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A STRONG CONVERGENCE THEOREM FOR FINITE FAMILIES OF BREGMAN QUASI-NONEXPANSIVE AND MONOTONE MAPPINGS IN BANACH SPACES

ENYINNAYA EKUMA-OKEREKE* AND ABIODUN TINUOYE OLADIPO**

*Department of Mathematics/Computer Science, Federal University of Petroleum Resources, Effurun, P.M.B 1221, Nigeria
E-mail: ekuma.okereke@fupre.edu.ng

**Department of Pure and Applied Mathematics, Ladoke Akintola University of Technology, Ogbomoso, P.M.B 4000, Nigeria

Abstract. We introduce a new iterative scheme and prove a strong convergence theorem for it. This iterative scheme finds a common point in the set of fixed points of a finite family of Bregman quasi-nonexpansive mappings and the common solution set of the variational inequality problem for continuous monotone mappings.
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1. Introduction

Throughout this paper, we let X denote a real reflexive Banach Space with the norm $\|.\|$, and X^* denote the dual space of X. We assume $f : X \to (-\infty, +\infty]$ to be proper, lower-semicontinuous and convex function and the domain of f be denoted as

$$domf = \{x \in X : f(x) < +\infty\}.$$

We let C be a nonempty, closed and convex subset of X. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \leq f(y); \forall y \in X\}. \tag{1.1}$$

Definition 1.1. A mapping $A : C \to X^*$ is said to be monotone if for each $x, y \in C$, the following inequality hold

$$\langle u - v, x - y\rangle \geq 0, \forall u \in Ax, \forall v \in Ay. \tag{1.2}$$

The class of monotone mappings includes the class of α-inverse strongly monotone (α-ism) mappings and $A : C \to X^*$ is said to be α-ism [15], if there exists a positive real number α such that

$$\langle u - v, x - y\rangle \geq \alpha||u - v||^2, \forall u \in Ax, \forall v \in Ay.$$

The problem of finding a point $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \forall y \in C \tag{1.3}$$

is called the variational inequality problem. The set of solution of the variational inequality is denoted by $VI(C, A)$. It is not difficult to check that when A is a continuous monotone mapping then the solution set of $VI(C, A)$ is closed and convex. To see this, let

$$A(x) = 1 - \frac{1}{x},\ x \in C,$$

then A is a continuous monotone mapping which is closed and convex.

We remark here that monotone variational inequalities were originally introduced in the work of [16], and have led to many researches on variational inequality problems being studied, see for e.g, [15], [18], [19], [40], [36], [38], [39], [2], [37], [27], [7], [14] and the references therein.

Let $T : C \to C$ be a nonlinear self mapping. T is said to be nonexpansive mapping if $||Tx - Ty|| \leq ||x - y||, \forall x, y \in C$, and T is said to quasi-nonexpansive mapping if $||Tx - p|| \leq ||x - p||, \forall x \in C, p \in F(T)$, where $F(T) = \{x \in C : Tx = x\}$ is the set of fixed point of a mapping T. A point $p \in C$, is called an asymptotic fixed point of a mapping T if C contains a sequence x_n with $x_n \rightharpoonup p$ such that $\|x_n - Tx_n\| = 0$. The set of asymptotic fixed point is denoted by $\widehat{F}(T)$, (see [25]).

Definition 1.2. A mapping $T : C \to C$ is said to be Bregman firmly nonexpansive (BFNE) (see for e.g.[29]) if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle, \forall x, y \in C$$

or equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

Definition 1.3. A mapping $T : C \to C$ is said to be Bregman quasi-nonexpansive (BQNE) (see [27]) if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, \forall p \in F(T) \tag{1.4}$$

Definition 1.4. A mapping $T : C \to C$ is said to be Bregman relatively-nonexpansive (BRNE) (see [27]) if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x), \forall x \in C, \forall p \in F(T) = \widehat{F}(T) \tag{1.5}$$

Definition 1.5. A function $f^* : X^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x),\ x \in X\}$$

is called the conjugate function of f. We see from the conjugate inequality that

$$f(x) \geq \langle x, x^*\rangle - f^*(x^*),\ \forall x \in X, x^* \in X^*,$$

(see [30]). A function is said to be cofinite if $dom f^* = X^*$. A function f on X is coercive (see [34]), if the sublevel set of f is bounded, equivalently

