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A COMMON MAXIMAL ELEMENT OF CONDENSING MAPPINGS

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Abstract. In this paper, we establish a general existence theorem of maximal elements of condensing mappings in the product $X := \prod_{\alpha \in I} X_\alpha$ of noncompact *l.c.*-spaces. As an application, we prove that a family of $\mathcal{L}_{\pi_\alpha}$-majorized Q_α-condensing mappings $T_\alpha : X \longrightarrow 2^{X_\alpha}$ admit a common maximal element under the mild condition that each $\{x \mid T_\alpha(x) \neq \emptyset\}$ is compactly open.

Key Words and Phrases: *l.c.*-space, Q_α-condensing mapping, maximal element, $\mathcal{L}_\theta$-majorized.
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1. Introduction and preliminary

In the last fifty years, the classical Arrow-Debreu result on the existence of Walrasian equilibria has been generalized in many directions. The equilibrium existence theory for various models have been extensively studied by many authors, and maximal element existence theorems are frequently used as the main tool for proving the existence of equilibria, e.g. see [8, 12] and references therein. For a set-valued mapping $T : X \longrightarrow 2^Y$, we say that a point $x \in X$ is a ***maximal element*** of T, if $T(x) = \emptyset$. The purpose of this paper is to apply two central theorems [14] below to obtain more general maximal element existence theorems for condensing mappings.

Theorem A. *Let $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)_{\alpha \in I}$ be a family of l.c.-spaces with precompact polytopes, $X := \prod_{\alpha \in I} X_\alpha$, and $T_\alpha : X \longrightarrow 2^{X_\alpha}$ be Q_α-condensing. Then there exists nonempty compact H-convex subset $K := \prod_{\alpha \in I} K_\alpha$ of X such that $T_\alpha(K) \subset K_\alpha$.*

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Theorem B. *Let $(X_\alpha,\mathcal{U}_\alpha,\mathcal{B}_\alpha)_{\alpha\in I}$ be a family of l.c.-spaces with precompact polytopes, $X := \prod_{\alpha\in I} X_\alpha$. For each $\alpha \in I$, let $T_\alpha : X \longrightarrow 2^{X_\alpha}$ be Q_α-condensing such that*

(1) *for each $x \in X$, $T_\alpha(x)$ is a nonempty H-convex subset of X_α,*
(2) *for each $x_\alpha \in X_\alpha$, $T_\alpha^{-1}(x_\alpha)$ contains a compactly open subset O_{x_α} of X such that $\bigcup_{x_\alpha\in X_\alpha} O_{x_\alpha} = X$ (where O_{x_α} may be empty for some x_α).*

Then $T := \prod_{\alpha\in I} T_\alpha$ has a fixed point.

Theorem A generalizes Mehta's Theorem [9] and Kim's result [7] to the general setting of *l.c.*-spaces. Whereas Theorem B is a variant result from Tarafdar's fixed point theorem [11].

We digest and list some definitions and notations as follows. Throughout this paper, all topological spaces are assumed to be Hausdorff. For a nonempty set X, we denote the set of all subsets of X by 2^X, and the set of all nonempty finite subsets of X by $\langle X\rangle$. In addition, for any subset C of a topological space X, the closure of C is denoted by $\mathrm{cl}_X C$.

An ***H-space*** is a topological space X, together with a family $\{\mathit{\Gamma}_D\}$ of some nonempty contractible subsets of a topological space X indexed by $D \in \langle X\rangle$ such that $\mathit{\Gamma}_D \subset \mathit{\Gamma}_{D'}$ whenever $D \subset D'$. The notion of H-space was introduced in 1988 by Bardaro and Ceppitelli [1]. Since then, there have appeared numerous applications and generalizations in the literature [3, 4, 10, 11, 12, 13]. Given an H-space $(X, \{\mathit{\Gamma}_D\})$, a nonempty subset C of X is said to be ***H-convex*** if $\mathit{\Gamma}_D \subset C$ for all $D \in \langle C\rangle$. For a nonempty subset C of X, we define the ***H-convex hull*** of C as $H\text{-}coC := \bigcap\{K \mid K \text{ is } H\text{-convex in } X \text{ and } C \subset K\}$. Moreover, for any $D \in \langle X\rangle$, H-*coD* is called a ***polytope***. We say that X is an ***H-space with precompact polytopes***, if any polytope of X is precompact. For example, a locally convex topological vector space X is an H-space with precompact polytopes, by setting $\mathit{\Gamma}_D = coD$ for all $D \in \langle X\rangle$.

