$ such that, for all $x, y \in A_0$ and $u \in \mathfrak{P}(x)$, one possesses $v \in \mathfrak{P}(y)$ satisfying*
$$\psi(d(u,v)) \le \gamma(d(x,y))\psi(d(x,y)),$$
where, $\psi : [0,\infty) \to [0,\infty)$ is an increasing continuous functions such that $t \le \psi(t)$ for each $t \ge 0$ and $\psi(0) = 0$.

Then T has a Picard iterated generalized best proximity point $x \in A$.

Proof. Under the hypothesis (1) the result is immediate by Theorem 3.3 since T is gp6-contractive with $\omega(t) = (\alpha + \beta + \gamma + 2\delta)t$.

The hypothesis (2) implies the gp3-contraction by taking $\widehat{\phi}(t) = t$ and $\overline{\phi} \in \tilde{\Phi}$.

To prove (3), we observe that the inverse of ψ, ψ^{-1}, exists and $\psi^{-1}(t) \le t$ for $t \ge 0$. Therefore, we have
$$d(u,v) \le \psi^{-1}(\gamma(d(x,y))\psi(d(x,y))).$$
As an analogous of the proof of Lemma 3.1, we deduce that $\mathfrak{P}$ satisfies
$$H(\mathfrak{P}(x), \mathfrak{P}(y)) \le \psi^{-1}(\gamma(d(x,y))\psi(d(x,y))).$$

It is sufficient to check that $\mathfrak{P}$ has a fixed point. Let $x_0 \in A_0$. If $x_0 \in \mathfrak{P}(x_0)$, our desire to achieve. Otherwise, by Lemma 2.6 there exists $x_1 \in \mathfrak{P}(x_0)$ with $x_1 \ne x_0$ such that $d(x_0, x_1) = d(x_0, \mathfrak{P}(x_0))$. Continuing this process, we can define a sequence $\{x_n\}$ in A_0 by $x_n \in \mathfrak{P}(x_{n-1})$ such that, for all $n \in \mathbb{N}$, either $x_n = x_{n-1}$ which completes our proof, or $x_n \ne x_{n-1}$ and
$$d(x_n, x_{n+1}) = d(x_n, \mathfrak{P}(x_n)).$$
It is easy to see
$$d(x_n, x_{n-1}) \le H(\mathfrak{P}(x_{n-2}), \mathfrak{P}(x_{n-1})) \le \psi^{-1}(\gamma(d(x_{n-2}, x_{n-1}))\psi(d(x_{n-2}, x_{n-1})))$$
for all $n \in \mathbb{N}$. Now following the line of arguments in Theorem 3.1 of [6], we get that $\{x_n\}$ is a Cauchy sequence. Since X is complete and A_0 is closed, then there exists $x \in A_0$ such that $\lim_{n\to\infty} x_n = x$.

We finally verify that x is a fixed point of $\mathfrak{P}$. Note that

$$d(x_n, \mathfrak{P}(x)) \leq H(\mathfrak{P}(x_{n-1}), \mathfrak{P}(x)) \leq \psi^{-1}(\gamma(d(x_{n-1}, x))\psi(d(x_{n-1}, x))) \leq d(x_{n-1}, x)$$

for all $n \in \mathbb{N}$, we obtain that $d(x, \mathfrak{P}(x)) = 0$ by letting $n \to \infty$ on two sides of the above inequality. By Lemma 2.6, there exists $u \in \mathfrak{P}(x)$ such that $d(x, u) = d(x, \mathfrak{P}(x)) = 0$ which reduces $x = u \in \mathfrak{P}(x)$. This proof is complete.

Remark 3.6. Corollary 3.5 extends and improves a lot of existing results. For example, (1) is an improvement of the main results in [14] which include Theorem 3.1 and its corollaries in [8, 9, 14, 17, 37], as well as, Theorem 3.3 and its corollaries in [10]; (2) generalizes Theorems 3.1 and 3.6 in [11], as well as, relaxes the hypothesis of P-property in [18]; (3) is an extension and improvement of main results in [6].

Example 3.7. Let the space X, the subsets A, B and the set-valued mapping T be given as Example 2.10. Then T has a Picard iterated generalized best proximity point $\left(\frac{1}{2}, -\frac{1}{2}\right) \in A$ by Theorem 3.3.

Example 3.8. Consider $X = \mathbb{R}$ with the usual metric,

$$A = \{-10, 10\} \quad \text{and} \quad B = \{-2, 2\}.$$

Then $T : A \to 2^B$ given by

$$T(-10) = \{-2, 2\} \quad \text{and} \quad T(10) = \{-2\}$$

has a Picard iterated best proximity point $-10 \in A$.

