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EXISTENCE RESULTS FOR A QUADRATIC INTEGRAL EQUATION OF FRACTIONAL ORDER BY A CERTAIN FUNCTION

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Abstract. The fractional integration of a function $f(t)$ by a function ϕ and some of its properties is presented in [23], [30] and [21]. As an application for this fractional integration we present some existence results for at least one continuous solution for a nonlinear quadratic functional integral equation of fractional (arbitrary) order. Also, some examples and remarks are illustrated. Finally, we prove the existence of maximal and minimal solutions for that equations.
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1. Introduction and preliminaries

Let $\mathbb{R}$ be the set of real numbers whereas $I = [a, b], L_1 = L_1[a, b]$ be the space of Lebesgue integrable functions on I with the stander norm

$$||f(t)|| = \int_a^b |f(t)|dt.$$

The main results of this paper will be based on the following fixed-point theorem and definition.

Theorem 1.1. (Schauder Fixed Point Theorem) [10]. *Let Q be a nonempty, convex, compact subset of a Banach space X, and $T : Q \to Q$ be a continuous map. Then T has at least one fixed point in Q.*

Lemma 1.2. *Assume that F is the superposition operator generated by the function $f : [a, b] \times R \to R$. Then F transform the space $C[a, b]$ into itself and is continuous if and only if the function f is continuous on the set $[a, b] \times R$.*

Let $L^1_\phi = L^1_\phi[a,b]$ be the space of all real functions defined on $[a,b]$ such that $\phi'(t)f(t) \in L^1$ and

$$\int_a^b |\phi'(t)f(t)|dt \leq \infty$$

where ϕ is increasing function and absolutely continuous on $[a,b]$ and we introduce the norm

$$||f(t)||_{L^1_\phi} = \int_a^b |\phi'(t)f(t)|dt.$$

Definition 1.3. The $\phi-$ fractional integral of order $\alpha \geq 0$ of the function $f(t) \in L^1_\phi$ is defined as

$$I^\alpha_\phi f(t) = \int_a^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)}\phi'(s)f(s)ds.$$

I^α_ϕ may be known as the fractional integral of the function $f(t)$ with respect to $\phi(t)$.

For more properties of this integral operator see [23], [30] and [21].

Quadratic integral equations occur more frequently in different research areas for examples, in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory [11].
The existence theorems for several classes of solutions of quadratic integral equations are studied in (see e.g. [1]-[7], [9] and [12]-[20]).

However, in most of the above literature, the main results are realized with the help of the technique associated with the measure of noncompactness [4]- [7]. Instead of using the technique of measure of noncompactness, Schauder fixed point theorem is applied to study the existence of continuous solutions [18] and [29].

Here, we prove the existence of at least one continuous solution for the quadratic functional integral equation of fractional order

$$x(t) = a(t) + g(t,x(t)) \int_0^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))\phi'(s)ds, t \in I, \alpha > 0 \quad (1.1)$$

and the existence of maximal and minimal solutions for (1.1) will be proved.
This result extends the results obtained by El-Sayed et al. [18]. For $\psi(t) = \phi(t) = t$, J. Banas and B. Rzepka [7] proved the existence of a nondecreasing continuous solution of (1.1) by using the technique of measure of noncompactness.

2. Main theorem

Let $I = [0,1]$. Equation (1.1) will be investigated under the assumptions:

(i) $a : I \to \mathbb{R}$ is continuous and $k_1 = \sup_{t\in I} |a(t)|$.

(ii) $g : I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded. We put $k_2 = \sup_{(t,x)\in I\times\mathbb{R}} |g(t,x)| < \infty$.

(iii) There exist two constants $l_i, i = 1,2$ respectively satisfying

$$|g(t,x) - g(s,y)| \leq l_1|t-s| + l_2|x-y|$$

for all $t, s \in I$ and $x, y \in \mathbb{R}$.

