# A NEW ITERATIVE ALGORITHM FOR A GENERALIZED MIXED EQUILIBRIUM PROBLEM AND A COUNTABLE FAMILY OF NONEXPANSIVE-TYPE MAPS WITH APPLICATIONS 

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#### Abstract

Let $C$ be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space with dual space $E^{*}$. In this paper, a new iterative algorithm of Krasnoselskiitype is constructed and used to approximate a common element of a generalized mixed equilibrium problem and a common fixed point of a countable family of generalized- $J$-nonexpansive maps. Applications of our theorem, in the case of real Hilbert spaces, complement and extend the results of Peng and Yao, (Taiwanese Journal of Mathematics Vol. 12, No. 6, pp. 1401-1432, September 2008); Nakajo and Takahashi, (J. Math. Anal. Appl. 273 (2003) 372-379); Martinez-Yanes and Xu, (Nonlinear Anal., 64 (2006), 2400-2411); Qin and Su, (J. Syst. Sci. and Complexity 21(2008) 474-482) Key Words and Phrases: Generalized mixed equilibrium problem, nonexpansive-type maps, monotone maps, strong convergence. 2010 Mathematics Subject Classification: 47H09, 47H10, 47J25 47J05, 47J20.


## 1. Introduction

Let $E$ be a uniformly convex and uniformly smooth real Banach space with dual space $E^{*}$. Let $C$ be a nonempty closed and convex subset of $E$ such that $J C$ is closed and convex where $J: E \rightarrow E^{*}$ is the normalized duality map on $E$. Let $\varphi$ be a map from $J C$ to $\mathbb{R}, f$ be a bifunction from $J C \times J C$ to $\mathbb{R}$ and $A$ be a nonlinear map from $C$ to $E^{*}$. The generalized mixed equilibrium problem is to find an element $v \in C$ such that

$$
\begin{equation*}
f(J v, J y)+\varphi(J y)-\varphi(J v)+\langle A v, y-v\rangle \geq 0 \forall y \in C \tag{1.1}
\end{equation*}
$$

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The set of solutions of the generalized mixed equilibrium problem is given by:

$$
G M E P(f, A, \varphi)=\{v \in C: f(J v, J y)+\varphi(J y)-\varphi(J v)+\langle A v, y-v\rangle \geq 0 \forall y \in C\}
$$

It is well known that the class of generalized mixed equilibrium problems contain, as special cases, numerous important classes of nonlinear problems such as variational inequality problems, optimization problems, equilibrium problems, and so on (see e.g., Browder et al. [5], Ezeora [11], Onjai-Uea and Kumam [23] and the references contained in them).

A map $T: C \rightarrow E$ is called $L$-Lipschitz if $\|T x-T y\|_{E} \leq L\|x-y\|_{E} \forall x, y$ in $C$, where $L \geq 0$. If $L=1$, then the map $T$ is called a nonexpansive map. We denote the fixed point set of $T$ by $F(T)$. Also, $T: C \times C \rightarrow C$ is called bivariate nonxpansive (see e.g., Suanoon et al., [28]) if

$$
\|T(x, y)-T(u, v)\| \leq \frac{1}{2}(\|x-u\|+\|y-v\|), \forall x, y, u, v \in C
$$

For several years, many authors have studied the problem of obtaining a common element in the set of solutions of equilibrium problems and the set of fixed points of nonexpansive maps from $E$ to $E$ in the setting of real Banach spaces. In 2008, Peng and Yao [24] studied the problem of obtaining a common element in the set of solutions of a generalized mixed equilibrium problem and a set of fixed points of a nonexpansive map in a real Hilbert space, $H$. To extend this result to classes of nonlinear maps more general than the class of nonexpansive maps, the concept of generalized nonexpansive maps and relatively nonexpansive maps have been introduced and studied by several authors (see e.g., Aoyama et al. [4], Chidume et al. [8], Matsushita et al. ([18]-[19]), Qin et al. ([26],[25]) and the references contained in them). Recently, Klin-earn et al. [14] studied a new and interesting monotone hybrid iterative method for generalized nonexpansive maps in a uniformly convex and uniformly smooth real Banach space.

Let $E$ be a real normed space with dual space $E^{*}$. A map $A$ from $E$ to $E^{*}$ is called monotone if $\langle A x-A y, x-y\rangle \geq 0 \forall x, y \in E$. Consider, for example, the following: Let $h: E \rightarrow \mathbb{R}$ be a convex functional. The subdifferential of $h, \partial h: E \rightarrow 2^{E^{*}}$, is given for each $x \in E$ by $\partial h(x)=\left\{x^{*} \in E^{*}:\left\langle y-x, x^{*}\right\rangle \leq h(y)-h(x) \forall y \in E\right\}$. It is easy to see that $\partial h$ is a monotone map on $E$ and that $0 \in \partial h(x)$ if and only if $x$ minimizes $h$. Setting $\partial h=A$, it follows that solving the inclusion $0 \in A x$, in this case, is searching for a minimizer of $h$.
A map $A$ from $E$ to $E$ is called accretive if $\langle A x-A y, j(x-y)\rangle \geq 0 \forall x, y \in E$ and for some $j(x-y) \in J(x-y)$. Numerous authors have studied extensively the class of accretive operators. For solving the equation $A x=0$, where $A: E \rightarrow E$ is an accretive operator, Browder introduced a map $T: E \rightarrow E$ given by $T:=I-A$, where $I$ is the identity map on $E$ and called it pseudocontractive. It is clear that solutions of $A x=0$, in this case, correspond to fixed points of $T$. Consequently, approximating zeros of accretive operators has been done by approximating fixed points of pseudocontractive maps. This fixed point technique obviously is not applicable in the case where $A$ : $E \rightarrow E^{*}$ is a monotone map.