An H-space $(X, \{\mathit{\Gamma}_D\})$ is called an ***l.c.-space*** (see [5]), if X is an uniform space whose topology is induced by its uniformity $\mathcal{U}$, and there is a base $\mathcal{B}$ consisting of symmetric entourages in $\mathcal{U}$ such that for each $V \in \mathcal{B}$, the set $V(E) := \{y \in X \mid (x, y) \in V \text{ for some } x \in E\}$ is H-convex whenever E is H-convex. We shall use the notation $(X,\mathcal{U},\mathcal{B})$ to stand for an *l.c.*-space. For details of uniform spaces, we refer to [6].

In an *l.c.*-space $(X,\mathcal{U},\mathcal{B})$, we define the ***measure of precompactness*** of a subset A in X by

$$Q(A) := \{V \in \mathcal{B} \mid A \subset \mathrm{cl}_X V(K) \text{ for some precompact set } K \text{ of } X\}.$$

Let $(X_\alpha,\mathcal{U}_\alpha,\mathcal{B}_\alpha)_{\alpha\in I}$ be a family of *l.c.*-spaces with precompact polytopes, where I is a finite or infinite index set, and let $X = \prod_{\alpha\in I} X_\alpha$ be the product H-space. For any nonempty subset E_β of X_β and arbitrarily fixed $\alpha \in I$, we define

$$\prod_{\beta\neq\alpha,\beta\in I} E_\beta \otimes E_\alpha := \left\{ x \in X \mid x = (y_\alpha, x_\alpha), y_\alpha \in \prod_{\beta\neq\alpha,\beta\in I} E_\beta \text{ and } x_\alpha \in E_\alpha \right\}.$$

For each $\alpha \in I$, let π_α be the projection of X onto X_α and Q_α be a measure of precompactness in X_α. We say that a set-valued mapping $T_\alpha : X \longrightarrow 2^{X_\alpha}$ is ***Q_α-condensing*** if $Q_\alpha(\pi_\alpha(C)) \subsetneq Q_\alpha(T_\alpha(C))$ for every C satisfying $\pi_\alpha(C)$ is a nonprecompact subset of X_α. It is easy to check that $T_\alpha : X \longrightarrow 2^{X_\alpha}$ is Q_α-condensing whenever X_α is compact. Also, in case $I = \{1\}$, the projection π_α is the identity on X. Thus, the above definition reduces to the usual Q-condensing mapping $T : X \longrightarrow 2^X$; see for example [7, 9].

2. Main results

Let $(X_\alpha, \Gamma^\alpha_{D_\alpha})_{\alpha\in I}$ be a family of H-spaces, where I is a finite or infinite index set. Tarafder [10] has shown that the product space $X = \prod_{\alpha\in I} X_\alpha$ with product topology is also an H-space, together with the family $\{\Gamma_D \mid D \in \langle X\rangle\}$, which is defined by

$$\Gamma_D = \prod_{\alpha\in I} \Gamma^\alpha_{\pi_\alpha(D)}.$$

Further, the product of H-convex sets is also H-convex. We begin with establishing some fundamental lemmas, which will be used to prove our main theorem.

Lemma 2.1. *The projection of an H-convex set in the product H-space $X = \prod_{\alpha\in I} X_\alpha$ is H-convex.*

Proof. Let K be an H-convex subset of X and $K_\alpha = \pi_\alpha(K)$ be the projection of K onto X_α. For any finite subset D_α of K_α, we have a correspondent finite subset D of K such that $D_\alpha = \pi_\alpha(D)$. Since $\Gamma_D \subset K$, it follows that

$$\Gamma^\alpha_{D_\alpha} = \pi_\alpha(\Gamma_D) \subset \pi_\alpha(K) = K_\alpha.$$

Thus, K_α is H-convex. □

Lemma 2.2. *Let $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)_{\alpha\in I}$ be a family of l.c.-spaces,*

$$X = \prod_{\alpha\in I} X_\alpha, \text{ and } \mathcal{U} = \prod_{\alpha\in I} \mathcal{U}_\alpha$$

be the product uniformity on X. Then there is a base $\mathcal{B}$ such that $(X, \mathcal{U}, \mathcal{B})$ forms an l.c.-space.

