Fixed Point Theory, 21(2020), No. 1, 151-166
DOI: 10.24193/fpt-ro.2020.1.11
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

MULTIVALUED ITERATED CONTRACTIONS

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Abstract. This paper focus on the class of *multivalued iterated contractions*, mappings which are contractive throughout the orbits. We show that the proof of Nadler's theorem still holds for these mappings whenever they satisfy a rather weak type of continuity, which gives us a new fixed point theorem. We show several types of mappings that properly contain Suzuki (C)-type generalized contraction mappings and for which our fixed point results apply. We conclude the paper showing some further examples of iterated contraction mappings which are, respectively, the mappings satisfying condition (B) and an extension to the multivalued case of mean iterated contractions and we also obtain fixed point results for these classes of mappings.

Key Words and Phrases: Fixed point, multivalued mappings, generalized contraction mappings, metric spaces.
2010 Mathematics Subject Classification: 47H04, 47H09, 47H10.

1. Introduction

The Banach Contraction Principle is perhaps the most widely applied fixed points theorem in nonlinear analysis. For this reason, there is a large amount of literature dealing with extensions and generalizations of Banach's theorem (see [12, Chapter 1] and [19] for references). One of the key fact behind the proof of Banach Contraction Principle result is the convergence of the Picard iterates. In the study of certain iterative processes, Ortega-Rheinboldt [17] realized that this convergence also holds when the contractiveness condition of the mapping is carried over its iterates.

Definition 1.1. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an iterated contraction if there exists a constant $k \in [0, 1)$ such that for all $x \in X$

$$d(Tx, T^2x) \leq kd(x, Tx).$$

An iterated contraction need not be continuous nor need its fixed points be unique. However, a continuous iterated contraction mapping always has a fixed point in a complete metric space [17, Chapter 12].

In 1969, Nadler [16] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. Since then, some classical theorems concerning the existence of fixed points for singlevalued mappings of contractive type have been generalized to the multivalued case ([5, 8, 15]). Among these it should highlight a result established by Kikkawa and Suzuki [11], where Nadler's theorem is generalized for a new type of contraction mappings.

Theorem 1.2. *Let (X, d) be a complete metric space and let $T : X \to 2^X$ be a closed bounded valued mapping. Assume that there exists $r \in [0, 1)$ such that*

$$\frac{1}{1+r} d(x, Tx) \leq d(x, y) \quad \textit{implies} \quad H(Tx, Ty) \leq rd(x, y),$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

This theorem is as well an extension of a remarkable result established by Suzuki [21, Theorem 2.1] in the single valued context. It is worthwhile to mention that the author in [21] not only generalizes Banach Contraction Principle for a new type of contractions but also characterizes the completeness of the underlying metric space. After that, some authors have considered wider classes of multivalued mappings of contractive type which meet Kikkawa-Suzuki's theorem ([3, 20]).

The purpose of this paper is to prove the existence of fixed point for several types of multivalued mappings which satisfy some contractiveness conditions. First of all we prove a fixed point theorem for mappings which are contractive throughout the orbits and satisfy a type of continuity weaker than the one used in [11]. We show that the proof of Nadler's theorem still holds for these mappings which will be called *multivalued iterated contractions*. We continue by studying several classes of multivalued mappings that properly contain Suzuki (C)-type generalized contractions proving that all these classes of mappings are multivalued iterated contractions and satisfy the weak continuity condition which is required in our fixed point theorem. We follow an indirect way, proving that these mappings satisfy the, so called, condition (E) and checking that condition (E) implies the required type of continuity. It is noteworthy that the types of mappings that we consider, do no satisfy, in general the continuity condition used in previous papers about this subject [3, 11, 20]. Finally, we introduce two new examples of iterated contraction mappings which are, respectively, the mappings satisfying condition (B) introduced in [13] and an extension to the multivalued case of mean iterated contractions [6, 7]. Again, we obtain fixed point results for these classes of mappings.

2. Multivalued iterated contractions

Let (X, d) be a metric space. In this paper we consider the following family of sets:

$\mathcal{P}(X) = \{D \subset X : D \text{ is nonempty}\}$,
$\mathcal{P}_{cl}(X) = \{D \subset X : D \text{ is nonempty and closed}\}$,
$\mathcal{P}_{cl,b}(X) = \{D \subset X : D \text{ is nonempty, closed and bounded}\}$,
$\mathcal{P}_{cp,cv}(X) = \{D \subset X : D \text{ is nonempty, compact and convex}\}$.

On $\mathcal{P}_{cl,b}(X)$ one defines the Hausdorff distance H

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\},$$

where $d(a, B) := \inf\{d(a, b) : b \in B\}$ is the usual distance from the point a to the subset B. It is said that $x \in C$ is a fixed point of T if and only if x belongs to Tx.

The following class of multivalued mappings has been studied in [4].