Motivated by the need to develop a fixed point technique for the equation $A x=0$ when $A$ is monotone, analogous to the fixed point theory for $A x=0$ when $A$ is accretive, a new notion of fixed points for maps from $E$ to $E^{*}$ called $J$-fixed points has recently been introduced and studied (see e.g., Zegeye [32], Liu [16], Chidume and Idu [7], Chidume et al. [9] and the references contained in them). This notion turns out to be very useful and applicable. For example, Chidume and Idu [7] introduced the concept of $J$-pseudocontractive maps and proved a strong convergence theorem for approximating $J$-fixed points of a $J$-pseudocontractive map. As an application of this theorem, they proved the following strong convergence theorem for approximating a zero of an $m$-accretive operator.
Theorem 1.1. (Chidume and Idu [7]) Let E be a uniformly smooth real Banach space with modulus of smoothness $\rho_{E}$, and let $A: E \rightarrow 2^{E}$ be a multi-valued bounded $m$-accretive operator with $D(A)=E$ such that the inclusion $0 \in A u$ has a solution. For arbitrary $x_{1} \in E$, define a sequence $\left\{x_{n}\right\}$ by:

$$
x_{n+1}=x_{n}-\lambda_{n} u_{n}-\lambda_{n} \theta_{n}\left(x_{n}-x_{1}\right), u_{n} \in A x_{n}, n \geq 1
$$

where $\left\{\lambda_{n}\right\}$ and $\left\{\theta_{n}\right\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim \theta_{n}=0,\left\{\theta_{n}\right\}$ is decreasing,
(ii) $\sum \lambda_{n} \theta_{n}=\infty, \sum \rho_{E}\left(\lambda_{n} M_{1}\right)<\infty$ for some constant $M_{1}>1$ and
(iii) $\lim \frac{\left[\frac{\theta_{n-1}}{\theta_{n}}-1\right]}{\lambda_{n} \theta_{n}}=0$.

There exists a constant $\gamma_{0}>0$ such that

$$
\frac{\rho_{E}\left(\lambda_{n} M_{1}\right)}{\lambda_{n}} \leq \gamma_{0} \theta_{n}
$$

Then, the sequence $\left\{x_{n}\right\}$ converges strongly to a zero of $A$.
Motivated by the works of these authors, it is our purpose in this paper to continue the study of $J$-fixed points and some of its applications. Here, we study a new iterative algorithm of Krasnoselskii-type and prove a strong convergence theorem for obtaining a common element between solutions of a generalized mixed equilibrium problem and common fixed points of a countable family of generalized- $J$-nonexpansive maps in a uniformly smooth and uniformly convex real Banach space. In the special case of a real Hilbert space, our theorem complements and extends the results of Pen and Yao [24], Nakajo and Takahashi [21], Martinez-Yanes and Xu [17], Qin and Su [27] and a host of other recent results.

## 2. Preliminaries

Let $E$ be a real normed linear space with dual space $E^{*}$. A map $J: E \rightarrow 2^{E^{*}}$ defined by

$$
J(x):=\left\{x^{*} \in E^{*}:\left\langle x, x^{*}\right\rangle=\|x\|^{2}=\left\|x^{*}\right\|^{2}\right\}
$$

is called the normalized duality map on $E$. For some properties of $J$ relevant to this work (see e.g., Chidume [6], Cioranescu [10] and the references contained in them).

In the sequel, we shall need the following definition and results. Let $E$ be a smooth real Banach space with dual space $E^{*}$. Consider a map $\phi: E \times E \rightarrow \mathbb{R}$ defined by

$$
\phi(x, y)=\|x\|^{2}-2\langle x, J y\rangle+\|y\|^{2} \text { for all } x, y \in E
$$

where $J$ is the normalized duality map from $E$ into $2^{E^{*}}$ will play a central role in the sequel. It was introduced by Alber and was first studied by Alber [1], Alber and Guerre-Delabriere [2], Kamimura and Takahashi [13].
Remark 1. If, in addition, $E$ is strictly convex, then the duality map $J$ is one-to-one. Hence, we have that

$$
\begin{equation*}
\phi(x, y)=0 \Longleftrightarrow x=y \tag{2.1}
\end{equation*}
$$

If $E=H$, a real Hilbert space, then we have that $\phi(x, y)=\|x-y\|^{2}$ for all $x, y \in H$. It is obvious from the definition of $\phi$ that

$$
(\|x\|-\|y\|)^{2} \leq \phi(x, y) \leq(\|x\|+\|y\|)^{2} \forall x, y \in E
$$

Definition 2.1. Let $C$ be a nonempty closed and convex subset of a real Banach space $E$ and $T$ be a map from $C$ to $E$. A map $T$ is called relatively nonexpansive if the fixed points set of $T$ denoted by $F(T) \neq \emptyset, \phi(p, T x) \leq \phi(p, x)$ for all $x \in C, p \in F(T)$ and $F(T)=\widehat{F}(T)$ where $\widehat{F}(T)$ is a set of asymptotic fixed points of $T$. A map $R$ from $E$ onto $C$ is said to be a retraction if $R^{2}=R$. A map $R$ is said to be sunny if $R(R x+t(x-R x))=R x$ for all $x \in E$ and $t \leq 0$. A nonempty closed subset $C$ of a smooth Banach space $E$ is said to be a sunny generalized nonexpansive retract of $E$ if there exists a sunny generalized nonexpansive retraction $R$ from $E$ onto $C$.
We now list some lemmas which will be used in the sequel.
Lemma 2.2. (Koshaka and Takahashi, [15]) Let $C$ be a nonempty closed and convex subset of a smooth, strictly convex and reflexive Banach space $E$. Then, the following are equivalent.
(i) $C$ is a sunny generalized nonexpansive retract of $E$;
(ii) $C$ is a generalized nonexpansive retract of $E$ and
(iii) $J C$ is closed and convex.

Lemma 2.3. (Ibaraki and Takahashi, [12]) Let $C$ be a nonempty closed and convex subset of a smooth and strictly convex Banach space $E$ such that there exists a sunny generalized nonexpansive retraction $R$ from $E$ onto $C$. Then, the following hold.
(i) $z=R x$ iff $\langle y-z, J z-J x\rangle \geq 0$ for all $y \in C$;
(ii) $\phi(x, R x)+\phi(R x, z) \leq \phi(x, z)$ for all $z \in C$.