Proof. It suffices to show that the product uniformity $\mathcal{U}$ has a base $\mathcal{B}$ consisting of symmetric entourages such that $V(E)$ is H-convex for each $V \in \mathcal{B}$ and for each H-convex set E. First, we define

$$\mathcal{S} := \{ \{(x, y) \in X \times X \mid (x_\alpha, y_\alpha) \in V_\alpha\} \mid \alpha \in I, V_\alpha \in \mathcal{B}_\alpha\}.$$

It is easy to check that $\mathcal{S}$ is a subbase of $\mathcal{U}$. Next, we let $\mathcal{B}$ be the base generated by $\mathcal{S}$; that is,

$$\mathcal{B} := \{V \mid V = \cap_{j=1}^{n} V^j, n \in \mathbb{N}\ ; V^j \in \mathcal{S}, j = 1, 2, \ldots, n\}.$$

Since each V^j is of the form $V^j = \{(x, y) \in X \times X \mid (x_{\alpha_j}, y_{\alpha_j}) \in V_{\alpha_j}\}$ for some $\alpha_j \in I$ and $V_{\alpha_j} \in \mathcal{B}_{\alpha_j}$, we obtain

$$V^j(E) = \prod_{\beta \in I, \beta \neq \alpha_j} X_\beta \otimes V_{\alpha_j}(\pi_{\alpha_j}(E)).$$

Notice that for each $V_\alpha \in \mathcal{B}_\alpha$, $V_\alpha(\pi_\alpha(E))$ is H-convex by Lemma 2.1. It follows that each $V^j(E)$ is H-convex. Therefore,

$$V(E) = \prod_{\beta \in I \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_n\}} X_\beta \otimes \prod_{j=1}^{n} V_{\alpha_j}(\pi_{\alpha_j}(E))$$

is H-convex. This yields that $(X, \mathcal{U}, \mathcal{B})$ forms an *l.c.*-space. □

Let X be a topological space, Y an H-space, $T : X \longrightarrow 2^Y$ a set-valued mapping, and $\theta : X \longrightarrow Y$ be a single-valued map.

(1) T is said to be ***of class*** $\mathcal{L}_\theta$, if
  (a) for each $x \in X$, $\theta(x) \notin H\text{-}coT(x)$,
  (b) for each $y \in Y$, $T^{-1}(y)$ is compactly open in X.
(2) A set-valued mapping $T_x : X \longrightarrow 2^Y$ is an $\mathcal{L}_\theta$***-majorant of T at x***, if there exists an open neighborhood N_x of x in X such that
  (a) for each $z \in N_x$, $T(z) \subset T_x(z)$ and $\theta(z) \notin H\text{-}coT_x(z)$,
  (b) for each $y \in Y$, $T_x^{-1}(y)$ is compactly open in X.
(3) T is said to be $\mathcal{L}_\theta$***-majorized***, if for each $x \in X$ with $T(x) \neq \emptyset$, there exists an $\mathcal{L}_\theta$-majorant of T at x.

In case $\theta : X \longrightarrow X$ is the identity map on X, with $Y = X$, all notations above are simplified to be *of class* $\mathcal{L}$, $\mathcal{L}$*-majorant*, and $\mathcal{L}$*-majorized*, respectively.

In [2], Ding et al. have shown the following lemma in the setting of locally convex topological vector spaces. Following the proof of Lemma 1 in [2], we have an extension in H-spaces as follows.

Lemma 2.3. *Let X be a regular topological space, Y an H-space, $\theta : X \longrightarrow Y$ a single-valued map, and $T : X \longrightarrow 2^Y$ be an $\mathcal{L}_\theta$-majorized mapping. If every open subset of X containing the set $\{x \in X \mid T(x) \neq \emptyset\}$ is paracompact, then there exists a set-valued mapping $S : X \longrightarrow 2^Y$ of class $\mathcal{L}_\theta$ such that $T(x) \subset S(x)$ for all $x \in X$.*

Proof. Let $R = \{x \in X \mid T(x) \neq \emptyset\}$. Since T is $\mathcal{L}_\theta$-majorized, for each $x \in R$, there exist an open neighborhood N_x of x in X and a set-valued mapping $T_x : X \longrightarrow 2^Y$ such that

(a) for each $z \in N_x$, $T(z) \subset T_x(z)$ and $\theta(z) \notin H\text{-}coT_x(z)$,
(b) for each $y \in Y$, $T_x^{-1}(y)$ is compactly open in X.