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$

It is said to be strongly coercive (see [30]), if

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in intdomf$ and $y \in X$, the right hand derivative of f at x in the direction of y is defined by

$$f^0(x,y) = \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}.$$

Definition 1.6. A function f is said to be Gâteaux differentiable at x if

$$\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$$

exists for any y. In this case, $f^0(x,y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in intdomf$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. f is said to be uniformly Fréchet differentiable on a subset C of X if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Definition 1.7. A function $f : X \to (-\infty, +\infty]$ is said to be a Legendre function (see [29]), if it satisfies the following two conditions:

(L1) $intdomf \neq \emptyset$, f is Gâteaux differentiable on $intdomf$ and

$$domf = intdomf;$$

(L2) $intdomf^* \neq \emptyset$, f^* is Gâteaux differentiable on $intdomf^*$ and

$$domf^* = intdomf^*.$$

Remark 1.8. (cf. [6], [4], [23], [24]). Since X is reflexive, then we have that

$$(\partial f^{-1}) = \partial f^*$$

and since f is Legendre, then ∂f is a bijection which satisfies

$$\nabla f = (\nabla f^*)^{-1},\ ran\nabla f = dom\nabla f^* = intdomf^*$$

and

$$ran\nabla f^* = dom\nabla f = intdomf.$$

f and f^* are strictly convex on their $intdomf$. If the subdifferential of f is single valued, it coincides with the gradient of f, that is $\partial f = \nabla f$.

Example of a Legendre function is

$$f(x) = \frac{1}{p}||x||^p,\ (1 < p < \infty).$$

If X is smooth and strictly convex Banach spaces, then in this case the gradient ∇f coincides with the generalised duality mapping of X, that is $\nabla f = J_p$. If the space is a Hilbert space, H then $\nabla f = I$, where I is the identity mapping in H. Throughout this paper, we assumed that f is Legendre.

Definition 1.9. Let $f : X \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of f at $x \in intdomf$ is the function

$$V_f(x,.) : intdomf \times [0,+\infty) \to [0,+\infty)$$

defined by

$$V_f(x,t) = inf\left\{D_f(y,x) : y \in domf, \|y-x\| = t\right\}. \quad (1.6)$$

The function f is called totally convex at x if $V_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at any point $x \in intdomf$. The function is said to be totally convex on bounded sets if $V_f(B,t) > 0$ for any nonempty bounded subset B of X and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $V_f : intdomf \times [0,+\infty) \to [0,+\infty)$ defined by

$$V_f(B,t) = inf\left\{V_f(x,t) : x \in B \cap domf\right\}. \quad (1.7)$$

Let $V_f : X \times X^* \to [0,+\infty)$ associated with f (see [10],[3],[12]) be defined by

$$V_f(x,x^*) = f(x) - \langle x, x^*\rangle + f^*(x^*), \forall x \in X, x^* \in X^*. \quad (1.8)$$

We see that $V_f(,) \geq 0$ and the relation

$$V_f(x,x^*) = D_f(x, \nabla f^*(x^*)), \quad (1.9)$$

holds. Moreover, by the subdifferential inequality, we obtain (see [17])

$$V_f(x,x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leq V_f(x, x^* + y^*), \forall x \in X, x^*, y^* \in X^*. \quad (1.10)$$

Definition 1.10. Let $f : X \to (-\infty,+\infty]$ be a Gâteaux differentiable function. The function $D_f : domf \times intdomf \to [0,+\infty)$ defined by

$$D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y-x\rangle, \quad (1.11)$$

is called the Bregman distance with respect to f (see [8],[12]). It is easy to see that Bregman distance function D_f does not satisfy the symmetric and triangle inequality associated with the properties of a classical distance function, but has some interesting properties like

$$D_f(y,x) = D_f(y,z) + D_f(z,x) + \langle \nabla f(z) - \nabla f(x), y-z\rangle.$$

Let $P_C^f : intdomf \to C$ be a mapping such that $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = inf\left\{D_f(y,x) : y \in C\right\}, \quad (1.12)$$

which is the Bregman Projection (see[8]) of $x \in intdomf$ onto a nonempty closed and convex set $C \subset domf$.