It is worth noting that the pair (A, B) in Example 3.8 does not have the (weak) P-property.

4. Results for α-admissible mappings

The notion of the α-admissible has recently been applied to establish the existence of fixed points and best proximity points. We refer to references [4, 28, 35]. For a set-valued mapping T, we first modify the concept of α-admissible.

Definition 4.1. Let $T : X \to 2^X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. T is called α-admissible with respect to η on X if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that, for any $u \in Tx$, there exists $v \in \mathcal{P}_{Ty}(y)$ such that $\alpha(u, v) \geq \eta(u, v)$.

Let $T : A \to 2^B$ and $\alpha : A \times A \to [0, \infty)$. T is called α-proximal admissible with respect to η on A if, for $x_1, x_2 \in A$ and $u_1 \in T^\sharp x_1$, there exists $u_2 \in \mathfrak{P}(x_2)$ with $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ such that $\alpha(u_1, u_2) \geq \eta(u_1, u_2)$.

Note that Definition 4.1 subsumes the α-proximal admissible notion given as [4, 28, 26] even if we take $\eta(x, y) = 1$, further, reduces to the α-admissible notion given as [35] if T is a single-valued mapping. Also, according to [35], T is said to be an η-subadmissible mapping if we take $\alpha(x, y) = 1$.

Definition 4.2. The set-valued mapping $T : X \to 2^X$ is called α-η-gi-contractive if the hypothesis of (gi) for $i = 1, 2, \ldots, 8$ is satisfied for any $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$.

The set-valued mapping $T : A \to 2^B$ is called α-η-gpi-contractive if the hypothesis of (gpi) for $i = 1, 2, \ldots, 8$ is satisfied for any $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$.

We remark that α-η-gi-contractions (α-η-gpi-contractions) for $i = 1, 2, \ldots, 8$ are equivalent.

Lemma 4.3. *Let the hypotheses of Lemma 3.1 hold. Then T is α-proximal admissible with respect to η on A if and only if $T^\sharp$ is α-admissible with respect to η on A_0. Moreover, $\mathfrak{P}$ is α-η- gi-contractive if T is α-η-gpi-contractive for $i = 1, 2, \ldots, 8$.*

We need the following fixed point theorem which is a generalization of [35].

Lemma 4.4. *Let X be complete, the set-valued mapping $S : X \to 2^X$ be α-admissible with respect to η on X and one of α-η-gi-contractions for $i = 1, 2, \ldots, 8$. Suppose that S has approximative values and the following assertions hold:*

(i) *there exist $x_0 \in X$ and $x_1 \in \mathcal{P}_{Sx_0}(x_0)$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;*
(ii) *for any sequence $\{x_n\} \subset X$ converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq \eta((x_n, x)$ for all $n \in \mathbb{N}$.*

Then S has a fixed point x.
Further, there exists a sequence $\{x_n\}$, defined by $x_{n+1} \in Sx_n$ for $n \geq 1$, that converges to the element x.
Proof. It is sufficient to assume that S is α-η-g6-contractive by virtue of the equivalence of contractions. Let $x_0, x_1 \in X$ be given in (i). Then $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and $d(x_0, x_1) = d(x_0, Sx_0)$. By the contractive condition, there exists $\omega \in \widehat{\Omega}$ such that $H(Sx_0, Sx_1) \leq \omega(M(x_0, x_1))$. Note that S is α-admissible with respect to η on X, for $x_1 \in Sx_0$, there exists $x_2 \in \mathcal{P}_{Sx_1}(x_1)$ such that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$. Applying the contractive condition again, we have $H(Sx_1, Sx_2) \leq \omega(M(x_1, x_2))$. Continuing this process, we can define a sequence $\{x_n\}$ in X by $x_n \in \mathcal{P}_{Sx_{n-1}}(x_{n-1})$ satisfying, for all $n \in \mathbb{N}$,

$$H(Sx_n, Sx_{n+1}) \leq \omega(M(x_n, x_{n+1})). \tag{6}$$

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}). \tag{7}$$

$$d(x_n, x_{n+1}) = d(x_n, Sx_n). \tag{8}$$

If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $x = x_n$ is a fixed point of S and the result is proved. Hence, we suppose that $x_{n+1} \neq x_n$, i.e, $x_n \notin Sx_n$ for all $n \in \mathbb{N}$. From (8) and definition of H it follows that

$$d(x_n, x_{n+1}) \leq H(Sx_{n-1}, Sx_n)$$

for all $n \in \mathbb{N}$. By means of (6) we have

$$d(x_n, x_{n+1}) \leq \omega(M(x_{n-1}, x_n)) \quad \text{for all } n \in \mathbb{N}. \tag{9}$$