Definition 2.1 ([4])**.** Let (X, d) be a metric space and $C \in \mathcal{P}(X)$. A mapping $T : C \to \mathcal{P}(X)$ is said to satisfy condition (E_μ) for some $\mu \geq 1$ if for each $x, y \in X$ and $u_x \in Tx$ there exists $u_y \in Ty$ such that

$$d(x, u_y) \leq \mu d(x, u_x) + d(x, y).$$

We say that T satisfies condition (E) on X whenever T satisfies (E_μ) for some $\mu \geq 1$.

Definition 2.2. Let (X, d) be a metric space, $C \in \mathcal{P}(X)$ and $T : C \to \mathcal{P}_{cl}(X)$. We will denote by $\phi_T : X \to \mathbb{R}$ the mapping $\phi_T(x) = d(x, Tx)$. The graph of ϕ_T is called strongly demiclosed at 0 if for every sequence $\{x_n\}$ in C convergent to $x \in C$ such that $\lim_n \phi_T(x_n) = 0$ one has that $x \in Tx$.

Inspired by the concept of iterated contraction for a single valued mapping in [17, Chapter 12], we introduce the class of multivalued iterated contractions.

Definition 2.3. Let (X, d) be a metric space. We say that a mapping $T : X \to \mathcal{P}(X)$ is a multivalued iterated contraction, (MIC in short), provided that there exists $k \in [0, 1)$ such that for every $x \in X$ and $u \in Tx$ there exists $v \in Tu$ such that

$$d(u, v) \leq k d(x, u).$$

Remark 2.4.

(1) Another natural extension of multivalued iterated contraction for a mapping $T : X \to \mathcal{P}(X)$ is the following: there exists $k \in [0, 1)$ such that for any $x \in X$ and $u \in Tx$

$$d(u, Tu) \leq k d(x, u).$$

It is easy to check that the above two classes of mappings are the same (possibly for a different constant k).

(2) When T is closed bounded valued and there exists $k \in [0, 1)$ such that

$$H(Tx, Ty) \leq k d(x, Tx) \qquad \text{for all } x \in X \ , \ y \in Tx,$$

we obtain a subclass of multivalued iterated contraction mappings according to Definition 2.3.

(3) Notice that a first attempt to generalize Definition 2.3 for multivalued mappings could be: there exists $k \in [0, 1)$ such that

$$H(Tx, T^2x) \leq k d(x, Tx) \qquad \text{for all } x \in X$$

It is clear that this condition implies the one in Definition 2.3. However, the class of MIC mappings is essentially wider.
Indeed, let $T : [0, 1] \to \mathcal{P}_{cp,cv}([0, 1])$ be the mapping defined by

$$Tx = \begin{cases} [0, \frac{1}{2}] & \text{if } x \in [0, \frac{1}{2}) \\ [0, 1] & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Clearly $T^2x = [0, 1]$, and if $x \in [0, \frac{1}{2})$ then $H(Tx, T^2x) = \frac{1}{2}$ and $d(x, Tx) = 0$. Hence $H(Tx, T^2x) > 0 = kd(x, Tx)$ for all $k \in [0, 1)$.
On the other hand, taking $x \in [0, 1]$, if $u \in Tx$ it follows that $v = u \in Tu$ an so $d(u, v) = 0 \leq kd(x, u)$, Consequently T is a multivalued iterated contraction for all $k \in [0, 1)$.

The following theorem states a fixed point result for MIC mappings.

Theorem 2.5. *Let (X, d) be a complete metric space and $T : X \to \mathcal{P}_{cl}(X)$ be a MIC mapping such that the graph of ϕ_T is strongly demiclosed at 0. Then, T has a fixed point.*

Proof. Following a similar argument to that in the proof of Nadler's fixed point theorem, we can obtain a sequence $\{x_n\}$ of X such that $x_n \in Tx_{n-1}$ and $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$. Indeed, take $x_0 \in X$ and $x_1 \in Tx_0$. Since T is a MIC mapping, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq kd(x_0, x_1).$$

By induction, for each $n \geq 1$ we construct a sequence $\{x_n\} \in X$ such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \leq k^2d(x_{n-2}, x_{n-1}) \leq \ldots \leq k^nd(x_0, x_1).$$

Since $k < 1$, $\{x_n\}$ is a Cauchy sequence. By completeness, $\{x_n\}$ converges to some fixed point $x \in X$. So we have

$$\phi_T(x_n) = d(x_n, T(x_n)) \leq d(x_n, x_{n+1}) \to 0$$

which implies that $x \in Tx$. $\square$

Remark 2.6. In [5] Y. Feng and S. Liu generalized Nadler's fixed point theorem for a multivalued iterated contraction mapping under the assumption of the lower semicontinuity of the mapping $x \mapsto d(x, Tx)$. A further extension is given in [8] for a class of set valued mappings slightly more general than MIC, also under the lower semicontinuity assumption.