We remark here that, if X is a smooth and strictly convex Banach spaces and $f(x) = \|x\|^2$, $\forall x \in X$, then we have that $f(x) = 2Jx$, $\forall x \in X$, where J is the normalized duality mapping. Clearly, we obtain that

$$\begin{aligned} D_f(y,x) &= f(y) - f(x) - \langle \nabla f(x), y-x\rangle \\ &= \|y\|^2 - \|x\|^2 - 2\langle y, Jx\rangle + 2\|x\|^2 \\ &= \|x\|^2 - 2\langle y, Jx\rangle + \|y\|^2 \\ &= \phi(y,x), \forall x, y \in X, \end{aligned}$$

which is the Lyapunov function introduced by [3] and has extensively been studied by various authors (see for e.g. [31], [35], [3]). We clearly see that $P_C^f(x)$ reduces to the generalized projection given as

$$\Pi_C(x) = argmin_{y\in C}\phi(y,x).$$

In addition, if X coincides with H, in Hilbert space then $J = I$ and

$$\begin{aligned} D_f(y,x) &= f(y) - f(x) - \langle \nabla f(x), y - x\rangle \\ &= \|x\|^2 - \|y\|^2 - 2\langle x,y\rangle + 2\|y\|^2 \\ &= \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle \\ &= \|x-y\|^2, \forall x,y \in X. \end{aligned}$$

Hence the Bregman Projection $P_C^f(x)$ reduces to metric projection of H onto C, $P_C(x)$.

The distance functionD_f introduced by Bregman [8] instead of norm have been studied and used by many authors over the past seven years as it opened a growing area of research (see e.g. [27], [25], [22], [5], [39]) and the references therein.

Recently, in 2016, [13] introduced an algorithm for finding fixed points of Bregman quasi-nonexpansive mappings and zeros of maximal monotone operators by using products of resolvents. The authors proved a strong convergence theorem for finding a common fixed point of infinitely countable family of Bregman quasi-nonexpansive mappings and a common zero of finitely many maximal monotone mappings in reflexive Banach spaces. In [32], the authors proved a new strong convergence theorem for finite family of quasi-Bregman nonexpansive mappings and system of equilibrium problem in real Banach space. In [1], the authors proved a strong convergence theorem for the common fixed point of finite family of quasi-Bregman nonexpansive mappings. Inspired and motivated by the works of [13], [32], [1], and the researches ongoing in this direction, we consider an iterative scheme which converges strongly to a common fixed point of a finite family of Bregman quasi-nonexpansive mappings and the common solution to a system of variational inequality problem for continuous monotone mappings in reflexive Banach spaces.

2. Preliminaries

In the sequel, we shall make use of the following lemmas.

Lemma 2.1. ([9]) *The function f is totally convex on bounded sets if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that the first one is bounded, then*

$$\lim_{n\to\infty} D_f(y_n,x_n) = 0 \Rightarrow \|y_n - x_n\| = 0.$$

Lemma 2.2. ([29]) *Let C be a nonempty, closed and convex subsets of $intdomf$ and $T : C \to C$ be a quasi-Bregman nonexpansive mapping with respect to f. Then $F(T)$ is closed and convex.*

Lemma 2.3. ([11]) *Let C be a nonempty, closed and convex subsets of X. Let $f : X \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function and let $x \in X$, then* *(i)* $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \forall y \in C,$

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \forall y \in C.$

Lemma 2.4. ([34]) *Let X be a reflexive Banach space and let $f : X \to R$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(i) f is bounded on bounded subsets and uniformly smooth on bounded subsets of X.

(ii) f^* is Fréchet differentiable and f^* is uniformly norm-to-norm continuous on bounded subsets of X^*.

(iii) dom$f^* = X^*$, f^* is strongly coercive and uniformly convex on bounded subsets of X^*.