(iv) $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheòdory condition (i.e. is measurable in t for all $x : I \to \mathbb{R}$ and continuous in x for all $t \in I$).
Moreover, there exist a function $m \in L_1$ and a non-negative constant b such that
$|f(t,x)| \leq m(t) + b|x|(\forall(t,x) \in I \times \mathbb{R})$ and $k_3 = \sup_{t\in I} I^{\beta} m(t)$ for any $\beta \leq \alpha$.

(v) $\psi : I \to I$ is continuous .

(vi) $\phi : I \to I$ is a nondecreasing function having a continuous derivative.

Theorem 2.1. *Let assumptions (i)-(vi) be satisfied. If* $k_2 b < \Gamma(1+\alpha)$, *then the quadratic functional integral equation (1.1) has at least one solution in the space* $C(I)$.

Proof. Let $C = C(I)$ be the Banach space of all real functions defined and continuous on interval I with the standard supremum norm.
Fix a number $r > 0$ and consider the ball S_r in the space $C(I)$ defined as

$$S_r = \{x \in C(I) : |x(t)| \leq r \text{ for } t \in I\}.$$

Let T be the operator defined on S_r by the formula

$$(Tx)(t) = a(t) + g(t,x(t)) \int_0^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s,x(\psi(s)))\phi'(s)ds, x \in S_r, t \in I.$$

Then, in view of our assumptions, for $x \in S_r$ and $t \in I$ we get

$$\begin{aligned}
|Tx(t)| &\leq |a(t)| + |g(t,x(t))| \int_0^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s,x(\psi(s)))|\phi'(s)ds \\
&\leq k_1 + k_2 I_\phi^{\alpha-\beta} I_\phi^{\beta} m(t) + k_2 b \int_0^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} |x(\psi(s))|\phi'(s)ds \\
&\leq k_1 + k_2 k_3 \int_0^t \frac{(\phi(t)-\phi(s))^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \phi'(s)ds + k_2 br \int_0^t \frac{(\phi(t)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} \phi'(s)ds \\
&\leq k_1 + \frac{k_2 k_3}{\Gamma(\alpha-\beta+1)} + \frac{k_2 br}{\Gamma(1+\alpha)}.
\end{aligned}$$

Hence, in view of the condition $k_2 b < \Gamma(1+\alpha)$, we have that T transforms the ball S_r into itself for

$$r = (k_1 + \frac{k_2 k_3}{\Gamma(\alpha-\beta+1)})(1 - \frac{k_2 b}{\Gamma(1+\alpha)})^{-1}.$$

Now, for $t_1 \text{and} t_2 \in I$ (without loss of generality assume that $t_1 < t_2$), we have

$$\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| = {} & |a(t_2) - a(t_1) \\
& + g(t_2,x(t_2)) I_\phi^{\alpha} f(t_2,x(\psi(t_2))) - g(t_1,x(t_1)) I_\phi^{\alpha} f(t_1,x(\psi(t_1))) \\
& + g(t_1,x(t_1)) I_\phi^{\alpha} f(t_2,x(\psi(t_2))) - g(t_1,x(t_1)) I_\phi^{\alpha} f(t_2,x(\psi(t_2)))| \\
\leq {} & |a(t_2) - a(t_1)| + |g(t_2,x(t_2)) - g(t_1,x(t_1))| I_\phi^{\alpha} |f(t_2,x(\psi(t_2)))| \\
& + |g(t_1,x(t_1))| |I_\phi^{\alpha} f(t_2,x(\psi(t_2))) - I_\phi^{\alpha} f(t_1,x(\psi(t_1)))|,
\end{aligned}$$