On the other hand, by (8) we get

$$\begin{aligned} & M(x_{n-1}, x_n) \\ = \quad & \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, Sx_{n-1}), d(x_n, Sx_n), \frac{d(x_{n-1}, Sx_n) + d(x_n, Sx_{n-1})}{2}\right\} \\ = \quad & \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}. \end{aligned}$$

By (9), this implies that

$$d(x_n, x_{n+1}) \leq \omega\left(\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}\right)$$

for all $n \in \mathbb{N}$. We claim that

$$d(x_{n-1}, x_n) \geq d(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

Suppose the contrary, then $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$. By virtue of the properties of ω, for all $n \in \mathbb{N}$, we get

$$d(x_n, x_{n+1}) \leq H(Sx_{n-1}, Sx_n) \leq \omega\left(d(x_n, x_{n+1})\right) < d(x_n, x_{n+1}),$$

a contradiction. Hence (10) is valid. Moreover, in view of the monotonicity of ω one has

$$d(x_{n-1}, x_n) \leq \omega(d(x_{n-2}, x_{n-1})) < d(x_{n-2}, x_{n-1})$$

for all $n \in \mathbb{N}$. Repeating this procedure, we have

$$d(x_n, x_{n+1}) \leq \omega(d(x_{n-1}, x_n)) \leq \cdots \leq \omega^n(d(x_0, x_1))$$

for all $n \in \mathbb{N}$, where ω^n denotes the n-time-repeated composition of ω with itself. We observe that $\lim\limits_{n\to\infty} \omega^n(t) = 0$ uniformly for $t > 0$ if and only if $\omega(t) < t$ for $\omega \in \widehat{\Omega}$ (see, e.g., [38]) which implies that $d(x_{n-1}, x_n) \to 0$ when $n \to \infty$. Now, analogous to the proof of lemma 3.1 in [23], we can verify that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$.

In what follows, we check that x is a fixed point of S. By means of (ii) and (7), we have $\alpha(x_n, x) \geq \eta((x_n, x)$ for all $n \in \mathbb{N}$. From the contractive condition it follows that $H(Sx_n, Sx) \leq \omega(M(x_n, x))$. In addition, evidently, $d(x_{n+1}, Sx) \leq H(Sx_n, Sx)$ for $n \in \mathbb{N}$. Consequently, one has

$$d(x_{n+1}, Sx) \leq \omega(M(x_n, x)).$$

Note that

$$M(x_n, x) \leq \max\left\{d(x_n, x), d(x_n, x_{n+1}), d(x, Sx), \tfrac{d(x_n, x_{n+1}) + d(x_n, x) + d(x_n, x) + d(x, Sx)}{2}\right\}.$$

By the continuity of the distance and ω, letting n go to infinity, we obtain

$$d(x, Sx) \leq \omega(d(x, Sx)).$$

This reduces that $d(x, Sx) = 0$. Finally, by virtue of the fact that S is approximative, there exists $u \in \mathcal{P}_{Sx}(x)$ such that $d(x, u) = 0$, i.e., $u = x$ and hence u is a fixed point of S.

Theorem 4.5. *Let X be complete, A_0 be closed, $T : A \to 2^B$ have proximal approximative values and g be isometry with $A_0 \subset g(A_0)$. If $T(A_0) \subset B_0$, (ii) and the following conditions hold*

(I) *there exist* $x_0 \in A_0$ *and* $x_1 \in \mathfrak{P}(x_0)$ *such that* $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$*;*

(II) T *is* α*-proximal admissible with respect to* η *on* A*;*

(III) T *is one of* α*-*η*-gpi-contractions for* $i = 1, 2, \ldots, 8$*;*

then T *has a generalized best proximity point* $x \in A$*. Further, there exists the sequence* $\{x_n\}$*, defined by* $d(gx_{n+1}, Tx_n) = d(A, B)$ *for* $n \geq 1$*, that converges to the element* x*.*

Proof. In the light of Lemma 2.8, it suffices to verify that $\mathfrak{P}$ has a fixed point. Lemma 4.3 guarantees that $\mathfrak{P}$ satisfies the contractive condition of Lemma 4.4. Moreover, (I) implies the validity of (i). Therefore, $\mathfrak{P}$ meets all conditions of Lemma 4.4 which guarantees that $\mathfrak{P}$ has a fixed point in A_0. This proof is complete.