Lemma 2.5. ([21]) *Let X be a Banach space, let $r > 0$ be a constant and $f : X \to R$ be a continuous and convex function which is uniformly convex on bounded subsets of X. Then*

$$f\left(\sum_{k=1}^{\infty} \alpha_k x_k\right) \leq \sum_{k=1}^{\infty} \alpha_k x_k f(x_k) - \alpha_i \alpha_j \rho_r(||x_i - x_j||),$$

$\forall\ i, j \in N \cup 0,\ x_k \in B_r,\ \alpha_k \in (0,1)$ *and* $k \in N \cup 0$ *with* $\sum_{k=1}^{\infty} \alpha_k = 1$, *where ρ_r is the gauge of uniform convexity of f.*

Lemma 2.6. ([26]) *If $f : X \to (-\infty, +\infty]$ is uniformly Fréchet differentiable and bounded on bounded subsets of X, then ∇f is uniformly continuous on bounded subsets of X from the strong topology of X to the strong topology of X^*.*

Lemma 2.7. ([20]) *Let $f : X \to (-\infty, +\infty]$ be a Gâteaux differentiable on intdom$f$ such that ∇f^* is bounded on bounded subsets of intdomf^*. Let $x_0 \in X$ and $\{x_n\}$ is a sequence in X. If $D_f(x_0, x_n)$ is bounded, then the sequence x_n is also bounded.*

Lemma 2.8. ([23]) *Let $f : X \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function, then $f^* : X^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $z \in X$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{n} t_i D_f(z, x_i).$$

Lemma 2.9. ([33]) *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n,\ n \geq n_0,$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$, $\{\delta_n\}$ is a sequence in R satisfying the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} sup \delta_n \leq 0.$$

Then $\lim_{n\to\infty} a_n = 0.$

Lemma 2.10. ([19]) *Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that there exists a nondecreasing subsequence $\{n_i\}$ of $\{n\}$ that is $a_{n_i} < a_{n_{i+1}}$ $\forall i \in N$. Then there exists a nondecreasing subsequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following* *properties are satisfied for all (sufficiently large number)* $k \in N$*:* $a_{m_k} \leq a_{m_{k+1}}$ *and* $a_k \leq a_{m_{k+1}}$*. In fact,* $m_k = max\{j \leq k : a_j \leq a_{j+1}\}$*.*

Lemma 2.11. ([28]) *Let* $f : X \to (-\infty, +\infty]$ *be a coercive Legendre function and* C *a nonempty closed and convex subset of* X*. Let the mapping* $A : C \to X^*$ *be a continuous monotone mapping. For* $r > 0$ *and* $x \in X$*, define the mapping* $G_r : X \to C$ *as follows:*

$$G_r x = \left\{ z \in C : \langle Az, y - z\rangle + \frac{1}{r}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0,\ \forall y \in C \right\}$$

for all $x \in X$*. Then the following hold:*

(1) G_r *is single valued;*
(2) $F(G_r) = VI(C, A)$*;*
(3) $D_f(p, G_r x) + D_f(G_r x, x) \leq D_f(p, x), \forall p \in F(G_r)$*;*
(4) $VI(C, A)$ *is closed and convex.*

3. Main results

Let C be a nonempty, closed and convex subset of X. Let the mappings $A_1, A_2, \cdots, A_d : C \to X^*$ be d continuous monotone mappings. For $r_n \subset (0, \infty), n \in N$ and $x \in X$, define the mapping $G_r : X \to C$ as follows:

$$G_{i,r_n} x = \left\{ z \in C : \langle A_i z, y - z\rangle + \frac{1}{r_n}\langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0,\ \forall y \in C \right\}$$

for all $x \in X$, for all $i = 1, 2, \cdots, d$. Then in what follows, we shall state and prove the following theorem:

Theorem 3.1. *Let* C *be a nonempty, closed and convex subset of* $intdomf$*, let* $f : X \to R$ *be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of a real reflexive Banach space* X*. Let* $A_1, A_2, \cdots, A_d : C \to X^*$ *be* d *continuous monotone mappings and let* $T_1, T_2, \cdots, T_m : C \to C$ *be* m *left Bregman quasi-nonexpansive mappings such that* $\widehat{F(T_i)} = F(T_i)$. Assume that

$$\digamma = \bigcap_{i=1}^{m} Fix(T_i) \cap \cap_{i=1}^{d} V(C, A_j) \neq \emptyset.$$