but

$$
\begin{aligned}
|I_\phi^\alpha f(t_2, x(\psi(t_2))) - I_\phi^\alpha f(t_1, x(\psi(t_1)))| &= \bigg| \int_0^{t_1} \frac{(\phi(t_2)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))\phi'(s)ds \\
&\quad + \int_{t_1}^{t_2} \frac{(\phi(t_2)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))\phi'(s)ds \\
&\quad - \int_0^{t_1} \frac{(\phi(t_1)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))\phi'(s)ds \bigg| \\
&\le \bigg| \int_0^{t_1} \frac{(\phi(t_1)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))\phi'(s)ds \\
&\quad + \int_{t_1}^{t_2} \frac{(\phi(t_2)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))\phi'(s)ds \\
&\quad - \int_0^{t_1} \frac{(\phi(t_1)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))\phi'(s)ds \bigg| \\
&\le \int_{t_1}^{t_2} \frac{(\phi(t_2)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(\psi(s)))\phi'(s)|ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
|I_\phi^\alpha f(t_2, x(\psi(t_2))) - I_\phi^\alpha f(t_1, x(\psi(t_1)))| &\le I_{t_1,\phi}^\alpha |f(t_2, x(\psi(t_2)))| \\
&\le I_{t_1,\phi}^\alpha m(t_2) + b I_{t_1,\phi}^\alpha |x(\psi(t_2))| \\
&\le I_{t_1,\phi}^{\alpha-\beta} I_{t_1,\phi}^{\beta} m(t_2) + b I_{t_1,\phi}^\alpha |x(\psi(t_2))| \\
&\le k_3 \frac{(\phi(t_2)-\phi(t_1))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + br \frac{(\phi(t_2)-\phi(t_1))^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| &\le |a(t_2) - a(t_1)| + [l_1|t_2 - t_1| \\
&\quad + l_2|x(t_2) - x(t_1)|] I_\phi^\alpha |f(t_2, x(\psi(t_2)))| \\
&\quad + |g(t_1, x(t_1))| \left(k_3 \frac{(\phi(t_2)-\phi(t_1))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + br \frac{(\phi(t_2)-\phi(t_1))^{\alpha}}{\Gamma(\alpha+1)} \right)
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| &\le |a(t_2) - a(t_1)| + [l_1|t_2 - t_1| \\
&\quad + l_2|x(t_2) - x(t_1)|] I_\phi^\alpha (m(t_2) + b|x(\psi(t_2))|) \\
&\quad + k_2 k_3 \frac{(\phi(t_2)-\phi(t_1))^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + k_2 br \frac{(\phi(t_2)-\phi(t_1))^{\alpha}}{\Gamma(\alpha+1)} \\
&\le |a(t_2) - a(t_1)| + \frac{k_3}{\Gamma(\alpha-\beta+1)} [l_1|t_2 - t_1| + l_2|x(t_2) - x(t_1)|] \\
&\quad + \frac{br}{\Gamma(\alpha+1)} [l_1|t_2 - t_1| + l_2|x(t_2) - x(t_1)|] + \frac{k_2 k_3}{\Gamma(\alpha-\beta+1)} (\phi(t_2)-\phi(t_1))^{\alpha-\beta} \\
&\quad + \frac{k_2 br}{\Gamma(\alpha+1)} (\phi(t_2)-\phi(t_1))^{\alpha} \to 0 \mathrm{as} t_2 \to t_1.
\end{aligned}
$$

This means that the functions from TS_r are equi-continuous on I. Then by the Arzela-Ascoli Theorem [10] the closure of TS_r is compact .
It is clear that the set S_r is nonempty, bounded, closed and convex.
Assumptions (ii) and (iv) imply that $T : S_r \to C(I)$ is a continuous operator in x.
Since all conditions of the Schauder fixed-point theorem hold, then T has a fixed point in S_r.

3. Examples and remarks

In this section, we present some examples of classical integral equations which are particular cases of equation (1.1) and consequently the existence of their solutions can be established by using Theorem 2.1.

Example 3.1. The equation (1.1) includes the fractional-order quadratic integral equation [7]

$$x(t) = a(t) + g(t, x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))ds.$$

To obtain this equation it sufficient to put $\psi(t) = \phi(t) = t$ in (1.1).