Lemma 2.4. (Xu, [31]) Let $E$ be a uniformly convex real Banach space. Let $r>0$. Then, there exists a strictly increasing continuous and convex function $g:[0, \infty) \rightarrow$ $[0, \infty)$ such that $g(0)=0$ and the following inequality holds:
$\|\lambda x+(1-\lambda) y\|^{2} \leq \lambda\|x\|^{2}+(1-\lambda)\|y\|^{2}-\lambda((1-\lambda)) g(\|x-y\|)$ for all $x, y \in B_{r}(0)$,
where $B_{r}(0):=\{v \in E:\|v\| \leq r\}$ and $\lambda \in[0,1]$.

Lemma 2.5. (Kamimura and Takahashi, [13]) Let $E$ be a uniformly convex and uniformly smooth real Banach space and $\left\{x_{n}\right\},\left\{y_{n}\right\}$ be sequences in $E$ such that either $\left\{x_{n}\right\}$ or $\left\{y_{n}\right\}$ is bounded. If $\lim _{n \rightarrow \infty} \phi\left(x_{n}, y_{n}\right)=0$, then, $\lim _{n \rightarrow \infty}\left\|x_{n}-y_{n}\right\|=0$.
Lemma 2.6. (Ibaraki and Takahashi, [12]) Let $C$ be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from $E$ onto $C$ is uniquely determined.

Remark 2. Let $C$ be a nonempty closed subset of a smooth, strictly convex and reflexive real Banach space $E$ with dual space $E^{*}$ such that $J C$ is closed and convex. Let $\varphi$ be a lower semicontinuous and convex function from $J C$ to $\mathbb{R}$. Let $A$, a nonlinear map from $C$ to $E^{*}$ be continuous and monotone. For solving the generalized equilibrium problems, we assume that the bifunctional $f: J C \times J C \rightarrow \mathbb{R}$ satisfies the following conditions:
$\left(A_{1}\right) f\left(x^{*}, x^{*}\right)=0$ for all $x^{*} \in J C$,
$\left(A_{2}\right) \mathrm{f}$ is monotone, i.e. $f\left(x^{*}, y^{*}\right)+f\left(y^{*}, x^{*}\right) \leq 0$ for all $x^{*}, y^{*} \in J C$,
$\left(A_{3}\right) \limsup _{t \downarrow 0} f\left(x^{*}+t\left(z^{*}-x^{*}\right), y^{*}\right) \leq f\left(x^{*}, y^{*}\right)$ for all $x^{*}, y^{*}, z^{*} \in J C$,
$\left(A_{4}\right)$ for all $x^{*} \in J C, f\left(x^{*}, \cdot\right)$ is convex and lower semi continuous.
NST-Condition. (Klin-eam et al. [14]) Let $\left\{T_{n}\right\}$ and $\Gamma$ be two families of generalized nonexpansive maps from $C$ into $E$ such that $\bigcap_{n=1}^{\infty} F\left(T_{n}\right)=F(\Gamma) \neq \emptyset$, where $F\left(T_{n}\right)$ is the set of fixed points of $T_{n}$ and $F(\Gamma)$ is the set of fixed points of $\Gamma$. A sequence $\left\{T_{n}\right\}$ from $C$ to $E$ is said to satisfy the NST-condition with $\Gamma$ if for each bounded sequence $\left\{x_{n}\right\} \subset C, \lim _{n \rightarrow \infty}\left\|x_{n}-T_{n} x_{n}\right\|=0 \Longrightarrow \lim _{n \rightarrow \infty}\left\|x_{n}-T x_{n}\right\|=0 \forall T \in \Gamma$.
Remark 3. In particular, if $T=\{T\}$, i.e., $T$ consists of one mapping $T$, then $\left\{T_{n}\right\}$ is said to satisfy the NST-condition with $T$. It is obvious that $\left\{T_{n}\right\}$ with $T_{n}=T$ for all $n \in \mathbb{N}$ satisfies NST-condition with $\Gamma=\{T\}$. For more examples of sequences with NTS-condition, see e.g., Klin-eam et al. [14].

## 3. Main Results

Let $C$ be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space $E$ with dual space $E^{*}$. Let $J$ be the normalized duality map on E and $J^{-1}$ be the normalized duality map on $E^{*}$. Obviously, $J^{-1}=J_{*}$ exists under this setting.
Definition 3.1. (Chidume and Idu [7]) Let $T: C \rightarrow E^{*}$ be a map. A point $x^{*} \in C$ is called a $J$-fixed point of $T$ if and only if $T x^{*}=J x^{*}$. The set of $J$-fixed points of $T$ will be denoted by $F_{J}(T)$.
Definition 3.2. A map $T: C \rightarrow E^{*}$ will be called generalized $J$-nonexpansive if $F_{J}(T) \neq \emptyset$ and $\phi\left(\left(J^{-1} o T\right) x, p\right) \leq \phi(x, p)$ for all $x \in C$, for all $p \in F_{J}(T)$.

NST-Condition. Let $\left\{T_{n}\right\}$ and $\Gamma$ be two families of generalized- $J$-nonexpansive maps from $C$ into $E^{*}$ such that $\cap_{n=1}^{\infty} F\left(T_{n}\right)=F(\Gamma) \neq \emptyset$, where $F\left(T_{n}\right)$ is the set of fixed points of $T_{n}$ and $(\Gamma)$ is the set of fixed points of $\Gamma$. A sequence $\left\{T_{n}\right\}$ from
$C$ to $E^{*}$ is said to satisfy the NST-condition with $\Gamma$ if for each bounded sequence $\left\{x_{n}\right\} \subset C, \lim _{n \rightarrow \infty}\left\|J x_{n}-T_{n} x_{n}\right\|=0 \Longrightarrow \lim _{n \rightarrow \infty}\left\|J x_{n}-T x_{n}\right\|=0, \forall T \in \Gamma$.
We now prove the following new lemmas which will be needed in the sequel.
Lemma 3.3. Let $C$ be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E, with dual space $E^{*}$, such that JC is closed and convex. Let $\varphi: J C \rightarrow \mathbb{R}$ be a lower semi-continuous and convex function. Let $A: C \rightarrow E^{*}$ be continuous and monotone, and $f: J C \times J C \rightarrow \mathbb{R}$ be a bifunction. Let $r>0$ and $x \in E$ be any point. Define a map $T_{r}: E \rightarrow C$ by

$$
\begin{aligned}
T_{r}(x)= & \{u \in C: f(J u, J z)+\varphi(J z)-\varphi(J u)+\langle A u, z-u\rangle \\
& \left.+\frac{1}{r}\langle u-x, J z-J u\rangle \geq 0, \forall z \in C\right\} .
\end{aligned}
$$