Example 4.6. Let $X = [0, +\infty) \times [0, \infty)$ be endowed with the usual metric. Suppose that $A = \{(1/2, x) : 0 \leq x < +\infty\}$ and $B = \{(0, x) : 0 \leq x < +\infty\}$. Then $T : A \to 2^B$ defined by

$$T\left(\frac{1}{2}, a\right) = \begin{cases} \left\{\left(0, \frac{x}{2}\right) : 0 \leq x \leq a\right\}, & a \in [0,1]/\{\frac{1}{n} : n \in \mathbb{N}\}, \\ \left(0, \frac{1}{a^2}\right), & a \in \{\frac{1}{n} : n \in \mathbb{N}\}, \\ \{(0, x^2) : 0 \leq x \leq a^2\}, & a > 1, \end{cases}$$

has a best proximity point in A.

Proof. Define $\alpha : A \times A \to [0, \infty)$ as follows

$$\alpha(x, y) = \begin{cases} 1, & x, y \in \left\{\left(\frac{1}{2}, a\right) : 0 \leq a \leq 1\right\}, \\ 0, & otherwise \end{cases} \quad \text{and} \quad \eta(x, y) \equiv \frac{1}{2}.$$

Notice that $d(A, B) = \frac{1}{2}, A_0 = A, B_0 = B$ and $T(A_0) \subset B_0$. Also,

$$T^\sharp\left(\frac{1}{2}, a\right) = \begin{cases} \left\{\left(\frac{1}{2}, u\right) : 0 \leq u \leq \frac{a}{2}\right\}, & a \in [0,1]/\{\frac{1}{n} : n \in \mathbb{N}\}, \\ \left(\frac{1}{2}, \frac{1}{a^2}\right), & a \in \{\frac{1}{n} : n \in \mathbb{N}\}, \\ \left\{\left(\frac{1}{2}, u\right) : 0 \leq u \leq a^2\right\}, & a \geq 1, \end{cases}$$

$$\mathfrak{P}\left(\frac{1}{2}, a\right) = \begin{cases} \left\{\left(\frac{1}{2}, \frac{a}{2}\right)\right\}, & a \in [0,1]/\{\frac{1}{n} : n \in \mathbb{N}\}, \\ \left(\frac{1}{2}, \frac{1}{a^2}\right), & a \in \{\frac{1}{n} : n \in \mathbb{N}\}, \\ \left\{\left(\frac{1}{2}, a\right)\right\}, & a \geq 1. \end{cases}$$

It is easy to check that T is α-proximal admissable with respect to η and (I) is valid with $x_0 = \left(\frac{1}{2}, \frac{3}{4}\right)$ and $x_1 = \left(\frac{1}{2}, \frac{3}{8}\right)$. Moreover, it is easy to see that T is an α-η-gp6-contraction with $\omega(t) = \frac{t}{2}$ for $t \geq 0$. Now all conditions of Theorem 4.5 are satisfied with $g = I$ and hence T satisfies the result of Theorem 4.5.

It is worth noting that T given in Example 4.6 is not α-proximal admissible in the sense of [4, 28, 26]. Consequently, Theorem 4.5 generalizes and improves the main results in [4, 28, 26]. In addition, if $\eta(x, y) = 1$ (resp. $\alpha(x, y) = 1$), then the condition (ii) is changed into

(IV) for any sequence $\{x_n\} \subset X$ converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ (resp. $\eta(x_n, x_{n+1}) \leq 1$) for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ (resp. $\eta(x_n, x) \leq 1$) for all $n \in \mathbb{N}$.

Corollary 4.7. *If $\eta(x,y) = 1$ (resp. $\alpha(x,y) = 1$), then, under Theorem 4.5 with (IV) instead of (ii), we obtain the result of Theorem 4.5.*

Corollary 4.7 extends and improves the main results in [22, 26] and implies the following results.

Corollary 4.8. *Let X be complete, A_0 be closed, $T : A \to 2^B$ have proximal approximative values and g be isometry with $A_0 \subset g(A_0)$. If $T(A_0) \subset B_0$, (I), (IV) and the following conditions hold*

(II') *T is α-proximal admissible;*

(III') *there exists $\omega \in \widehat{\Omega}$ such that, for any $x, y \in A$ and $u \in \mathfrak{P}(x)$, one possesses at least an element $v \in \mathfrak{P}(y)$ such that*

$$a(x,y)d(u,v) \leq \omega(N(x,y,u,v));$$

then the result of Theorem 4.5 holds.

We observe that Corollary 4.8 is an extension and improvement in the contractive sense of main results in [3, 31]. In addition, it is interesting to deduce the following especial corollaries.