For any fixed $u, x_0 \in C$, let $\{x_n\}$ be a sequence of C generated by the following algorithm:

$$\begin{cases} y_n = \nabla f^*(\beta_n \nabla f(x_n) + (1 - \beta_n)\nabla f(T_i x_n)), \\ u_{i,n} = G_{i,r_n} y_n, i = 1, 2, \cdots, d, \\ x_{n+1} = P_C^f \nabla f^* \left(\alpha_n \nabla f\left(u\right) + (1 - \alpha_n)\nabla f\left(u_{i,n}\right)\right), n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset (0, 1), \{\beta_n\} \subset [c, d] \subset (0, 1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to a point of $\digamma$.

Proof. Now by Lemma 2.2 and Lemma 2.11, we obtain that $\digamma$ is closed and convex. Let $p \in \digamma$. From Lemma 2.4 and since f is bounded and uniformly smooth on bounded subsets of X, so f^* is uniformly convex on bounded subsets of X^*. Then using Lemma 2.5, the properties of D_f and T_i, for each $i = 1, 2, \cdots, m$ and from (3.1), (1.8), (1.9), we obtain that

$$\begin{aligned} D_f(p, y_n) &= D_f(p, \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T_i x_n))) \\ &= V_f(p, \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T_i x_n)) \\ &\leq f(p) - \langle p, \beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T_i x_n)\rangle \\ &+ f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(T_i x_n)) \\ &\leq \beta_n f(p) + (1-\beta_n) f(p) - \beta_n \langle p, \nabla f(x_n)\rangle \\ &+ (1-\beta_n)\langle p, \nabla f(T_i x_n)\rangle + \beta_n f^*(\nabla f(x_n)) \\ &+ (1-\beta_n) f^*(\nabla f(T_i x_n)) - \beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n) - \nabla f(T_i x_n)||) \\ &= \beta_n V_f(p, \nabla f(x_n)) + (1-\beta_n) V_f(p, \nabla f(T_i x_n)) \\ &- \beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n) - \nabla f(T_i x_n)||) \\ &= \beta_n D_f(p, x_n) + (1-\beta_n) D_f(p, T_i x_n) \\ &- \beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n) - \nabla f(T_i x_n)||) \\ &\leq D_f(p, x_n) - \beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n) - \nabla f(T_i x_n)||) \\ &\leq D_f(p, x_n) \end{aligned} \tag{3.2}$$

Again, from Lemma 2.11 and (3.2), we obtain

$$\begin{aligned} D_f(p, u_{i,n}) = D_f(p, G_{i,r_n} y_n) &\leq D_f(p, y_n) \\ &\leq D_f(p, x_n) - \beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n) - \nabla f(T_i x_n)||) \\ &\leq D_f(p, x_n). \end{aligned} \tag{3.3}$$

Setting $h_n = \nabla f^* \left(\alpha_n \nabla f\left(u\right) + \left(1-\alpha_n\right) \nabla f\left(u_{i,n}\right)\right)$, we obtain from Lemma 2.3, Lemma 2.8, (3.1) and (3.3) that

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, P_C^f h_n) \\ &\leq D_f(p, h_n) \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(u_{i,n}))) \\ &\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, u_{i,n}) \\ &\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) \\ &- \beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n) - \nabla f(T_i x_n)||) \\ &\leq \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n). \end{aligned} \tag{3.4}$$

Thus by induction, we obtain that

$$D_f(p, x_{n+1}) \leq \max\{D_f(p, u), D_f(p, x_0)\}, \forall n \geq 0,$$

which implies that $\{D_f(p, x_n)\}$ and hence $\{D_f(p, T_i x_n)\}$ are bounded. Thus we get from Lemmas 2.6, 2.7 that $\{x_n\}$, $\{y_n\}$,$\{u_{i,n}\}$ and $\{h_n\}$ are all bounded.