It is easy to check that the graph of the mapping ϕ_T is strongly demiclosed at 0 if either the mapping $x \mapsto d(x, Tx)$ is lower semicontinuous or the mapping T is upper semicontinuous. According to [1, Corollary 1.4.17], $x \mapsto d(x, Tx)$ is lower semicontinuous if T is upper semicontinuous with compact values. However, the last two conditions are different: indeed, the mapping $T : [-1, 1] \to \mathcal{P}_{cp,cv}([-1, 1])$ defined by

$$Tx = \begin{cases} 0 & \text{if } x = 0 \\ [-1,1] & \text{if } x \neq 0 \end{cases}$$

is not upper semicontinuous at zero, but $x \mapsto d(x, Tx)$ is lower semicontinuous because $d(x, Tx) = 0$.

We will show later (Example 2.18) that the assumption of ϕ_T being strongly demiclosed at 0 is more general.

In the remainder of this section we shall consider several classes of multivalued iterated contractions.

KARAPINAR - TAS's SKC MAPPINGS

In 2010 Karapinar and Tas (see [10]) proposed the so called SKC mapping for a single valued mapping. Inspired on multivalued mappings of Suzuki type considered in the paper [11], the authors in [20] adapted the definition of SKC mappings to the multivalued case. Next we introduce a class of SKC multivalued mappings more general than those studied in [20].

Definition 2.7. Let (X, d) be a metric space. A mapping $T : X \to \mathcal{P}(X)$ is said to be a multivalued k-SKC if for some $k \in [0, 1)$ and for each $x, y \in X$ and $u_x \in Tx$ such that

$$\frac{1}{1+k} d(x, u_x) \leq d(x, y),$$

there exists $u_y \in Ty$ such that

$$d(u_x, u_y) \leq kN(x, y),$$

where

$$N(x, y) = \max\left\{d(x, y), \frac{1}{2}\big(d(x, u_x) + d(y, u_y)\big), \frac{1}{2}\big(d(x, u_y) + d(y, u_x)\big)\right\}.$$

Proposition 2.8. *Let (X, d) be a metric space. Then every multivalued k-SKC mapping $T : X \to \mathcal{P}(X)$ is a MIC mapping.*

Proof. Let $x \in X$ and $u_x \in Tx$. Since $\frac{1}{1+k} d(x, u_x) \leq d(x, u_x)$ there exists $v_x \in Tu_x$ such that

$$d(u_x, v_x) \leq kN(x, u_x),$$

where

$$N(x, u_x) = \max\left\{d(x, u_x), \frac{1}{2}\big(d(x, u_x) + d(u_x, v_x)\big), \frac{1}{2}\big(d(x, v_x) + d(u_x, u_x)\big)\right\}.$$

This implies that

$$d(u_x, v_x) \leq kd(x, u_x).$$

Indeed, if $N(x, u_x) = d(x, u_x)$, then we have done. If

$$N(x, u_x) = \frac{1}{2}\big(d(x, u_x) + d(u_x, v_x)\big)$$

then

$$d(u_x, v_x) \leq \frac{k}{2}\big(d(x, u_x) + d(u_x, v_x)\big),$$

implies

$$d(u_x, v_x) \leq \frac{k}{2-k} d(x, u_x) \leq k d(x, u_x).$$

If

$$N(x, u_x) = \frac{1}{2} d(x, v_x),$$

then

$$d(u_x, v_x) \leq \frac{k}{2} d(x, v_x) \leq \frac{k}{2}\big(d(x, u_x) + d(u_x, v_x)\big),$$

which implies T is MIC mapping. □

KANNAN AND CHATTERJEA TYPE MAPPINGS

We introduce a class of MIC mapping, which is an extension to the multivalued case of generalized contraction mappings defined in [14].

Definition 2.9. Let (X, d) be a metric space. A mapping $T : X \to \mathcal{P}(X)$ is said to be (C)-type contractive, if for some $a, b, c \geq 0$ with $a + 2b + 2c < 1$, and for all $x, y \in X$ and $u_x \in Tx$ such that

$$\frac{1}{2} d(x, u_x) \leq d(x, y),$$

there exists $u_y \in Ty$ such that

$$d(u_x, u_y) \leq a d(x, y) + b\big(d(x, u_x) + d(y, u_y)\big) + c\big(d(x, u_y) + d(y, u_x)\big).$$

Remark 2.10. Obviously every multivalued (C)-type contractive mapping is a multivalued k-SKC mapping. Take $k := a + 2b + 2c < 1$, if $x, y \in X$, $u_x \in Tx$ and

$$\frac{1}{1+k} d(x, u_x) \leq d(x, y),$$

we have

$$\frac{1}{2} d(x, u_x) \leq \frac{1}{1+k} d(x, u_x) \leq d(x, y).$$

Thus there exists $u_y \in Ty$ such that

$$\begin{aligned} d(u_x, u_y) &\leq a d(x, y) + b\big(d(x, u_x) + d(y, u_y)\big) + c\big(d(x, u_y) + d(y, u_x)\big) \\ &\leq a N(x, y) + 2b N(x, y) + 2c N(x, y) \\ &= (a + 2b + 2c) N(x, y). \end{aligned}$$

GENERALIZED α-CONTRACTIVE MAPPINGS

Inspired by the definition of a generalized α -nonexpansive mapping introduced in [18], we present a new class of multivalued generalized contractions of Suzuki's type.