Corollary 4.9. *Let X be complete, A_0 be closed, $T : A \to 2^B$ have proximal approximative values and g be isometry with $A_0 \subset g(A_0)$. If $T(A_0) \subset B_0$, (I), (II'), (IV) and the following condition hold: for any $x, y \in A$ and $u \in \mathfrak{P}(x)$, one possesses at least an element $v \in \mathfrak{P}(y)$ such that*

$$(a(x,y) + c)^{d(u,v)} \leq (1 + c)^{\omega(N(x,y,u,v))}$$

with $\omega \in \widehat{\Omega}$ and the constant $c > 0$, then there exists an element $x \in A$ such that $d(gx, Tx) = d(A, B)$.

Proof. Let $\alpha(x,y) \geq 1$. Then our hypothesis implies that

$$(1 + c)^{d(u,v)} \leq (a(x,y) + c)^{d(u,v)} \leq (1 + c)^{\omega(d(x,y))}.$$

This yields that $d(u,v) \leq \omega(N(x,y,u,v))$ and hence T is α-1-gp6-contractive. Now the conditions of Corollary 4.7 hold and hence T has a generalized best proximity point.

Corollary 4.10. *Let X be complete, A_0 be closed, $T : A \to 2^B$ have proximal approximative values and g be isometry with $A_0 \subset g(A_0)$. If $T(A_0) \subset B_0$, (I), (II'), (IV) and the following condition hold: for any $x, y \in A$ and $u \in \mathfrak{P}(x)$, one possesses at least an element $v \in \mathfrak{P}(y)$ such that*

$$(d(u,v) + c)^{a(x,y)} \leq \omega(N(x,y,u,v)) + c$$

with $\omega \in \widehat{\Omega}$ and the constant $c > 0$, then T has a generalized best proximity point in A_0.

Lemma 4.11. *Suppose that $g : A \to A$ is an isometry, $\alpha, \eta : X \times X \to [0, +\infty)$, $T : A \to 2^B$ satisfies $T(A_0) \subset B_0$ and*

(V) *there exist* $\omega \in \widehat{\Omega}$ *and* $\beta \in [0, \infty)$ *such that for all* $x, y \in A$ *and* $u \in \mathfrak{P}(x)$*, one possesses an element* $v \in \mathfrak{P}(y)$ *such that*

$$\eta(x, Tx) =: \inf_{u \in Tx} \eta(x, u) \leq \alpha(x, y) \text{ implies } d(u, v) \leq \omega(d(x, y)) + \beta[d(Tx, gy) - d(A, B)]. \tag{11}$$

Then $\mathfrak{P}$ *satisfies the following conclusion*

(iii) *for any* $x, y \in A$ *and* $\alpha(x, y) \geq \eta(x, Tx)$ *implies*

$$H(\mathfrak{P}(x), \mathfrak{P}(y)) \leq \omega(d(x, y)) + \beta[d(Tx, gy) - d(A, B)]$$

with $\beta \geq 0$ *and* $\omega \in \widehat{\Omega}$

and vice versa if T^{-1} *has approximative values on* A_0*.*

Proof. Let $x, y \in A_0$ be any given. For any $\varepsilon > 0$, there exists $u \in \mathfrak{P}(x)$ such that (3) is valid. By means of (V) for $\omega \in \widehat{\Omega}$ and $\beta \geq 0$, one possesses an element $v \in \mathfrak{P}(y)$ such that (11) holds. Therefore, by (3) and the nondecreasing of ω we have

$$\omega\left(\sup_{u^* \in \mathfrak{P}(x)} d(u^*, \mathfrak{P}(y))\right) \leq \omega(d(u, \mathfrak{P}(y)) + \varepsilon) \leq \omega(d(u, v) + \varepsilon). \tag{12}$$

Note that the continuity of ω, we can put $\omega(d(u, v) + \varepsilon) = \omega(d(u, v)) + o(\varepsilon)$, where $o(\varepsilon)$ stands for the infinitesimal of ε. substituting this for (12), we have

$$\begin{aligned} \omega\left(\sup_{u^* \in \mathfrak{P}(x)} d(u^*, \mathfrak{P}(y))\right) &\leq \omega(d(x, y)) + o(\varepsilon) \\ &\leq \omega(d(x, y)) + \beta[d(Tx, gy) - d(A, B)] + o(\varepsilon). \end{aligned}$$

Let $\varepsilon \to 0$, we have

$$\omega\left(\sup_{u^* \in \mathfrak{P}(x)} d(u^*, \mathfrak{P}(y))\right) \leq \omega(d(x, y)) + \beta[d(Tx, gy) - d(A, B)].$$

We can similarly infer

$$\omega\left(\sup_{v^* \in \mathfrak{P}(y)} d(v^*, \mathfrak{P}(x))\right) \leq \omega(d(x, y)) + \beta[d(Tx, gy) - d(A, B)].$$

Hence

$$\widehat{\phi}(H(\mathfrak{P}(x), \mathfrak{P}(y))) \leq \omega(d(x, y)) + \beta[d(Tx, gy) - d(A, B)].$$

This yields that $\mathfrak{P}$ satisfies (iii).