Since X is regular, for each $x \in R$, there exists an open neighborhood G_x of x in X such that $\mathrm{cl}_X G_x \subset N_x$. Let $G = \bigcup_{x \in R} G_x$. Then G is an open subset of X containing

R so that G is paracompact by assumption. Hence the open cover $\{G_x\}$ of G has a locally finite subcover $\{G'_x\}$. For each $x \in R$, we define $S_x : G \longrightarrow 2^Y$ by

$$S_x(z) = \begin{cases} T_x(z), & \text{if } z \in G \cap \mathrm{cl}_X G'_x, \\ Y, & \text{if } z \in G \setminus \mathrm{cl}_X G'_x. \end{cases}$$

We claim that for each $y \in Y$, $S_x^{-1}(y) = \{z \in G \mid y \in S_x(z)\}$ is compactly open in X. Indeed,

$$\begin{aligned} S_x^{-1}(y) &= \{z \in G \cap \mathrm{cl}_X G'_x \mid y \in S_x(z)\} \cup \{z \in G \setminus \mathrm{cl}_X G'_x \mid y \in S_x(z)\} \\ &= \{z \in G \cap \mathrm{cl}_X G'_x \mid y \in T_x(z)\} \cup \{z \in G \setminus \mathrm{cl}_X G'_x \mid y \in Y\} \\ &= \big(G \cap \mathrm{cl}_X G'_x \cap T_x^{-1}(y)\big) \cup (G \setminus \mathrm{cl}_X G'_x) \\ &= \big(G \cap T_x^{-1}(y)\big) \cup (G \setminus \mathrm{cl}_X G'_x)\,. \end{aligned}$$

It follows that for each nonempty compact subset K of X,

$$S_x^{-1}(y) \cap K = \big(G \cap T_x^{-1}(y) \cap K\big) \cup ((G \setminus \mathrm{cl}_X G'_x) \cap K)$$

is open in K; i.e., $S_x^{-1}(y)$ is compactly open in X for all $y \in Y$.

Next, we define $S : X \longrightarrow 2^Y$ by

$$S(z) = \begin{cases} \bigcap_{x \in R} S_x(z) & , \text{ if } z \in G, \\ \emptyset & , \text{ if } z \in X \setminus G. \end{cases}$$

We now prove that S is a set-valued mapping of class $\mathcal{L}_\theta$. Indeed, if $z \in X \setminus G$, then $S(z) = \emptyset$ so that $\theta(z) \notin H\text{-}coS(z)$. If $z \in G$, then $z \in G \cap \mathrm{cl}_X G'_x$ for some $x \in R$ so that $S_x(z) = T_x(z)$ and hence $S(z) \subset T_x(z)$. Since $\theta(z) \notin H\text{-}coT_x(z)$, it follows that $\theta(z) \notin H\text{-}coS(z)$. Thus, $\theta(z) \notin H\text{-}coS(z)$ for all $z \in X$.

On the other hand, for any $y \in Y$ with $S^{-1}(y) \neq \emptyset$, we let K be any compact subset of X, and fix any $u \in S^{-1}(y) \cap K$. Note that

$$S^{-1}(y) = \{z \in X \mid y \in S(z)\} = \{z \in G \mid y \in S(z)\}.$$

Since $\{G'_x\}$ is a locally finite subcover, there exists an open neighborhood M_u of u in G such that

$$\{x \in R \mid M_u \cap G'_x \neq \emptyset\} = \{x_1^{(u)}, x_2^{(u)}, \ldots, x_{n(u)}^{(u)}\}.$$

Note that for each $x \in R$, with $x \notin \{x_1^{(u)}, x_2^{(u)}, \ldots, x_{n(u)}^{(u)}\}$, we have

$$M_u \cap \mathrm{cl}_X G'_x = M_u \cap G'_x = \emptyset,$$

and hence $S_x(z) = Y$ for all $z \in M_u$. Thus we have

$$S(z) = \bigcap_{x \in R} S_x(z) = \bigcap_{i=1}^{n(u)} S_{x_i^{(u)}}(z)$$

for all $z \in M_u$. It follows that

$$\begin{aligned} S^{-1}(y) = \{z \in X \mid y \in S(z)\} &= \left\{ z \in X \mid y \in \bigcap_{x \in R} S_x(z) \right\} \\ &\supset \left\{ z \in M_u \mid y \in \bigcap_{x \in R} S_x(z) \right\} \\ &= \left\{ z \in M_u \mid y \in \bigcap_{i=1}^{n(u)} S_{x_i^{(u)}}(z) \right\} \\ &= M_u \cap \left[\bigcap_{i=1}^{n(u)} (S_{x_i^{(u)}})^{-1}(y) \right]. \end{aligned}$$