Definition 2.11. Let (X,d) be a metric space. For $\alpha \in [0,1)$, a mapping $T : X \to \mathcal{P}(X)$ is said to be a multivalued generalized α- contraction ($G\alpha C$ in short), if there exists $k \in [0,1)$ such that for all $x,y \in X$ and $u_x \in T(x)$ verifying

$$\frac{1}{1+k} d(x,u_x) \leq d(x,y),$$

there exists $u_y \in T(y)$ such that

$$d(u_x,u_y) \leq k(\alpha d(x,u_y) + \alpha d(u_x,y) + (1-2\alpha) d(x,y)).$$

Proposition 2.12. *Let X be a metric space and $T : X \to \mathcal{P}(X)$ be a $G\alpha C$ mapping. Then T is a MIC.*

Proof. Let $x \in X$ and $u_x \in T(x)$. Because $\frac{1}{1+k} d(x,u_x) \leq d(x,u_x)$, there exists $v_x \in T(u_x)$ such that

$$\begin{aligned} d(u_x,v_x) &\leq k(\alpha d(x,v_x) + \alpha d(u_x,u_x) + (1-2\alpha) d(x,u_x)) \\ &\leq k(\alpha d(x,u_x) + \alpha d(u_x,v_x) + (1-2\alpha) d(x,u_x)). \end{aligned}$$

This implies that

$$d(u_x,v_x) \leq k d(x,u_x),$$

and so T is a MIC. $\square$

Now we study some direct relationship between the above two classes of mappings and condition (E).

Lemma 2.13. *Let X be a metric space and $T : X \to \mathcal{P}(X)$ be a multivalued k-SKC mapping. Then T satisfies condition (E_7).*

Proof. Let $x,y \in X$ and $u_x \in Tx$. Since T is MIC, there exists $v_x \in Tu_x$ such that

$$d(u_x,v_x) \leq k d(x,u_x). \tag{2.1}$$

We prove that either

$$\frac{1}{1+k} d(x,u_x) \leq d(x,y) \tag{2.2}$$

or

$$\frac{1}{1+k} d(u_x,v_x) \leq d(u_x,y) \tag{2.3}$$

holds. Suppose

$$\frac{1}{1+k} d(x,u_x) > d(x,y) \quad \text{and} \quad \frac{1}{1+k} d(u_x,v_x) > d(u_x,y).$$

Then, using (3.1) we obtain the following contradiction:

$$d(x,u_x) \leq d(x,y) + d(y,u_x) < \frac{1}{1+k} d(x,u_x) + \frac{1}{1+k} d(u_x,v_x) \leq d(x,u_x).$$

Hence, if (2.2) holds, then there exists $u_y \in Ty$ such that $d(u_x,u_y) \leq kN(x,y)$, where

$$N(x,y) = \max\{d(x,y), \frac{1}{2}(d(x,u_x) + d(y,u_y)), \frac{1}{2}(d(x,u_y) + d(y,u_x))\}.$$

If $N(x,y)=d(x,y)$, then we have

$$d(x,u_y)\le d(x,u_x)+d(u_x,u_y)\le d(x,u_x)+kd(x,y)\le d(x,u_x)+d(x,y).$$

For $N(x,y)=\frac{1}{2}(d(x,u_x)+d(y,u_y))$, one can observe

$$\begin{aligned}d(x,u_y)\le d(x,u_x)+d(u_x,u_y)&\le d(x,u_x)+\frac{k}{2}\big(d(x,u_x)+d(y,u_y)\big)\\&\le\frac{2+k}{2}d(x,u_x)+\frac{k}{2}\big(d(x,y)+d(x,u_y)\big).\end{aligned}$$

Since $k<1$, we have

$$d(x,u_y)\le 3d(x,u_x)+d(x,y).$$

If $N(x,y)=\frac{1}{2}\big(d(x,u_y)+d(y,u_x)\big)$ then

$$\begin{aligned}d(x,u_y)\le d(x,u_x)+d(u_x,u_y)&\le d(x,u_x)+\frac{k}{2}\big(d(x,u_y)+d(y,u_x)\big)\\&\le d(x,u_x)+\frac{1}{2}\big(d(x,u_y)+d(x,u_x)+d(x,y)\big).\end{aligned}$$