Conversely, if $T^\sharp$ has approximative values and is g1-contractive, then $\mathfrak{P}$ has approximative values and, for any $u \in \mathfrak{P}(x)$, there exists $v \in T^\sharp y$ such that $d(u, v) = d(u, \mathfrak{P}(y))$. On the other hand, one has

$$\omega(d(u, \mathfrak{P}(y))) \leq \omega\left(\sup_{u^* \in \mathfrak{P}(x)} d(u^*, \mathfrak{P}(y))\right) \leq \omega(d(x, y)) + \beta[d(Tx, gy) - d(A, B)].$$

Therefore,

$$\omega(d(u, v)) \leq \omega(d(x, y)) + \beta[d(Tx, gy) - d(A, B)].$$

This implies that T meets (V).

Lemma 4.12. *Let X be complete, A_0 be closed, g and T satisfy hypotheses of Lemma 4.11 with $\alpha, \eta : X \times X \to [0, +\infty)$ such that the following assertions hold:*

(i') *there exist $x_0 \in A_0$ and $x_1 \in \mathfrak{P}(x_0)$ such that $\alpha(x_0, x_1) \geq \eta(x_0, Tx_0)$;*

(ii') *for any sequence $\{x_n\} \subset A_0$ with $x_n \in \mathfrak{P}(x_{n-1})$, $\lim_{n\to\infty} x_n = x \in A_0$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, Tx_n)$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq \eta((x, Tx)$ for all $n \in \mathbb{N}$;*

(iv) *if $\alpha(x, y) \geq \eta(x, Tx)$ for $x, y \in A_0$, then, for any $u \in \mathfrak{P}(x)$, there exists $v \in \mathfrak{P}(y)$ such that $\alpha(u, v) \geq \eta(u, Ty)$.*

Then $\mathfrak{P}$ has a fixed point x. Moreover, there exists a sequence $\{x_n\}$ which satisfies

$$d(gx_{n+1}, Tx_n) = d(A, B) \text{ and } \lim_{n\to\infty} x_n = x.$$

Proof. Let $x_0, x_1 \in A_0$ be given in (i'), i.e., $x_1 \in \mathfrak{P}(x_0), \alpha(x_0, x_1) \geq \eta(x_0, Tx_0)$ and $d(x_0, x_1) = d(x_0, \mathfrak{P}(x_0))$. By Lemma 4.11, $\mathfrak{P}$ satisfies the contractive condition (iii) and hence there exists $\omega \in \widehat{\Omega}$ such that

$$H(\mathfrak{P}(x_0), \mathfrak{P}(x_1)) \leq \omega(d(x_0, x_1)) + \beta[d(gx_1, Tx_0) - d(A, B)] = \omega(d(x_0, x_1)).$$

By (iv), there exists $x_2 \in \mathfrak{P}(x_1)$ such that $\alpha(x_1, x_2) \geq \eta(x_1, Tx_1)$. Applying again (iii), we have

$$H(\mathfrak{P}(x_1), \mathfrak{P}(x_2)) \leq \omega(d(x_1, x_2)) + \beta[d(Tx_1, gx_2) - d(A, B)] = \omega(d(x_1, x_2)).$$

Proceeding this manner, we can construct a sequence $\{x_n\}$ in A_0 with $x_n \in \mathfrak{P}(x_{n-1})$ satisfying $H(\mathfrak{P}(x_n), \mathfrak{P}(x_{n+1})) \leq \omega(d(x_n, x_{n+1})), \alpha(x_n, x_{n+1}) \geq \eta(x_n, Tx_n)$ and $d(x_n, x_{n+1}) = d(x_n, Tx_n)$. Note that A_0 is complete, as an analogy of the proof of Lemma 4.4, we have verified that there exists $x \in A_0$ such that $\lim_{n\to\infty} x_n = x$. In view of $\alpha(x_n, x_{n+1}) \geq \eta(x_n, Tx_n)$ and (ii') we have

$$\alpha(x_n, x) \geq \eta(x, Tx), \quad n = 1, 2, \ldots. \tag{13}$$

In order to check that x is a fixed point of $\mathfrak{P}$, By means of (13) and the contractive condition (iii) it follows that

$$H(\mathfrak{P}(x_n), \mathfrak{P}(x)) \leq \omega(d(x_n, x)) + \beta[d(gx_n, Tx) - d(A, B)] \leq \omega(d(x_n, x)) + \beta d(x_{n+1}, x).$$