Since each $(S_{x_i^{(u)}})^{-1}(y)$ is a compactly open set in X, the set

$$M'_u := M_u \cap \left(\bigcap_{i=1}^{n(u)} (S_{x_i^{(u)}})^{-1}(y) \right) \cap K$$

is an open neighborhood of u in K such that $M'_u \subset S^{-1}(y) \cap K$. This shows that for each $y \in Y$, $S^{-1}(y)$ is compactly open in X. Therefore, $S : X \longrightarrow 2^Y$ is a set-valued mapping of class $\mathcal{L}_\theta$.

It remains to show that $T(w) \subset S(w)$ for all $w \in X$. For any $w \in X$ with $T(w) \neq \emptyset$, we have $w \in G$. Let $x \in R$. If $w \in G \setminus \mathrm{cl}_X G'_x$, then $S_x(w) = Y \supset T(w)$, and if $w \in G \cap \mathrm{cl}_X G'_x$, we have $w \in \mathrm{cl}_X G'_x \subset \mathrm{cl}_X G_x \subset N_x$ so that $T(w) \subset T_x(w) = S_x(w)$. It follows that $T(w) \subset S_x(w)$ for all $x \in R$, and hence $T(w) \subset \bigcap_{x \in B} S_x(z) = S(w)$.

□

We are ready to establish our main result.

Theorem 2.4. *Let* $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)_{\alpha \in I}$ *be a family of l.c.-spaces with precompact polytopes, and* $X := \prod_{\alpha \in I} X_\alpha$. *If for each* $\alpha \in I$, $T_\alpha : X \longrightarrow 2^{X_\alpha}$ *is an* $\mathcal{L}_{\pi_\alpha}$*-majorized* Q_α*-condensing mapping, then there exists* $\widehat{x} \in X$ *such that* $T(\widehat{x}) := \prod_{\alpha \in I} T_\alpha(\widehat{x}) = \emptyset$.

Proof. Suppose the contrary, i.e., for any $\alpha \in I$, $T_\alpha(x) \neq \emptyset$ for all $x \in X$. Since for each $\alpha \in I$, $T_\alpha : X \longrightarrow 2^{X_\alpha}$ is Q_α-condensing, by Theorem A, there exists a nonempty compact H-convex subset $K := \prod_{\alpha \in I} K_\alpha$ of X such that $T_\alpha(K) \subset K_\alpha$. Then the set $\{x \in K \mid T_\alpha(x) \neq \emptyset\} = K$ is compact. Note that a compact space is normal, so X is regular. By Lemma 2.3, there exists $S_\alpha : K \longrightarrow 2^{K_\alpha}$ of class $\mathcal{L}_{\pi_\alpha}$ such that $T_\alpha(x) \subset S_\alpha(x)$ for each $x \in K$. Define $S^*_\alpha : K \longrightarrow 2^{K_\alpha}$ by

$$S^*_\alpha(x) := H\text{-}coS_\alpha(x) \text{ for all } x \in X.$$

Then, by Theorem B, there exists $\bar{x} \in K$ such that $\bar{x}_\alpha \in S^*_\alpha(\bar{x}) = H\text{-}coS_\alpha(\bar{x})$, which leads to a contradiction, since S_α is of class $\mathcal{L}_{\pi_\alpha}$. This completes the proof. □

As an immediate result, we have the following consequence.

Corollary 2.5. *If* $(X,\mathcal{U},\mathcal{B})$ *is an l.c.-space with precompact polytopes, and* $T : X \longrightarrow 2^X$ *is an* $\mathcal{L}$*-majorized* Q*-condensing mapping, then there exists a maximal element of* T.

Theorem 2.4 implies that one of T_α's has a maximal element. Furthermore, the following theorem provides a sufficient condition of existence of common maximal elements.