Thus, we have

$$d(x,u_y)\le 3d(x,u_x)+d(x,y).$$

If (2.3) holds, then there exists $u_y\in Ty$ such that $d(v_x,u_y)\le kN(u_x,y)$, where

$$N(u_x,y)=\max\{d(u_x,y),\frac{1}{2}\big(d(u_x,v_x)+d(y,u_y)\big),\frac{1}{2}\big(d(u_x,u_y)+d(y,v_x)\big)\}.$$

If $N(u_x,y)=d(u_x,y)$, using (2.1) and since $k<1$ we have

$$\begin{aligned}d(x,u_y)&\le d(x,u_x)+d(u_x,v_x)+d(v_x,u_y)\\&\le d(x,u_x)+kd(x,u_x)+k(d(u_x,x)+d(x,y))\\&\le 3d(x,u_x)+d(x,y).\end{aligned}$$

For the case

$$N(u_x,y)=\frac{1}{2}\big(d(u_x,v_x)+d(y,u_y)\big),$$

one can observe

$$\begin{aligned}d(x,u_y)&\le d(x,u_x)+d(u_x,v_x)+d(v_x,u_y)\\&\le(1+k)d(x,u_x)+\frac{k}{2}\big(d(u_x,v_x)+d(y,u_y)\big)\\&\le(1+k)d(x,u_x)+\frac{k}{2}\big(kd(x,u_x)+d(x,y)+d(x,u_y)\big)\\&\le\frac{k^2+2k+2}{2}d(x,u_x)+\frac{k}{2}\big(d(x,y)+d(x,u_y)\big).\end{aligned}$$

This implies that

$$d(x,u_y)\le 5d(x,u_x)+d(x,y).$$

For the last case,

$$N(u_x,y)=\frac{1}{2}\big(d(u_x,u_y)+d(y,v_x)\big),$$

one can obtain

$$\begin{aligned} d(x,u_y) &\le d(x,u_x)+d(u_x,v_x)+d(v_x,u_y)\\ &\le d(x,u_x)+kd(x,u_x)+\frac{k}{2}\big(d(u_x,u_y)+d(y,v_x)\big)\\ &\le (1+k)d(x,u_x)+\frac{k}{2}\big(d(x,u_x)+d(x,u_y)\big)\\ &+\frac{k}{2}\big(d(x,y)+d(x,u_x)+d(v_x,u_x)\big). \end{aligned}$$

Thus we have

$$d(x,u_y)\le 7d(x,u_x)+d(x,y).$$ □

Lemma 2.14. *Let (X,d) be a metric space and $T: X\to \mathcal{P}(X)$ be a $G\alpha C$ mapping. Then T satisfies condition (E).*

Proof. Let $x,y\in X$ and $u_x\in T(x)$. Since T is MIC there exists $v_x\in T(u_x)$ such that

$$d(u_x,v_x)\le kd(x,u_x).$$

Note either

$$\frac{1}{2}d(x,u_x)\le d(x,y)\quad \text{or}\quad \frac{1}{2}d(u_x,v_x)\le d(u_x,y)$$

holds. Indeed, otherwise we have the contradiction:

$$d(x,u_x)\le d(x,y)+d(u_x,y)<\frac{1}{2}d(x,u_x)+\frac{1}{2}d(u_x,v_x)\le d(x,u_x).$$

Consider the first case. Then there exists $u_y\in T(y)$ such that

$$d(u_x,u_y)\le k(\alpha d(x,u_y)+\alpha d(y,u_x)+(1-2\alpha)d(u_x,y)),$$

so we have

$$\begin{aligned} d(x,u_y) &\le d(x,u_x)+d(u_x,u_y)\\ &\le d(x,u_x)+k(\alpha d(x,u_y)+\alpha d(y,u_x)+(1-2\alpha)d(x,y))\\ &\le d(x,u_x)+k\alpha d(x,u_y)+k\alpha d(y,x)+k\alpha d(x,u_x)+k(1-2\alpha)d(x,y)\\ &\le (1+k\alpha)d(x,u_x)+k(1-\alpha)d(x,y)+k\alpha d(x,u_y). \end{aligned}$$

Since $k<1$, this implies that

$$d(x,u_y)\le \frac{1+\alpha}{1-\alpha}d(x,u_x)+d(x,y).$$

In other case, there exists $u_y\in Ty$ such that

$$d(v_x,u_y)\le \alpha d(u_x,u_y)+\alpha d(y,v_x)+(1-2\alpha)d(u_x,y).$$

Since $d(u_x, v_x) \leq kd(x, u_x)$ we have that

$$\begin{aligned}
d(x, u_y) &\leq d(x, u_x) + d(u_x, v_x) + d(v_x, u_y) \\
&\leq (1 + k)d(x, u_x) + k(\alpha d(u_x, u_y) + \alpha d(y, v_x) + (1 - 2\alpha)d(u_x, y)) \\
&\leq (1 + (1 + \alpha)k)d(x, u_x) + k\alpha d(x, u_y) \\
&+ k\alpha d(y, u_x) + k\alpha d(u_x, v_x) + k(1 - 2\alpha)d(u_x, y) \\
&\leq (1 + (1 + \alpha)k)d(x, u_x) + k\alpha d(x, u_y) \\
&+ k(1 - \alpha)d(x, u_x) + k(1 - \alpha)d(x, y) + k\alpha d(x, u_x).
\end{aligned}$$