Then, the following conclusions hold:
(a) $T_{r}$ is single-valued,
(b) $T_{r}$ is a firmly nonexpansive-type map, i.e.,

$$
\forall x, y \in E,\left\langle T_{r} x-T_{r} y, J T_{r} x-J T_{r} y\right\rangle \leq\left\langle x-y, J T_{r} x-J T_{r} y\right\rangle
$$

(c) $F\left(T_{r}\right)=G M E P(f, \varphi, A)$,
(d) $\operatorname{GMEP}(f, \varphi, A)$ is closed and $J(G M E P(f, \varphi, A))$ is closed and convex,
(e) $\phi\left(x, T_{r} x\right)+\phi\left(T_{r} x, q\right) \leq \phi(x, q), \forall q \in F\left(T_{r}\right), x \in E$.

Proof. (a) Let $x \in E$ and $r>0$. Let $u_{1}, u_{2} \in T_{r}(x)$. Then, we have that

$$
\begin{aligned}
& f\left(J u_{1}, J u_{2}\right)+\varphi\left(J u_{2}\right)-\varphi\left(J u_{1}\right)+\left\langle A u_{1}, u_{2}-u_{1}\right\rangle+\frac{1}{r}\left\langle u_{1}-x, J u_{2}-J u_{1}\right\rangle \geq 0 \\
& f\left(J u_{2}, J u_{1}\right)+\varphi\left(J u_{1}\right)-\varphi\left(J u_{2}\right)+\left\langle A u_{2}, u_{1}-u_{2}\right\rangle+\frac{1}{r}\left\langle u_{2}-x, J u_{1}-J u_{2}\right\rangle \geq 0
\end{aligned}
$$

From the above inequalities and condition $\left(A_{2}\right)$ and the monotonicity of $A$, we have that

$$
\begin{equation*}
\frac{1}{r}\left\langle u_{1}-u_{2}, J u_{2}-J u_{1}\right\rangle \geq 0 \tag{3.1}
\end{equation*}
$$

From monotonicity of $J$ and strict convexity of $E$, we have that $u_{1}=u_{2}$, which implies that $T_{r}$ is single-valued.
(b) For any $x, y \in C$, we have that

$$
\begin{aligned}
& f\left(J T_{r} x, J T_{r} y\right)+\varphi\left(J T_{r} y\right)-\varphi\left(J T_{r} x\right)+\left\langle A T_{r} x, T_{r} y-T_{r} x\right\rangle \\
& +\frac{1}{r}\left\langle T_{r} x-x, J T_{r} y-J T_{r} x\right\rangle \geq 0 \\
& f\left(J T_{r} y, J T_{r} x\right)+\varphi\left(J T_{r} x\right)-\varphi\left(J T_{r} y\right)+\left\langle A T_{r} y, T_{r} x-T_{r} y\right\rangle \\
& +\frac{1}{r}\left\langle T_{r} y-y, J T_{r} x-J T_{r} y\right\rangle \geq 0 .
\end{aligned}
$$

From the above inequalities, condition $\left(A_{2}\right)$ and the monotonicity of $A$, we conclude that

$$
\left\langle T_{r} x-T_{r} y, J T_{r} x-J T_{r} y\right\rangle \leq\left\langle x-y, J T_{r} x-J T_{r} y\right\rangle
$$

(c) Claim. $F\left(T_{r}\right)=G M E P(f, \varphi, A)$. Let $u \in F\left(T_{r}\right)$.

$$
\begin{aligned}
& \Longleftrightarrow \quad u=T_{r} u \\
& \Longleftrightarrow \quad f(J u, J z)+\varphi(J z)-\varphi(J u)+\langle A u, z-u\rangle+\frac{1}{r}\langle u-u, J z-J u\rangle \geq 0, z \in C \\
& \Longleftrightarrow \quad f(J u, J z)+\varphi(J z)-\varphi(J u)+\langle A u, z-u\rangle \geq 0, \forall z \in C \\
& \Longleftrightarrow \quad u \in G M E P(f, \varphi, A)
\end{aligned}
$$

(d) Claim. $\operatorname{GMEP}(f, \varphi, A)$ is closed, and $J(G M E P(f, \varphi, A))$ is closed and convex. Clearly, $\operatorname{GMEP}(f, \varphi, A)$ is closed. Let $\left\{u_{n}^{*}\right\} \subset J(G M E P(\Theta, \chi, B))$ such that $u_{n}^{*} \rightarrow u^{*}$, for some $u^{*} \in E^{*}$. Since $J C$ is closed, we have that $u^{*} \in J C$. Hence, there exist $u \in C$ and $\left\{u_{n}\right\} \subset(G M E P(f, \varphi, A))$ such that $u^{*}=J u$ and $u_{n}^{*}=J u_{n}, \forall n \in \mathbb{N}$. Utilizing the definitions of $f, A, \varphi$ and the fact that $J^{-1}$ is uniformly continuous on bounded subset of $E^{*}$, we have:

$$
\begin{aligned}
\varphi\left(u^{*}\right) \leq \liminf \varphi\left(u_{n}^{*}\right) & \leq \liminf \left[f\left(u_{n}^{*}, J y\right)+\varphi(J y)+\left\langle A J^{-1} u_{n}^{*}, y-J^{-1} u_{n}^{*}\right\rangle\right] \\
& \leq \limsup \left[f\left(u_{n}^{*}, J y\right)+\varphi(J y)+\left\langle A J^{-1} u_{n}^{*}, y-J^{-1} u_{n}^{*}\right\rangle\right] \\
& \leq f\left(u^{*}, J y\right)+\varphi(J y)+\left\langle A J^{-1} u^{*}, y-J^{-1} u^{*}\right\rangle
\end{aligned}
$$