Theorem 2.6. *Let* $(X_\alpha,\mathcal{U}_\alpha,\mathcal{B}_\alpha)_{\alpha\in I}$ *be a family of l.c.-spaces with precompact polytopes,* $X := \prod_{\alpha\in I} X_\alpha$*, and* $T_\alpha : X \longrightarrow 2^{X_\alpha}$ *be* $\mathcal{L}_{\pi_\alpha}$*-majorized* Q_α*-condensing. If for each* $\alpha \in I$*, the set* $\{x \in X \mid T_\alpha(x) \neq \emptyset\}$ *is compactly open in* X*, then there exists* $\widehat{x} \in X$ *such that* $T_\alpha(\widehat{x}) = \emptyset$ *for all* $\alpha \in I$*; that is,* $\widehat{x}$ *is a common maximal element of* $\{\, T_\alpha \mid \alpha \in I\}$.

Proof. Since each $T_\alpha : X \longrightarrow 2^{X_\alpha}$ is a Q_α-condensing mapping, by Theorem A, there exists a nonempty compact H-convex subset $K := \prod_{\alpha\in I} K_\alpha$ of X such that $T_\alpha(K) \subset K_\alpha$. For each $x \in K$, we let $I(x) := \{\alpha \in I \mid T_\alpha(x) \neq \emptyset\}$, and define

$$T'_\alpha(x) = \prod_{\beta\in I,\beta\neq\alpha} K_\beta \otimes T_\alpha(x)$$

for each $\alpha \in I(x)$. Thus, we can define a set-valued mapping $T : K \longrightarrow 2^K$ by

$$T(x) := \begin{cases} \bigcap_{\alpha\in I(x)} T'_\alpha(x) & \text{, if } I(x) \neq \emptyset, \\ \emptyset & \text{, if } I(x) = \emptyset, \end{cases}$$

Then for each $x \in K$ with $I(x) \neq \emptyset$, $T(x) \neq \emptyset$.

Let $x \in K$ be such that $T(x) \neq \emptyset$. Then $T'_\alpha(x) \neq \emptyset$ for all $\alpha \in I(x)$. Since each T_α is $\mathcal{L}_{\pi_\alpha}$-majorized, for a fixed $\alpha \in I(x)$, there exist an open neighborhood N_x of x in K and a set-valued mapping $S_\alpha : K \longrightarrow 2^{K_\alpha}$ such that

(a) for each $z \in N_x$, $T_\alpha(z) \subset S_\alpha(z)$ and $z_\alpha \notin H\text{-}coS_\alpha(z)$,
(b) for each $y_\alpha \in K_\alpha$, $S_\alpha^{-1}(y_\alpha)$ is (compactly) open in K.

Since $\{z \in X \mid T_\alpha(z) \neq \emptyset\}$ is compactly open in X, the set $\{z \in K \mid T_\alpha(z) \neq \emptyset\}$ is open in K. Thus, we may assume $N_x \subset \{z \in K \mid T_\alpha(z) \neq \emptyset\}$, so that $T_\alpha(z) \neq \emptyset$ for all $z \in N_x$. Next, we define $S_x : K \longrightarrow 2^K$ by

$$S_x(z) := \prod_{\beta\in I,\beta\neq\alpha} K_\beta \otimes S_\alpha(z) \text{ for all } z \in K.$$

We claim that S_x is an $\mathcal{L}$-majorant of T at x. Indeed, for all $z \in N_x$, $T_\alpha(z) \neq \emptyset$, which implies $\alpha \in I(z)$. By (a), we obtain

$$T(z) = \bigcap_{\beta\in I(z)} T'_\beta(z) \subset T'_\alpha(z) \subset S_x(z) \quad\text{and}\quad z \notin H\text{-}coS_x(z).$$

On the other hand, for each $y \in K$, we have

$$S_x^{-1}(y) = \{z \in K \mid y \in S_x(z)\} = \{z \in K \mid y_\alpha \in S_\alpha(z)\} = S_\alpha^{-1}(y_\alpha).$$

It follows that $S_x^{-1}(y)$ is (compactly) open in K.

Therefore, S_x is an $\mathcal{L}$-majorant of T at x. This shows that T is $\mathcal{L}$-majorized. By Corollary 2.5, there exists a point $\widehat{x} \in K$ such that $T(\widehat{x}) = \emptyset$. By the definition of T, we have $I(\widehat{x}) = \emptyset$ and hence $T_\alpha(\widehat{x}) = \emptyset$ for all $\alpha \in I$. This completes the proof. □

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