Since $k < 1$, this implies that

$$d(x, u_y) \leq \frac{3 + \alpha}{1 - \alpha} d(x, u_x) + d(x, y). \qquad \square$$

Proposition 2.15. *Let (X, d) be a metric space and $T : X \to \mathcal{P}_{cl}(X)$ be a mapping satisfying condition (E). Then, the graph of ϕ_T is strongly demiclosed at 0.*

Proof. Assume $\{x_n\}$ is a sequence in X convergent to $x \in X$ such that

$$\lim_n d(x_n, Tx_n) = 0.$$

It is easily seen that for every $n \in \mathbb{N}$ we can find $v_n \in Tx_n$ such that $d(x_n, v_n) \to 0$. By condition (E) there exists $u_n \in Tx$ such that

$$d(x_n, u_n) \leq \mu d(x_n, v_n) + d(x_n, x) \to 0.$$

Thus $d(x_n, Tx) \to 0$ and so, $d(x, Tx) = 0$. $\square$

In the example below we find a mapping for which the converse of the above result does not hold.

From Proposition 2.15, Theorem 2.5, Proposition 2.8 and Lemma 2.13 we easily obtain:

Corollary 2.16. *Let (X, d) be a complete metric space and $T : X \to \mathcal{P}_{cl}(X)$ be a k-SKC mapping. Then, T has a fixed point.*

From Proposition 2.15, Theorem 2.5, Proposition 2.12 and Lemma 2.14 we obtain:

Corollary 2.17. *Let (X, d) be a complete metric space and $T : X \to \mathcal{P}_{cl}(X)$ be a $G\alpha C$ mapping. Then, T has a fixed point.*

The following example shows that the assumption of ϕ_T being strongly demiclosed at 0 is more general than the lower semicontinuity of the function $x \to d(x, Tx)$. Moreover, it shows that the existence of a fixed point for k-SKC mappings or $G\alpha C$ mappings cannot be derived of this latter semicontinuity condition.

Example 2.18. Let $T : [0, 3] \to \mathcal{P}_{cl}([0, 3])$ be defined as

$$Tx = \begin{cases} \{3\} & \text{if } x = 0 \\ [2, 3] & \text{if } x \in (0, 3]. \end{cases}$$

If we take (x_n) in $[0, 3]$ convergent to a point x such that $\lim_n d(x_n, Tx_n) = 0$ then $x \in [2, 3]$. Hence $d(x, Tx) = 0$ and ϕ_T is strongly demiclosed at 0. However, if we

have a non-constant sequence $x_n \to 0$, then $\lim_n d(x_n, Tx_n) = 2$, but $d(0, T(0)) = 3$. Consequently, the mapping $x \mapsto d(x, Tx)$ fails to be lower semicontinuous. We claim that T is a k-SKC mapping with $k = \frac{2}{3}$.
Indeed, we split in three cases: (a) Let $x, y \in (0, 3]$. Then $Tx = Ty$ and for a given $u_x \in [2, 3]$, choose $u_y = u_x$. We have

$$d(u_x, u_y) = 0 \leq kN(x, y).$$

(b) Let $x = 0$ and $y \in (0, 3]$, then $u_x = 3$. By choosing $u_y = u_x = 3$, we have

$$d(u_x, u_y) = 0 \leq kN(x, y).$$

(c) Let $x \in (0, 3]$ and $y = 0$. Then, $u_x \in [2, 3]$, $u_y = 3$ and we have

$$d(u_x, u_y) \leq 1 \leq \frac{2}{3} N(x, y),$$

since

$$\begin{aligned} N(x,y) &= \max\{d(x,y), \frac{1}{2}\big(d(x,u_x) + d(y,u_y)\big), \frac{1}{2}\big(d(x,u_y) + d(y,u_x)\big)\} \\ &\geq \frac{1}{2} d(y,u_y) = \frac{3}{2}. \end{aligned}$$

Furthermore, by a similar argument it is easy to check that T is a generalized α-contractive mapping with $k = \frac{1}{5\alpha}$, where $\frac{1}{5} < \alpha \leq \frac{1}{3}$.
Indeed, in the nontrivial case corresponding to $x \in (0, 3]$ and $y = 0$, we have $u_x \in [2, 3]$, $u_y = 3$. Since $\alpha d(x, u_y) + \alpha d(u_x, y) + (1 - 2\alpha) d(x, y) \geq 5\alpha$, we obtain

$$d(u_x, u_y) \leq 1 \leq \frac{1}{5\alpha} (\alpha d(x, u_y) + \alpha d(u_x, y) + (1 - 2\alpha) d(x, y)).$$

A class of (single valued) iterated nonexpansive mappings (the so-called mappings satisfying condition (B)) were defined in the framework of a Hilbert space in [13]. In a recent paper [2], the authors have generalized Kirk's fixed point theorem for such class of mappings.