Hence, $J(G M E P(f, \varphi, A))$ is closed.
Let $u_{1}^{*}, u_{2}^{*} \in J(G M E P(f, \varphi, A))$. Then, $u_{1}^{*}=J u_{1}, u_{2}^{*}=J u_{2}$, for some $u_{1}, u_{2} \in C$. For $\lambda, t \in(0,1]$, let $u_{\lambda}^{*}=\lambda u_{1}^{*}+(1-\lambda) u_{2}^{*} \in J C$. For any $y \in C$, set $z_{t}^{*}=t J y+(1-t) u_{\lambda}^{*}$. By conditions $\left(A_{1}\right)$ to $\left(A_{4}\right)$, we have that

$$
\begin{aligned}
0= & f\left(z_{t}^{*}, z_{t}^{*}\right)+\varphi\left(z_{t}^{*}\right)-\varphi\left(z_{t}^{*}\right)+\left\langle A\left(J^{-1} z_{t}^{*}\right), y-J^{-1} z_{t}^{*}\right\rangle-\left\langle A\left(J^{-1} z_{t}^{*}\right), y-J^{-1} z_{t}^{*}\right\rangle \\
\leq & f\left(z_{t}^{*}, J y\right)+\varphi(J y)-\varphi\left(z_{t}^{*}\right)+\left\langle A\left(J^{-1} z_{t}^{*}\right), y-J^{-1} z_{t}^{*}\right\rangle \\
= & f\left(u_{\lambda}^{*}+t\left(J y-u_{\lambda}^{*}\right), J y\right)+\varphi(J y)-\varphi\left(u_{\lambda}^{*}+t\left(J y-u_{\lambda}^{*}\right)\right) \\
& +\left\langle A J^{-1}\left(u_{\lambda}^{*}+t\left(J y-u_{\lambda}^{*}\right)\right), y-J^{-1}\left(u_{\lambda}^{*}+t\left(J y-u_{\lambda}^{*}\right)\right)\right\rangle .
\end{aligned}
$$

Applying condition $\left(A_{3}\right)$ we conclude that

$$
f\left(u_{\lambda}^{*}, J y\right)+\varphi(J y)-\varphi\left(u_{\lambda}^{*}\right)+\left\langle A\left(J^{-1} u_{\lambda}^{*}\right), y-J^{-1} u_{\lambda}^{*}\right\rangle \geq 0
$$

Hence, $u_{\lambda}^{*} \in J(G M E P(f, \varphi, A))$. Therefore, $J(G M E P(\Theta, \chi, B))$ is convex.
(e) Claim. $\phi\left(x, T_{r} x\right)+\phi\left(T_{r} x, q\right) \leq \phi(x, q), \forall q \in F\left(T_{r}\right), x \in E$. Let $x, y \in C$. Then, we have:

$$
\begin{align*}
& \phi\left(T_{r} x, T_{r} y\right)+\phi\left(T_{r} y, T_{r} x\right)=2\left\langle T_{r} x-T_{r} y, J T_{r} x-J T_{r} y\right\rangle  \tag{3.2}\\
& \phi\left(x, T_{r} y\right)+\phi\left(y, T_{r} x\right)-\phi\left(x, T_{r} x\right)-\phi\left(y, T_{r} y\right)=2\left\langle x-y, J T_{r} x-J T_{r} y\right\rangle . \tag{3.3}
\end{align*}
$$

Applying Lemma 3.3 (b), equations (3.2) and (3.3), we have that
$\phi\left(T_{r} x, T_{r} y\right)+\phi\left(T_{r} y, T_{r} x\right) \leq \phi\left(x, T_{r} y\right)+\phi\left(y, T_{r} x\right)-\phi\left(x, T_{r} x\right)-\phi\left(y, T_{r} y\right), \forall x, y \in C$.
For $y=u \in F\left(T_{r}\right)$, we have that

$$
\begin{equation*}
\phi\left(T_{r} x, u\right)+\phi\left(u, T_{r} x\right) \leq \phi(x, u)+\phi\left(u, T_{r} x\right)-\phi\left(x, T_{r} x\right)-\phi(u, u), \forall x \in C . \tag{3.5}
\end{equation*}
$$

Hence, we conclude that

$$
\begin{equation*}
\phi\left(x, T_{r} x\right)+\phi\left(T_{r} x, u\right) \leq \phi(x, u), \forall x, \in C, u \in F\left(T_{r}\right) \tag{3.6}
\end{equation*}
$$

This proof is complete.
Lemma 3.4. Let $E$ be a uniformly convex and uniformly smooth real Banach space with dual space $E^{*}$. Let $C$ be a closed subset of $E$ such that JC is closed and convex. Let $T$ be a generalized $J$-nonexpansive map from $C$ to $E^{*}$ with $F_{J}(T) \neq \emptyset$. Then, $F_{J}(T)$ is closed and $J F_{J}(T)$ is closed and convex.