Furthermore, from (3.1), Lemma 2.3, (1.9) and (1.10), we obtain

$$\begin{aligned}
D_f(p,x_{n+1}) &= D_f(p,P_C^f h_n)\\
&\le D_f(p,h_n)\\
&= D_f(p,\nabla f^*(\alpha_n\nabla f(u)+(1-\alpha_n)\nabla f(u_{i,n})))\\
&= V_f(p,\alpha_n\nabla f(u)+(1-\alpha_n)\nabla f(u_{i,n}))\\
&\le V_f(p,\alpha_n\nabla f(u)+(1-\alpha_n)\nabla f(u_{i,n})-\alpha_n(\nabla f(u)-\nabla f(p)))\\
&+\alpha_n\langle\nabla f(u)-\nabla f(p),h_n-p\rangle\\
&= V_f(p,\alpha_n\nabla f(p)+(1-\alpha_n)\nabla f(u_{i,n}))+\alpha_n\langle\nabla f(u)-\nabla f(p),h_n-p\rangle\\
&= D_f(p,\nabla f^*(\alpha_n\nabla f(p)+(1-\alpha_n)\nabla f(u_{i,n})))\\
&+\alpha_n\langle\nabla f(u)-\nabla f(p),h_n-p\rangle\\
&\le \alpha_n D_f(p,p)+(1-\alpha_n)D_f(p,u_{i,n})+\alpha_n\langle\nabla f(u)-\nabla f(p),h_n-p\rangle\\
&\le (1-\alpha_n)D_f(p,u_{i,n})+\alpha_n\langle\nabla f(u)-\nabla f(p),h_n-p\rangle\\
&\le (1-\alpha_n)D_f(p,x_n)-(1-\alpha_n)\beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n)-\nabla f(T_ix_n)||)\\
&+\alpha_n\langle\nabla f(u)-\nabla f(p),h_n-p\rangle \qquad (3.5)\\
&\le (1-\alpha_n)D_f(p,x_n)+\alpha_n\langle\nabla f(u)-\nabla f(p),h_n-p\rangle. \qquad (3.6)
\end{aligned}$$

We now consider two cases.

Case I. Suppose that there exists $n_0 \in N$ such that $\{D_f(p,x_n)\}$ is monotone non-increasing for all $n \ge n_0$. Then we get that $\{D_f(p,x_n)\}$ is convergent and

$$D_f(p,x_n)-D_f(p,x_{n+1})\to 0,$$

so that from (3.5), we obtain for

$$M=\sup\{\beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n)-\nabla f(T_ix_n)||)-D_f(p,x_n)\}$$

that

$$\beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n)-\nabla f(T_ix_n)||)\le D_f(p,x_n)-D_f(p,x_{n+1})+\alpha_n M, \qquad (3.7)$$

where

$$M=\sup\{\beta_n(1-\beta_n)\rho_r^*(||\nabla f(x_n)-\nabla f(T_ix_n)||)-D_f(p,x_n)\}<\infty$$

since $D_f(p,x_n)$ is bounded and ρ_s^* is nondecreasing.
Hence by this and since $\{\beta_n\}\subset[c,d]\subset(0,1)$, we get as $n\to\infty$

$$\nabla f(x_n)-\nabla f(T_ix_n)\to 0. \qquad (3.8)$$

Since f is strongly coercive and uniformly convex on bounded subsets of X, f^* is uniformly Fréchet differentiable on bounded subsets of X^* and by Lemma 2.4, we get that ∇f^* is uniformly continuous. So we obtain as $n\to\infty$ that

$$x_n-T_ix_n\to 0, i=1,2,\cdots,m. \qquad (3.9)$$

Moreover, from Lemma 2.8 and condition (i), we obtain that

$$\begin{aligned}
D_f(u_{i,n},h_n) &= D_f(u_{i,n},\nabla f^*(\alpha_n\nabla f(u)+(1-\alpha_n)\nabla f(u_{i,n})))\\
&\le \alpha_n D_f(u_{i,n},u)+(1-\alpha_n)D_f(u_{i,n},u_{i,n})\to 0, \qquad (3.10)
\end{aligned}$$

as $n \to \infty$, and by Lemma 2.1, we obtain as $n \to \infty$ that

$$u_{i,n} - h_n \to 0, \forall i = 1, 2, \cdots, d. \tag{3.11}$$

Furthermore, we obtain as $n \to \infty$

$$||\nabla f(x_n) - \nabla f(y_n)|| = (1 - \beta_n)||\nabla f(x_n) - \nabla f(T_i x_n)|| \to 0.$$