Definition 2.19. Let C be a subset of a Banach space $(X, \|\cdot\|)$.
A mapping $T : C \to C$ is said to satisfy condition (B) if for all $x, y \in C$ such that

$$\frac{1}{2} \|x - T(x)\| \leq \|x - y\|,$$

then,

$$\|T(x) - T(y)\|^2 + \|x - T(y)\|^2 \leq \|T(x) - y\|^2 + \|x - y\|^2.$$

Next we introduce a multivalued version of contractive mappings enjoying condition (B).

Definition 2.20. Let (X, d) be a metric space. A mapping $T : X \to \mathcal{P}(X)$ is said to satisfy k-condition (B) for some $k \in [0, 1)$, if for each $x, y \in X$ and $u_x \in Tx$ such that

$$\frac{1}{1+k} d(x, u_x) \leq d(x, y)$$

there exists $u_y \in Ty$ such that

$$d(u_x, u_y)^2 + d(x, u_y)^2 \leq k^2\big(d(y, u_x)^2 + d(x, y)^2\big).$$

Proposition 2.21. *Let (X, d) be a metric space and let $T : X \to \mathcal{P}(X)$ satisfy k-condition (B). Then T is a MIC with constant k.*

Proof. Let $x \in X$ and $u \in Tx$. Because $\frac{1}{1+k}d(x, u) \leq d(x, u)$ there exists $v \in Tu$ such that

$$d(u, v)^2 + d(x, v)^2 \leq k^2(d(u, u)^2 + d(x, u)^2).$$

So,

$$d(u, v)^2 \leq k^2 d(x, u)^2 - d(x, v)^2 \leq k^2 d(x, u)^2$$

implies that $d(u, v) \leq kd(x, u)$; that is, T is MIC. □

Theorem 2.22. *Let (X, d) be a complete metric space.*
Every mapping $T : X \to \mathcal{P}_{cl}(X)$ satisfying k-condition (B) has a fixed point.

Proof. Since T is a MIC mapping, we appeal to the proof of Theorem 2.5 to get a sequence $\{x_n\}$ in X convergent to a point $x \in X$ such that $x_{n+1} \in Tx_n$ and

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1}).$$

We prove that either

(a) $\dfrac{1}{1+k}d(x_n, x_{n+1}) \leq d(x_n, x)$,

or

(b) (b) $\dfrac{1}{1+k}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x)$,

holds. Suppose that for some $n \in \mathbb{N}$,

$$\frac{1}{1+k}d(x_n, x_{n+1}) > d(x_n, x) \quad \text{and} \quad \frac{1}{1+k}d(x_{n+1}, x_{n+2}) > d(x_{n+1}, x).$$

Bearing in mind that T is a MIC mapping, we obtain the following contradiction,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x) + d(x_{n+1}, x) \\ &< \frac{1}{1+k}d(x_n, x_{n+1}) + \frac{1}{1+k}d(x_{n+1}, x_{n+2}) \\ &\leq \frac{1}{1+k}d(x_n, x_{n+1}) + \frac{k}{1+k}d(x_n, x_{n+1}) \\ &= d(x_n, x_{n+1}). \end{aligned}$$

Thus, we can split $\mathbb{N} = \mathbb{A} \cup \mathbb{B}$ such that for $n \in \mathbb{A}$ we have that there exists $z_n \in Tx$ such that

$$d(x_{n+1}, z_n)^2 + d(x_n, z_n)^2 \leq k^2\big(d(x_{n+1}, x)^2 + d(x_n, x)^2\big)$$

and for $n \in \mathbb{B}$ there exists $\tilde{z}_n \in Tx$ such that

$$d(x_{n+2}, \tilde{z}_n)^2 + d(x_{n+1}, \tilde{z}_n)^2 \leq k^2\big(d(x_{n+2}, x)^2 + d(x_{n+1}, x)^2\big).$$

The convergence of $x_n \to x$ implies that for every $\varepsilon > 0$ there exists n_0 such that $d(x, z_n) < \varepsilon$ if $n \in \mathbb{A} \cap [n_0, +\infty)$ and $d(x, \tilde{z}_n) < \varepsilon$ if $n \in \mathbb{B} \cap [n_0, +\infty)$. Thus, x is a cluster point of Tx which implies that $x \in Tx$. □

3. Mean iterated contractions

In [6] and [7] the authors introduce a class of Lipschitzian mappings which are defined involving the mapping and a finite number of its iterates. It is referred to as the class of mean Lipschitzian mappings. The study of this class of mappings has lead to interesting results in metric fixed point theory (see [6] and [7]). For instance, it is shown that a mean contraction is in fact a contraction with respect to some equivalent metric. Following this idea, we introduce the following class of set-valued mappings.