Proof. First, we prove that $J F_{J}(T)$ is convex. Let $u^{*}, v^{*} \in J F_{J}(T)$ and $\alpha, \beta \in(0,1)$ with $\alpha+\beta=1$. Then, using the definition of $\phi$, we compute as follows:

$$
\begin{aligned}
& \phi\left(\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), J^{-1}\left(\alpha u^{*}+\beta v^{*}\right)\right) \\
= & \left\|\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right)\right\|^{2}-2\left\langle\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), \alpha u^{*}+\beta v^{*}\right\rangle \\
& +\left\|\alpha u^{*}+\beta v^{*}\right\|^{2}+\alpha\|u\|^{2}+\beta\|v\|^{2}-\left(\alpha\|u\|^{2}+\beta\|v\|^{2}\right) \\
= & \alpha\left(\left\|\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right)\right\|^{2}-2\left\langle\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), J u\right\rangle+\|u\|^{2}\right) \\
& +\beta\left(\left\|\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right)\right\|^{2}+\left\|\alpha u^{*}+\beta v^{*}\right\|^{2}-\left(\alpha\|u\|^{2}+\beta\|v\|^{2}\right)\right. \\
& \left.-2\left\langle\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), J v\right\rangle+\|v\|^{2}\right) \\
= & \alpha \phi\left(\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), u\right)+\beta \phi\left(\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), v\right) \\
& +\left\|\alpha u^{*}+\beta v^{*}\right\|^{2}-\left(\alpha\|u\|^{2}+\beta\|v\|^{2}\right) \\
\leq \quad & \alpha \phi\left(\left(J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), u\right)+\beta \phi\left(\left(J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), v\right)+\left\|\alpha u^{*}+\beta v^{*}\right\|^{2}\right.\right. \\
& \left.-\alpha\|u\|^{2}+\beta\|v\|^{2}\right) \\
= & \alpha\left(\left\|\alpha u^{*}+\beta v^{*}\right\|^{2}-2\left\langle\left(J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), J u\right\rangle+\|u\|^{2}\right)+\beta\left(\left\|\alpha u^{*}+\beta v^{*}\right\|^{2}\right.\right. \\
& -2\left\langle\left(J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), J v\right\rangle+\|v\|^{2}\right)+\left\|\alpha u^{*}+\beta v^{*}\right\|^{2}-\left(\alpha\|u\|^{2}+\beta\|v\|^{2}\right) \\
= & 2\left\|\alpha u^{*}+\beta v^{*}\right\|^{2}-2\left\langle\left(J^{-1}\left(\alpha u^{*}+\beta v^{*}\right), \alpha u^{*}+\beta v^{*}\right\rangle=0 .\right.
\end{aligned}
$$

By Remark 1, we get that $\left(J^{-1} o T\right) J^{-1}\left(\alpha u^{*}+\beta v^{*}\right)=J^{-1}\left(\alpha u^{*}+\beta v^{*}\right)$. This implies that $J^{-1}\left(\alpha u^{*}+\beta v^{*}\right) \in F_{J}(T)$. Thus, $\alpha u^{*}+\beta v^{*} \in J F_{J}(T)$. Hence, $J F_{J}(T)$ is convex.

Next, we prove that $F_{J}(T)$ and $J F_{J}(T)$ are closed. Obviously $F_{J}(T)$ is closed. Let $\left\{v_{n}^{*}\right\} \subset J F_{J}(T)$ such that $v_{n}^{*} \rightarrow v^{*}$ for some $v^{*} \in E^{*}$. Since $J C$ is closed, we have that $v^{*} \in J C$. Hence, there exist $v \in C$ and $\left\{v_{n}\right\} \subset F_{J}(T)$ such that $v^{*}=J v$ and $v_{n}^{*}=J v_{n} \forall n \in \mathbb{N}$. Utilizing the definition of $T$, we have that

$$
\begin{aligned}
\phi\left(\left(J^{-1} o T\right) v, v\right) & =\lim _{n \rightarrow \infty} \phi\left(\left(J^{-1} o T\right) v, v_{n}\right) \leq \lim _{n \rightarrow \infty}\left(\|v\|^{2}-2\left\langle v, J v_{n}\right\rangle+\left\|v_{n}\right\|^{2}\right) \\
& =\lim _{n \rightarrow \infty}\left(\|v\|^{2}-2\left\langle v, v_{n}^{*}\right\rangle+\left\|v_{n}^{*}\right\|^{2}\right)=\phi(v, v)=0
\end{aligned}
$$

By Remark 1, we have that $v^{*}=J v \in J F_{J}(T)$. Hence, $J F_{J}(T)$ is closed.
Using Lemmas 2.2 and 2.3, we obtain the following lemma.
Lemma 3.5. Let $E$ be a smooth, strictly convex and reflexive real Banach with dual space $E^{*}$. Let $C$ be a closed subset of $E$ such that $J C$ is closed and convex. Let $T$ be
a generalized-J-nonexpansive map from $C$ to $E^{*}$ such that $F_{J}(T) \neq \emptyset$. Then, $F_{J}(T)$ is a sunny generalized-J-nonexpansive retract of $E$.

Remark 4. From Lemmas 3.3 and 3.4, we have that $J F_{J}(T)$ and $J G M E P$ are closed and convex. Since $E$ is strictly convex, we have that

$$
J\left(F_{J}(T) \cap G M E P(f, A, \varphi)\right)=J F_{J}(T) \cap J G M E P(f, A, \varphi)
$$

By Lemma 2.2, we obtain that $F_{J}(T) \cap \operatorname{GMEP}(f, A, \varphi)$ is a sunny generalized- $J$ nonexpansive retract of $E$.

We now prove the following theorem.
Theorem 3.6. Let $E$ be a uniformly convex and uniformly smooth real Banach space with dual space $E^{*}$. Let $C$ be a nonempty closed and convex subset of $E$ such that JC is closed and convex. Let $\varphi: J C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let $A: C \rightarrow E^{*}$ be a continuous and monotone map. Let $f: J C \times J C \rightarrow \mathbb{R}$ be a bifunction satisfying conditions $\left(A_{1}\right)-\left(A_{4}\right)$. Let $T_{n}: C \rightarrow E^{*}$ be a countable family of generalized-J-nonexpansive maps and $\Gamma$ be a family of closed and generalized- $J$ nonexpansive maps from $C$ to $E^{*}$ such that

$$
\bigcap_{n=1}^{\infty} F_{J}\left(T_{n}\right) \cap \operatorname{GMEP}(f, A, \varphi)=F_{J}(\Gamma) \cap G M E P(f, A, \varphi) \neq \emptyset
$$