Example 3.2. Let the assumptions of Theorem 2.1 be satisfied (with $\phi(t) = t^m$, $m > 0$ and $\psi(t) = t$), then the fractional-order quadratic integral equation

$$x(t) = a(t) + g(t, x(t)) \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) m s^{m-1} ds$$

has at least one solution $x \in C$

which is the same result proved in [22]

Example 3.3. Let the assumptions of Theorem 2.1 be satisfied (with $g(t, x) = 1$, $\phi(t) = t^m$, $m > 0$ and $\psi(t) = t$), then the fractional-order integral equation [21]

$$x(t) = a(t) + \int_0^t \frac{(t^m - s^m)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) m s^{m-1} ds$$

has at least one solution $x \in C$.

Example 3.4. Let the assumptions of Theorem 2.1 be satisfied (with $g(t, x) = 1$ and $\psi(t) = t$), then the fractional-order integral equation [21]

$$x(t) = a(t) + \int_0^t \frac{(\phi(t) - \phi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \phi'(s) ds$$

has at least one solution $x \in C$.

Example 3.5. Let the assumptions of Theorem 2.1 be satisfied (with $g(t, x) = 1$, $\phi(t) = t$), then the fractional-order integral equation

$$x(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))ds \tag{3.1}$$

has at least one solution $x \in C$.

Letting $b = 0$ in assumption (iv) and $\psi(t) = t$ in Eqn. (3.1), we obtain the same result as was proved in [13].
For the initial value problem for the nonlinear fractional-order differential equation

$$_RD^{\alpha}x(t) = f(t, x(\psi(t))),\ t \in I \text{ and } x(0) = 0,\ \alpha \in (0,1) \tag{3.2}$$

(where $_RD^{\alpha}$ is the Riemann-Liouville fractional order derivative).
As a consequence of Theorem 2.1 (with $a(t) = 0, \phi(t) = t$ and $g(t, x(t)) = 1$), the Cauchy type problem (3.2) is equivalent to the integral equation

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(\psi(s)))ds,\ t \in I$$

has at least one solution $x \in C$.
Now letting $\alpha, \beta \to 1$, we obtain

Example 3.6. Let the assumptions of Theorem 2.1 be satisfied (with $g(t, x) = 1$, $a(t) = x_0$ and letting $\alpha, \beta \to 1$), then the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(\psi(s)))ds.$$

has at least one solution $x \in C$ which is equivalent to the initial value problem

$$\frac{dx(t)}{dt} = f(t, x(\psi(t))), x(0) = x_0, \tag{3.3}$$

Letting $b = 0$ in assumption (v) and $\psi(t) = t$ in 3.3 we obtain Carathéodory Theorem (proved in [10]).

Example 3.7. Consider the following quadratic functional integral equation

$$\begin{aligned} x(t) = t/6 + [\sqrt{t^2+5} + t(|\log(x(t)+3)| + 1)] \\ \times \int_0^t \frac{(\phi(t)-\phi(s))^{1/2}}{\Gamma(5/2)}[1 + \frac{1}{3+s}x(\sin(s^2+4s))]ds, t \in [0,1]. \end{aligned} \tag{3.4}$$

Taking

$$\begin{aligned} &a(t) = t/6, \\ &g(t, x) = \sqrt{t^2+5} + t(|\log(x(t)+3)| + 1), \\ &f(t, x) = t + \frac{1}{3+t}x \end{aligned}$$

then easily we can deduce that:

- $|f(t,x)| \leq 1 + 1/2|x|$ and

$$\begin{aligned}|g(t,z)-g(s,y)| &= |\sqrt{t^2+5}+t(|\log(z(t)+3)|+1)-\sqrt{s^2+5} \\ &- s(|\log(y(s)+3)|+1)| \\ &\leq |\sqrt{t^2+5}-\sqrt{s^2+5}|+t|(|\log(z(t)+3)|+1) \\ &- (|\log(y(s)+3)|+1)| \\ &+ |t(|\log(y(s)+3)|+1)-s(|\log(y(s)+3)|+1)| \\ &\leq \frac{2}{5}|t-s|+\frac{1}{10}|z-y|+|t-s|+3|t-s| \\ &\leq \frac{11}{5}|t-s|+\frac{1}{10}|z-y|\end{aligned}$$

- $k_1 = 1/6, k_2 = 8, \beta = 1, k_3 = 1, b = 1/2$ and $\psi(t) = \sin(t^2+4t), m(t) = 1$.