Definition 3.1. Let (X, d) be a metric space.
We say that a mapping $T : X \to \mathcal{P}_{cl,b}(X)$ is a mean multivalued iterated contraction, (MMIC), if there exist $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 > 0$ and $k \in [0, 1)$ such that for all $x \in X$, $u \in Tx$ and $v \in Tu$ we have

$$\alpha_1 H(Tx, Tu) + \alpha_2 H(Tu, Tv) \leq kd(x, u).$$

In the single valued case, the Banach Contraction Principle is valid for mean contractions as it is indicated in [7]. A generalization of this result can be found in the proposition below.

Proposition 3.2. *Let (X, d) be a metric space. Assume that $T : C \to \mathcal{P}_{cl,b}(X)$ is a MMIC. Then there exists a metric $\rho \geq d$ such that T is a MIC mapping with respect to ρ.*

Proof. Let $\alpha = (\alpha_1, \alpha_2)$ and let $T : X \to X$ be a MMIC with constant $k < 1$.
Define ρ by

$$\rho(x, u) = d(x, u) + \alpha_2 H(Tx, Tu).$$

It is readily seen that ρ is a metric on X and we have

$$d(x, u) \leq \rho(x, u) \leq \big(1 + \frac{\alpha_2 k}{\alpha_1}\big)d(x, u) = \frac{\alpha_1 + \alpha_2 k}{\alpha_1} d(x, u).$$

Note that by adding $\alpha_2 H(Tx, Tu)$ the inequality in Definition 3.1 can be rewritten in the form

$$H(Tx, Tu) + \alpha_2 H(Tu, Tv) \leq d(x, u) + \alpha_2 H(Tx, Tu) - (1 - k)d(x, u). \tag{3.1}$$

From (3.1) we have for all $v \in Tu$,

$$\begin{aligned} H(Tx, Tu) + \alpha_2 H(Tu, Tv) &\leq \rho(x, u) - (1 - k)d(x, u) \\ &\leq \rho(x, u) - \frac{\alpha_1(1 - k)}{\alpha_1 + \alpha_2 k}\rho(x, u) \\ &= \frac{k}{\alpha_1 + \alpha_2 k}\rho(x, u). \end{aligned} \tag{3.2}$$

Consider $a > 1$ such that

$$k\left[\frac{1}{\alpha_1 + \alpha_2 k} + (a - 1)\frac{1}{\alpha_1}\right] < 1.$$

Since $a > 1$ we can choose $w \in Tu$ such that

$$d(u,w) \leq ad(u,Tu) \leq aH(Tx,Tu).$$

Choosing $w = v$ in (3.2) we obtain

$$\begin{aligned}\rho(u,w) &= d(u,w) + \alpha_2 H(Tu,Tw) \leq aH(Tx,Tu) + \alpha_2 H(Tu,Tw)\\ &= H(Tx,Tu) + \alpha_2 H(Tu,Tw) + (a-1)H(Tx,Tu)\\ &\leq \frac{k}{\alpha_1 + \alpha_2 k}\rho(x,u) + (a-1)\frac{k}{\alpha_1}\rho(x,u)\\ &= k\left[\frac{1}{\alpha_1+\alpha_2 k} + (a-1)\frac{1}{\alpha_1}\right]\rho(x,u).\end{aligned}$$

Therefore, T is ρ-MIC. □

Theorem 3.3. *Let (X,d) be a complete metric space. Assume that $T : C \to \mathcal{P}_{cl,b}(X)$ is a MMIC such that the graph of the mapping ϕ_T is demiclosed. Then, T has a fixed point.*

Proof. Consider the metric ρ defined above for which T is MIC. According to the proof of Theorem 2.5, we can find a sequence $\{x_n\}$ of X and a point $x \in X$ such that $x_{n+1} \in Tx_n$ and $\lim_n \rho(x_n,x) = 0$. By definition of the metric ρ, for each $n \in \mathbb{N}$,

$$\rho(x_n,x) = d(x_n,x) + \alpha_2 H(Tx_n,Tx).$$

Hence, $\lim_n d(x_n,x) = 0$ and $\lim_n H(Tx_n,Tx) = 0$. Since $d(x_{n+1},Tx) \leq H(Tx_n,Tx)$, it follows that $x \in Tx$. □

Remark 3.4. It is worth noting that in the two last classes of mappings above we have obtained fixed point without using condition (E).

Acknowledgement. The first and second authors are partially supported by MINECO, Grant MTM2015-65242-C2-1-P and Andalusian Regional Government Grant FQM-127. We are indebted to M. A. Japón and the anonymous referee for their valuable suggestions to improve this manuscript. This research was done while the third and fourth authors were visiting the University of Seville. They are grateful to the Department of Mathematical Analysis for its hospitality.

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Received: July 13, 2018; Accepted: March 19, 2019.