$\alpha \in(0,1)$ and $\left\{r_{n}\right\} \subset[a, \infty)$ for some $a>0$. Let $\left\{x_{n}\right\}$ be a sequence generated by:

$$
\left\{\begin{array}{l}
x_{1}=x \in C, C_{1}=C  \tag{3.7}\\
y_{n}=\alpha x_{n}+(1-\alpha) J^{-1} o T_{n} x_{n}, v_{n}=K_{r_{n}} y_{n} \\
C_{n+1}=\left\{v \in C_{n}: \phi\left(v_{n}, v\right) \leq \phi\left(x_{n}, v\right)\right\} \\
x_{n+1}=R_{C_{n+1}} x, n \geq 1
\end{array}\right.
$$

Assume that $\left\{T_{n}\right\}$ satisfies the NST-condition with $\Gamma$, then $\left\{x_{n}\right\}$ converges strongly to $R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)}$ x, where $R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)}$ is the sunny generalized- $J$ nonexpansive retraction of $E$ onto $F_{J}(\Gamma) \cap G M E P(f, A, \varphi)$.
Proof. Step 1. The sequence $\left\{x_{n}\right\}$ is well defined and $F_{J}(\Gamma) \cap G M E P(f, A, \varphi) \subset C_{n}$. First, we show that $J C_{n}$ is closed and convex. Clearly $J C_{1}=J C$ is closed and convex. Assume that $J C_{n}$ is closed and convex for some $n \geq 1$, utilizing the definition of $C_{n+1}$, it is easy to see that

$$
C_{n+1}=\left\{v \in C_{n}: 2\left\langle x_{n}-v_{n}, J v\right\rangle \leq\left\|x_{n}\right\|^{2}-\left\|v_{n}\right\|^{2}\right\}
$$

Thus, $J C_{n+1}$ is closed and convex, for each $n \geq 1$. Hence $J C_{n}$ is closed and convex. By Lemma 2.2, $C_{n}$ is a sunny generalized- $J$-nonexpansive retract of $E$. Hence, $\left\{x_{n}\right\}$ is well defined.
Next, we prove that

$$
F_{J}(\Gamma) \cap G M E P(f, A, \varphi) \subset C_{n}, \forall n \geq 1
$$

Clearly $F_{J}(\Gamma) \cap \operatorname{GMEP}(f, A, \varphi)$ is a subset of $C_{1}$. Assume that

$$
F_{J}(\Gamma) \cap G M E P(f, A, \varphi) \subset C_{n}
$$

for some $n \geq 1$. Let $p \in F_{J}(\Gamma) \cap G M E P(f, A, \varphi)$. By a Lemma $3.3(e)$ and definition of $T_{n}$, We have that

$$
\begin{align*}
\phi\left(v_{n}, p\right)= & \phi\left(K_{r_{n}} y_{n}, p\right) \leq \phi\left(y_{n}, p\right) \\
= & \left\|\alpha x_{n}+(1-\alpha) J^{-1} o T_{n} x_{n}\right\|^{2}-2\left\langle\alpha x_{n}+(1-\alpha) J^{-1} o T_{n} x_{n}, J p\right\rangle+\|p\|^{2} \\
\leq & \alpha\left(\left\|x_{n}\right\|^{2}-2\left\langle x_{n}, J p\right\rangle+\|p\|^{2}\right)+(1-\alpha)\left(\left\|J^{-1} o T_{n} x_{n}\right\|^{2}\right. \\
& \left.-2\left\langle J^{-1} o T_{n} x_{n}, J p\right\rangle+\|p\|^{2}\right)-\alpha(1-\alpha) g\left(\left\|x_{n}-J^{-1} o T_{n} x_{n}\right\|\right) \\
\leq & \phi\left(x_{n}, p\right)-\alpha(1-\alpha) g\left(\left\|x_{n}-J^{-1} o T_{n} x_{n}\right\|\right) \leq \phi\left(x_{n}, p\right) \tag{3.8}
\end{align*}
$$

Which implies that $p \in C_{n+1}$. Hence, $F_{J}(\Gamma) \cap \operatorname{GMEP}(f, A, \varphi) \subset C_{n} \forall n \geq 1$.
Step 2. $\lim _{n \rightarrow \infty}\left\|x_{n}-v_{n}\right\|=\lim _{n \rightarrow \infty}\left\|x_{n}-y_{n}\right\|=\lim _{n \rightarrow \infty}\left\|v_{n}-y_{n}\right\|=0$.
First, we prove that $\left\{x_{n}\right\}$ is bounded. From the definition of $\left\{x_{n}\right\}$ and Lemma 2.3, we have that

$$
\phi\left(x, x_{n}\right)=\phi\left(x, R_{C_{n}} x\right) \leq \phi(x, p)-\phi\left(R_{C_{n}} x, p\right) \leq \phi(x, p)
$$

$\forall p \in F_{J}(\Gamma) \cap G M E P(f, A, \varphi) \subset C_{n}$. This implies that $\left\{\phi\left(x, x_{n}\right)\right\}$ is bounded. It follows from the definition of $\phi$ that $\left\{x_{n}\right\}$ is bounded. Since

$$
x_{n+1}=R_{C_{n+1}} x \in C_{n+1} \subset C_{n}
$$

and $x_{n}=R_{C_{n}} x$, we have that $\phi\left(x, x_{n}\right) \leq \phi\left(x, x_{n+1}\right)$ and this implies that $\left\{\phi\left(x, x_{n}\right)\right\}$ is nondecreasing. Hence $\lim _{n \rightarrow \infty} \phi\left(x, x_{n}\right)$ exists. Also, from Lemma 3.3 and $x_{n}=R_{C_{n}} x$, we have that

$$
\begin{aligned}
\phi\left(x_{n}, x_{m}\right)=\phi\left(R_{C_{n}} x, R_{C_{m}} x\right) & \leq \phi\left(x, R_{C_{m}} x\right)-\phi\left(x, R_{C_{n}} x\right) \\
& =\phi\left(x, x_{m}\right)-\phi\left(x, x_{n}\right) \rightarrow 0 \text { as } n \rightarrow \infty
\end{aligned}
$$