4. Maximal and minimal solutions

Definition 4.1. [24] Let $q(t)$ be a solution $x(t)$ of (1.1) Then $q(t)$ is said to be a maximal solution of (1.1) if every solution of (1.1) on I satisfies the inequality $x(t) \leq q(t),\ t \in I$. A minimal solution $s(t)$ can be defined in a similar way by reversing the above inequality i.e. $x(t) \geq s(t), t \in I$.

We need the following lemma to prove the existence of maximal and minimal solutions of (1.1).

Lemma 4.2. *Let $g(t,x), f(t,x)$ satisfy the assumptions in Theorem 2.1 and let $x(t), y(t)$ be continuous functions on I satisfying*

$$\begin{aligned}x(t) &\leq a(t)+g(t,x(t))I_\phi^\alpha f(t,x(\psi(t))) \\ y(t) &\geq a(t)+g(t,y(t))I_\phi^\alpha f(t,y(\psi(t)))\end{aligned}$$

where one of them is strict.
Suppose $f(t,x)$ is nondecreasing function in x. Then

$$x(t) < y(t),\ t \in I. \tag{4.1}$$

Proof. Let the conclusion (4.1) be false; then there exists t_1 such that

$$x(t_1) = y(t_1), t_1 > 0$$

and

$$x(t) < y(t), 0 < t < t_1.$$

From the monotonicity of the function f in x, we get

$$\begin{aligned}x(t_1) &\leq a(t_1)+g(t_1,x(t_1))I_\phi^\alpha f(t_1,x(\psi(t_1))) \\ &= a(t_1)+g(t_1,x(t_1))\int_0^{t_1}\frac{(\phi(t_1)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)}f(s,x(\psi(s)))\phi'(s)ds \\ &< a(t_1)+g(t_1,y(t_1))\int_0^{t_1}\frac{(\phi(t_1)-\phi(s))^{\alpha-1}}{\Gamma(\alpha)}f(s,y(\psi(s)))\phi'(s)ds \\ &< y(t_1).\end{aligned}$$

This contradicts the fact that $x(t_1) = y(t_1)$;then

$$x(t) < y(t),\ t \in I. \qquad \square$$

As particular cases of Lemma 4.2 we remark the following:

- For $\phi(t) = t^m, m > 0$, all the assumptions of Lemma 4.2 are satisfied and

$$\begin{aligned} x(t) &\leq a(t) + g(t, x(t)) I_m^\alpha f(t, x(\psi(t))) \\ y(t) &\geq a(t) + g(t, y(t)) I_m^\alpha f(t, y(\psi(t))) \end{aligned}$$

  where one of them is strict. Then

$$x(t) < y(t), t \in I.$$

- For $\phi(t) = t$, all the assumptions of Lemma 4.2 are satisfied and

$$\begin{aligned} x(t) &\leq a(t) + g(t, x(t)) I^\alpha f(t, x(\psi(t))) \\ y(t) &\geq a(t) + g(t, y(t)) I^\alpha f(t, y(\psi(t))) \end{aligned}$$

  where one of them is strict. Then

$$x(t) < y(t) \text{ for } t \in I.$$

Theorem 4.3. *Let the assumptions of Theorem 2.1 be satisfied. Furthermore, if $f(t, x)$ is nondecreasing functions in x, then there exist maximal and minimal solutions of (1.1).*