In addition, evidently, $d(x_{n+1}, \mathfrak{P}(x)) \leq H(\mathfrak{P}(x_n), \mathfrak{P}(x))$ for $n \in \mathbb{N}$. Consequently, one has

$$d(x_{n+1}, \mathfrak{P}(x)) \leq \omega(d(x_n, x)) + \beta d(x_{n+1}, x).$$

By the continuity of the distance and ω, letting n go to infinity, we obtain

$$d(x, \mathfrak{P}(x)) = 0.$$

Finally, by virtue of the fact that $\mathfrak{P}$ is approximative, there exists $u \in \mathfrak{P}(x)$ such that $d(x, u) = 0$, i.e., $u = x$ and hence x is a fixed point of $\mathfrak{P}$.

Now Lemma 2.8 immediately infers the following result.

Theorem 4.13. *Under the hypotheses of Lemma 4.12, T has a generalized best proximity point x. Moreover, there exists a sequence $\{x_n\}$ which satisfies*

$$d(gx_{n+1}, Tx_n) = d(A, B) \text{ and } \lim_{n\to\infty} x_n = x.$$

Corollary 4.14. *Let X be complete, $g : A \to A$ be an isometry with $A_0 \subset g(A_0)$ and A_0 bounded closed. If $T : A \to 2^B$ satisfies that, for all $x, y \in A$ and $u \in \mathfrak{P}(x)$, one possesses an element $v \in \mathfrak{P}(y)$ such that*

$$\frac{1}{1+a+\beta}[d(x,Tx)-d(A,B)] \leq d(x,y) \text{ implies } d(u,v) \leq \alpha d(x,y)+\beta[d(Tx,gy)-d(A,B)]$$

with $a \in (0,1)$ and $\beta \in (0,\infty)$, then T has a generalized best proximity point x. Moreover, for any element $\xi \in A_0$, there exists a sequence $\{x_n\}$ that satisfies

$$x_0 = \xi, d(gx_n, Tx_{n-1}) = d(A,B)$$

for $n = 1, 2, \dots$ and $\lim_{n\to\infty} x_n = x$.

Proof. Note that $g(A_0)$ is bounded closed, too. For $x, y \in X$, define

$$\eta(x,y) = \begin{cases} \frac{1}{1+a+\beta}[d(x,y) - d(A,B)], & x \in A, y \in B, \\ d(x,y), & otherwise, \end{cases}$$

$$\alpha(x,y) = \begin{cases} \max\{d(x,gy) : x, y \in A_0\}, & x, y \in A_0, \\ d(x,gy), & otherwise \end{cases}$$

and $\omega(t) = at$ with $a \in (0,1)$ and $t \geq 0$. For $x, y \in A_0$, it is easy to see that

$$\alpha(x,y) \geq d(x,gy) \geq d(x,Tx) - d(Tx,gy) = d(x,Tx) - d(A,B) \geq \eta(x,Tx)$$

for each $x \in A_0$ and each $y \in T^\sharp x$ which can infer that α and η satisfy (i') and (iv). To check (ii'), taking the sequence $\{x_n\}$ and x given in (ii'), we have to check $\eta(x,Tx) \leq \alpha(x_n,x)$. In fact, we have $Tx \subset B_0$, which deduces that there exists $u \in A_0$ such that $d(u,Tx) = d(A,B)$. Since $A_0 \subset g(A_0)$, there exists $z \in A_0$ such that $u = gz$. We obtain

$$d(x,Tx) - d(A,B) \leq d(x,u) + d(u,Tx) - d(A,B) = d(x,gz) \leq \alpha(x_n,x).$$

This guarantees that $\eta(x,Tx) \leq \alpha(x_n,x)$. In addition, our assumptions guarantee that T satisfies (V) for a given $\beta \in (0,\infty)$. Consequently, all conditions of Theorem 4.13 are satisfied. Thus, x is a best proximity point of T by Theorem 4.13. This completes the proof.

Corollary 4.14 includes Theorem 3.1 of [20, 21].

Example 4.15. Let $X = \mathbb{R}$ with the usual metric and $g = I$. Suppose $A = \{0,1,4\}$ and $B = \{-1,2,3\} \cup [5,+\infty)$. Then, A and B are nonempty and A is bounded closed subsets of X, $A_0 = A$ and $B_0 = \{-1,2,3,5\}$. We note that, $d(A,B) = 1$. Let $T : A \to 2^B$ be a set-valued mapping defined by

$$Tx = \begin{cases} [5,+\infty), & x = 0, \\ \{-1,2\}, & x = 1,4. \end{cases}$$

Then $T^\sharp(0) = \mathfrak{P}(0) = \{4\}$, $T^\sharp(1) = T^\sharp(4) = \{0,1\}$ and $\mathfrak{P}(1) = \mathfrak{P}(4) = \{1\}$.
If $(x,y) = (0,1)$ and $(u,v) = (4,1)$, then, we have $d(u,Tx) = d(v,Ty) = d(A,B)$. Now, if $\alpha = \frac{1}{2}, \beta = 2$, then

$$\frac{1}{1+\alpha+\beta}[d(x,Tx) - d(A,B)] = \frac{2}{7} \times 4 > 1 = d(x,y),$$

i.e., T satisfies the proximal contraction in Corollary 4.14. Thus, T has a best proximity point $x = 1$.