Hence, we get as $n \to \infty$ that

$$x_n - y_n \to 0. \tag{3.12}$$

Also, from Lemma 2.11, we have

$$\begin{aligned} D_f(y_n, u_{i,n}) &= D_f(y_n, G_{i,r_n} y_n) \\ &\leq D_f(p, G_{i,r_n} y_n) - D_f(p, y_n) \\ &\leq D_f(p, y_n) - D_f(p, y_n) \to 0. \end{aligned}$$

as $n \to \infty$.
Thus we have from Lemma 2.1 as $n \to \infty$ that

$$y_n - u_{i,n} \to 0, \forall i = 1, 2, \cdots, d. \tag{3.13}$$

Also, from Lemma 2.3, we have

$$\begin{aligned} D_f(y_n, P_C^f h_n) \leq D_f(y_n, h_n) & \\ & D_f(y_n, \nabla f^*(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(u_{i,n})) \\ & \alpha_n D_f(y_n, u) + (1 - \alpha_n) D_f(y_n, u_{i,n}) \\ & \alpha_n D_f(y_n, u) + (1 - \alpha_n) D_f(y_n, y_n) \to 0. \end{aligned}$$

as $n \to \infty$.
So that from Lemma 2.1, we have as $n \to \infty$

$$y_n - h_n \to 0. \tag{3.14}$$

Hence from 3.12 and 3.14, we obtain as $n \to \infty$

$$x_n - h_n \to 0. \tag{3.15}$$

Similarly, from 3.12 and 3.13, we obtain as $n \to \infty$

$$x_n - u_{i,n} \to 0. \tag{3.16}$$

Since f is strongly coercive and uniformly convex on bounded subsets of X, f^* is uniformly Fréchet differentiable on bounded subsets of X^* and by Lemma 2.4 we get that ∇f^* is uniformly continuous and from 3.16, we obtain as $n \to \infty$

$$\nabla f(x_n) - \nabla f(u_{i,n}) \to 0, . \tag{3.17}$$

Now since X is reflexive and $\{h_n\}$ is bounded, there exists a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ such that $h_{n_i} \rightharpoonup h \in C$, and

$$\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), h_n - p\rangle = \limsup_{i\to\infty} \langle \nabla f(u) - \nabla f(p), h_{n_i} - p\rangle.$$

Hence, we obtain from 3.15 and 3.16, that $x_{n_i} \rightharpoonup h$. Using 3.9 and the fact that $\widehat{F(T_i)} = F(T_i)$, we obtain that $h \in \cap_{i=1}^m F(T_i)$.

Now, we show that $h \in VI(C, A_i)$ for each $i = 1, 2, \cdots, d$. Recalling by definition, we have that

$$\langle A_i u_{i,n}, y - u_{i,n}\rangle + \langle \frac{\nabla f(u_{i,n}) - \nabla f(x_n)}{r_n}, y - u_{i,n}\rangle \geq 0, \forall y \in C,$$

and hence

$$\langle A_i u_{i,n_j}, y - u_{i,n_j}\rangle + \langle \frac{\nabla f(u_{i,n_j}) - \nabla f(x_{n_j})}{r_{n_j}}, y - u_{i,n_j}\rangle \geq 0, \forall y \in C, \tag{3.18}$$

Letting $v_t = ty + (1-t)h$ for all $t \in (0, 1]$ and $y \in C$. Consequently, we obtain that $v_t \in C$. From (3.18), it then follows that

$$\begin{aligned}\langle A_i v_t, v_t - u_{i,n_j}\rangle &\geq \langle A_i v_t, v_t - u_{i,n_j}\rangle \\ &- \langle A_i u_{i,n_j}, v_t - u_{i,n_j}\rangle - \left\langle \frac{\nabla f(u_{i,n_j}) - \nabla f(x_{n_j})}{r_{n_j}}, v_t - u_{i,n_j}\right\rangle \\ &= \langle A_i v_t - A_i u_{i,n_j}, v_t - u_{i,n_j}\rangle - \left\langle \frac{\nabla f(u_{i,n_j}) - \nabla f(x_{n_j})}{r_{n_j}}, v_t - u_{i,n_j}\right\rangle.\end{aligned}$$