Proof. Firstly, we shall prove the existence of maximal solution of (1.1). Let $\varepsilon > 0$ be given. Now consider the fractional-order quadratic functional integral equation

$$x_\varepsilon(t) = a(t) + g_\varepsilon(t, x_\varepsilon(t)) I_\phi^\alpha f_\varepsilon(t, x_\varepsilon(\psi(t))), \tag{4.2}$$

where

$$f_\varepsilon(t, x_\varepsilon(\psi(t))) = f(t, x_\varepsilon(\psi(t))) + \varepsilon$$

and

$$g_\varepsilon(t, x_\varepsilon(t)) = g(t, x_\varepsilon(t)) + \varepsilon.$$

Clearly the functions $f_\varepsilon(t, x_\varepsilon)$ and $g_\varepsilon(t, x_\varepsilon)$ satisfy assumptions (ii), (iv) and

$$|g_\varepsilon(t, x_\varepsilon)| \leq M + \varepsilon = M'.$$

$$|f_\varepsilon(t, x_\varepsilon)| \leq m(t) + \varepsilon + b|x| = m'(t) + b|x|.$$

Therefore, equation (4.2) has a continuous solution $x_\varepsilon(t)$ according to Theorem 2.1. Let ε_1 and ε_2 be such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$. Then

$$x_{\varepsilon_1}(t) = a(t) + g_{\varepsilon_1}(t, x_{\varepsilon_1}(t)) I_\phi^\alpha f_{\varepsilon_1}(t, x_{\varepsilon_1}(\psi(t))),$$

$$\begin{aligned} x_{\varepsilon_1}(t) &= a(t) + (g(t, x_{\varepsilon_1}(t)) + \varepsilon_1) I_\phi^\alpha (f(t, x_{\varepsilon_1}(\psi(t))) + \varepsilon_1), \\ &> a(t) + (g(t, x_{\varepsilon_1}(t)) + \varepsilon_2) I_\phi^\alpha (f(t, x_{\varepsilon_1}(\psi(t))) + \varepsilon_2), \end{aligned} \tag{4.3}$$

$$x_{\varepsilon_2}(t) = a(t) + (g(t, x_{\varepsilon_2}(t)) + \varepsilon_2) I_\phi^\alpha (f(t, x_{\varepsilon_2}(\psi(t))) + \varepsilon_2). \tag{4.4}$$

Applying Lemma 4.2, then (4.3) and (4.4) imply that

$$x_{\varepsilon_2}(t) < x_{\varepsilon_1}(t) fort \in I.$$

As shown before in the proof of Theorem 2.1, the family of functions $x_\varepsilon(t)$ defined by (4.2) is uniformly bounded and of equi-continuous functions. Hence by the Arzela-Ascoli Theorem, there exists a decreasing sequence ε_n such that $\varepsilon_n \to 0$ as $n \to \infty$, and $\lim\limits_{n\to\infty} x_{\varepsilon_n}(t)$ exists uniformly in I. We denote this limit by $q(t)$. From the continuity of the functions f_{ε_n} and g_{ε_n} in the second argument, we get

$$q(t) = \lim_{n\to\infty} x_{\varepsilon_n}(t) = a(t) + g(t, q(t)) I_\phi^\alpha f(t, q(\psi(t)))$$

which proves that $q(t)$ is a solution of (1.1).
Finally, we shall show that $q(t)$ is maximal solution of (1.1). To do this, let $x(t)$ be any solution of (1). Then

$$\begin{array}{rcl} x_\varepsilon(t) & = & a(t) + g_\varepsilon(t, x_\varepsilon(t)) I_\phi^\alpha f_\varepsilon(t, x_\varepsilon(\psi(t))) \\ & > & a(t) + g(t, x_\varepsilon(t)) I_\phi^\alpha f(t, x_\varepsilon(\psi(t))) \end{array}$$

and

$$x(t) = a(t) + g(t, x(t)) I_\phi^\alpha f(t, x(\psi(t))).$$

Applying Lemma 4.2, we get

$$x_\varepsilon(t) > x(t) fort \in I.$$

From the uniqueness of the maximal solution (see [24], [27]), it is clear that $x_\varepsilon(t)$ tends to $q(t)$ uniformly in $t \in I as \varepsilon \to 0$.
In a similar way we can prove that there exists a minimal solution of (1.1). □

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