It is interesting to note that the non-self set-valued mapping T is not a proximal gi-contraction since $T^\sharp$ is not single-valued by Lemma 2.9(a3).

5. Applications

In this section, as an application of our results, we will discuss the existence of best proximity points in a partially ordered complete metric space $(X, d, \leq)$. To wit, we first recall the following notions:

Definition 5.1. For two subsets X_1, X_2 of X, we write $X_1 \leq X_2$ if for each $x \in X_1$ and each $y \in X_2$ it follows that $x \leq y$.

A multivalued mapping $T : X \to 2^X$ is said to be nondecreasing (nonincreasing) if $x \leq y$ implies that $Tx \leq Ty$ $(Ty \leq Tx)$ for all $x, y \in X$.

T is said to be monotone if T is nondecreasing or nonincreasing.

Definition 5.2. A mapping $T : A \to 2^B$ is said to be proximally nondecreasing (nonincreasing) if

$$\left.\begin{array}{r} x_1 \leq x_2 \\ d(y_1, Tx_1) = d(A, B) \\ d(y_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow y_1 \leq y_2 (y_1 \geq y_2),$$

where $x_1, x_2, y_1, y_2 \in A$.

T is said to be proximally monotone if T is proximally nondecreasing or proximally nonincreasing.

It is obvious that proximally monotone property deduces the monotone property when $A = B$.

Lemma 5.3. *Suppose that $g : A \to A$ is an surjective isometry and its inverse nondecreasing, as well as, $A_0 \subset g(A_0)$ and $T : A \to 2^B$ satisfies $T(A_0) \subset B_0$. If T is proximally nondecreasing (resp. nonincreasing), $T^\sharp$ is nondecreasing (resp. nonincreasing). Vice versa if g is an identity.*

The following theorem extends and improves the results in [12, 32, 33].

Theorem 5.4. *Let $(X, d, \leq)$ be a partially ordered complete metric space. Let $A, B \subset X$ be nonempty, A_0 nonempty closed and $g : A \to A$ an isometry with $A_0 \subset g(A_0)$. Let $T : A \to 2^B$ with $T(A_0) \subset B_0$ have proximal approximative values and satisfy the following conditions.*

- (o1) *T is proximally nondecreasing.*
- (o2) *One of the hypotheses (gpi) for $i = 1, 2, \ldots, 8$ holds when $x, y \in A$ with $x \leq y$ instead of any $x, y \in A$.*
- (o3) *There exist elements $x_0 \in A_0$ and $x_1 \in \mathfrak{P}(x_0)$ such that $x_0 \leq x_1$.*
- (o4) *If $\{x_n\}$ is a nondecreasing sequence in A such that $x_n \to x$, then $x_n \leq x$.*

Then T has a generalized best proximity point $x \in A$. Further, the sequence $\{x_n\}$, defined by $d(gx_{n+1}, Tx_n) = d(A,B)$ for $n \geq 1$, converges to the element x.

Proof. Let functions $\alpha, \eta : A \times A \to [0, +\infty)$ be defined by, respectively,

$$\alpha(x,y) = \begin{cases} 1, & x \leq y, \\ 0, & otherwise \end{cases} \quad \text{and} \quad \eta(x,y) \equiv 1.$$

Then we clearly obtain the following equivalences:

1. (I) in Theorem 4.5 and (o3);
2. (II) in Theorem 4.5 and (o1);
3. (III) in Theorem 4.5 and (o2);
4. (ii) in Lemma 4.4 and (o4).

Now Theorem 4.5 guarantees that the desired result holds which completes the proof.

Remark 5.5. As mentioned in Remark 2.7, if T is a single-valued mapping and set pair (A,B) satisfies (weak) P-property, then each gi-contraction implies proximal gi-contraction for $i = 1, 2, \ldots, 8$. In this sense, Theorem 5.4 includes the main results in [15, 27]. Moreover, some existence theorems, as follows, Theorem 18 in [27], Theorem 2.1 and Theorem 2.2 in [15] are equivalent in the sense that deal with the same class of mappings.

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