Using (3.17) and the fact that A_i for each $i = 1, 2, \cdots, d$ is monotone, implies that

$$0 \leq \lim_{j\to\infty} \langle A_i v_t, v_t - u_{i,n_j}\rangle = \langle A_i v_t, v_t - h\rangle.$$

Hence we get $\langle A_i v_t, y - v\rangle \geq 0$, $\forall y \in C, i = 1, 2, \cdots, d$. Letting $t \to 0$, and the continuity of A_i for each $i = 1, 2, \cdots, d$ implies that $\langle A_i h, y - h\rangle \geq 0$, $\forall y \in C, i = 1, 2, \cdots, d$. This shows that

$$h \in \bigcap_{i=1}^{d} VI(C, A_i)$$

and hence

$$h \in \bigcap_{i=1}^{m} Fix(T_i) \cap \cap_{i=1}^{d} VI(C, A_i) = F.$$

Thus by Lemma 2.3, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle \nabla f(u) - \nabla f(p), h_n - p\rangle &= \limsup_{i\to\infty} \langle \nabla f(u) - \nabla f(p), h_{n_i} - p\rangle, \\ &= \nabla f(u) - \nabla f(p), h - p\rangle \leq 0.\end{aligned} \tag{3.19}$$

It therefore follows from (3.6), (3.18) and Lemma 2.9, that $D_f(p, x_n) \to 0$ as $n \to \infty$. Consequently, from Lemma 2.1, we obtain that $x \to p = P_F^f(u)$.

Case II. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$D_f(p, x_{n_i}) < D_f(p, x_{n_{i+1}}), \forall i \in N. \tag{3.20}$$

Then by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$, and $D_f(p, x_{m_k}) \leq D_f(p, x_{m_{k+1}})$ and $D_f(p, x_k) \leq D_f(p, x_{m_{k+1}})$, $\forall k \in N$. Then from (3.9) and the fact that $\alpha_{m_k} \to 0$, we obtain as $n \to \infty$ that

$$\rho_s^*(||\nabla f(x_{m_k}) - \nabla f(T_i x_{m_k})||) \to 0. \tag{3.21}$$

Thus we get from the same method of proof in **CaseI** that

$$x_{m_k} - T_i x_{m_k} \to 0, x_{m_k} - y_{m_k} \to 0, x_{m_k} - h_{m_k} \to 0, x_{m_k} - u_{i,m_k} \to 0, \tag{3.22}$$

as $n \to \infty$ and also we obtain

$$\limsup_{k\to\infty}\langle \nabla f(u) - \nabla f(p), h_{m_k} - p\rangle \leq 0. \tag{3.23}$$

Now from (3.5) we obtain that

$$D_f(p, x_{m_{k+1}}) \leq (1-\alpha_{m_k})D_f(p, x_{m_k}) + \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), h_{m_k} - p\rangle$$
$$\alpha_{m_k}D_f(p, x_{m_k}) \leq D_f(p, x_{m_k}) - D_f(p, x_{m_{k+1}}) + \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), h_{m_k} - p\rangle.$$

Since, $D_f(p, x_{m_k}) \leq D_f(p, x_{m_{k+1}})$, we have

$$\alpha_{m_k}D_f(p, x_{m_k}) \leq \alpha_{m_k}\langle \nabla f(u) - \nabla f(p), h_{m_k} - p\rangle. \tag{3.24}$$

Using (3.23), then (3.24) implies as $n \to \infty$

$$D_f(p, x_{m_k}) \to 0. \tag{3.25}$$

Consequently,as $n \to \infty$

$$D_f(p, x_{m_{k+1}}) \to 0. \tag{3.26}$$

But $D_f(p, x_k) \leq D_f(p, x_{m_{k+1}})$ for all $k \in N$. Thus we obtain that $D_f(p, x_k) \to 0$ as $n \to \infty$. Hence, by Lemma 2.1, we have $x_k \to p$ as $k \to \infty$. Therefore, from the above two cases, we conclude that the sequence $\{x_n\}$ converges strongly to $p = P_F^f(u)$ and that completes the proof of our theorem. □

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