So $\lim _{n \rightarrow \infty} \phi\left(x_{n}, x_{m}\right)=0$. It follows from Lemma 2.5 that $\lim _{m, n \rightarrow \infty}\left\|x_{n}-x_{m}\right\|=0$. Hence $\left\{x_{n}\right\}$ is Cauchy. Thus, there exists $x^{*} \in C$ such that $\lim _{n \rightarrow \infty} x_{n}=x^{*}$. From inequality (3.8) it follows that $\phi\left(v_{n}, x_{m}\right) \leq \phi\left(x_{n}, x_{m}\right) \rightarrow 0$ as $n \rightarrow \infty$. By lemma 2.5, we have

$$
\begin{equation*}
\lim _{m, n \rightarrow \infty}\left\|v_{n}-x_{m}\right\|=0 . \text { Hence, } \lim _{n \rightarrow \infty}\left\|v_{n}-x_{n}\right\|=0 \tag{3.9}
\end{equation*}
$$

From inequality (3.8), Lemma 3.3 (e) and inequality (3.9), we get that

$$
\begin{aligned}
\phi\left(y_{n}, v_{n}\right) & =\phi\left(y_{n}, k_{r_{n}} y_{n}\right) \leq \phi\left(y_{n}, p\right)-\phi\left(v_{n}, p\right) \\
& \leq \phi\left(x_{n}, p\right)-\phi\left(v_{n}, p\right) \leq\left\|v_{n}-x_{n}\right\|\left(\left\|x_{n}\right\|+\left\|v_{n}\right\|\right)+2\left\|v_{n}-x_{n}\right\|\|p\| \rightarrow 0
\end{aligned}
$$

By Lemma 2.5, it follows that

$$
\lim _{n \rightarrow \infty}\left\|v_{n}-y_{n}\right\|=0
$$

Using this and equation (3.9), we conclude that

$$
\lim _{n \rightarrow \infty}\left\|x_{n}-v_{n}\right\|=\lim _{n \rightarrow \infty}\left\|v_{n}-y_{n}\right\|=\lim _{n \rightarrow \infty}\left\|x_{n}-y_{n}\right\|=0
$$

Step 3. $\lim _{n \rightarrow \infty}\left\|J x_{n}-T x_{n}\right\|=0 \forall T \in \Gamma$.
From inequality (3.2), and setting $\mu=\alpha(1-\alpha)$, we obtain that

$$
\begin{equation*}
\mu g\left(\left\|x_{n}-J^{-1} o T_{n} x_{n}\right\|\right) \leq \phi\left(x_{n}, p\right)-\phi\left(v_{n}, p\right) \leq\left\|x_{n}-v_{n}\right\| M+2\left\|x_{n}-v_{n}\right\|\|p\| . \tag{3.10}
\end{equation*}
$$

First, we observe that $\left\{\left(J^{-1} o T_{n}\right) x_{n}\right\}$ is bounded in $E$. Using step 2 and property of $g$ in inequality (3.10), we obtain that

$$
\lim _{n \rightarrow \infty}\left\|x_{n}-J^{-1} o T_{n} x_{n}\right\|=0
$$

By uniform continuity of $J$ on bounded subset of $E$, we get that

$$
\lim _{n \rightarrow \infty}\left\|J x_{n}-T_{n} x_{n}\right\|=0
$$

Since $\left\{T_{n}\right\}$ satisfies the NST-condition with $\Gamma$, we conclude that

$$
\lim _{n \rightarrow \infty}\left\|J x_{n}-T x_{n}\right\|=0
$$

Step 4. $x^{*} \in F_{J}(\Gamma) \cap \operatorname{GMEP}(f, A, \varphi)$.
From step 3, we have that $\lim _{n \rightarrow \infty}\left\|J x_{n}-T x_{n}\right\|=0 \forall T \in \Gamma$. We also proved that $x_{n} \rightarrow x^{*} \in C$. Since $T$ is closed, we conclude that $x^{*} \in F_{J}(\Gamma)$. Furthermore, from step 2, we get that

$$
\lim _{n \rightarrow \infty}\left\|J y_{n}-J v_{n}\right\|=0
$$

Since $\left\{r_{n}\right\} \subset[a, \infty)$ by assumption, we obtain that

$$
\lim _{n \rightarrow \infty} \frac{\left\|J y_{n}-J v_{n}\right\|}{r_{n}}=0
$$

Since $v_{n}=K_{r_{n}} y_{n}$ in equation (3.7) and by Lemma 3.3, we have that

$$
\begin{equation*}
F\left(J v_{n}, J y\right)+\frac{1}{r_{n}}\left\langle y-v_{n}, J v_{n}-J y_{n}\right\rangle \geq 0 \forall y \in C \tag{3.11}
\end{equation*}
$$

By $A_{2}$ of Remark 2, we have that

$$
\frac{1}{r_{n}}\left\langle y-v_{n}, J v_{n}-J y_{n}\right\rangle \geq F\left(J y, J v_{n}\right)
$$

Since $y \mapsto F(J u, J y)$ is convex and lower semicontinuous, we obtain from the above inequality that $0 \geq F\left(J y, J x^{*}\right) \forall y \in C$.
For $t \in(0,1]$ and $y \in C$, letting $y_{t}^{*}=t J y+(1-t) J x^{*}$, then $y_{t}^{*} \in J C$ since $J C$ is closed and convex. Hence, $0 \geq F\left(y_{t}^{*}, J x^{*}\right) \forall y \in C$. By $A_{1}$ of Remark 2, we have that

$$
0=F\left(y_{t}^{*}, y_{t}^{*}\right) \leq t F\left(y_{t}^{*}, J y\right)+(1-t) F\left(y_{t}^{*}, J x^{*}\right) \leq F\left(J x^{*}+t\left(J y-J x^{*}\right), J y\right)
$$

Letting $t \downarrow 0$, by $A_{3}$ of Remark 2, we obtain that $F\left(J x^{*}, J y\right) \geq 0$. Hence,

$$
x^{*} \in G M E P(f, A, \varphi)
$$

Using this and the fact that $x^{*} \in F_{J}(T)$, we conclude $x^{*} \in F_{J}(\Gamma) \cap \operatorname{GMEP}(f, A, \varphi)$.
Step 5. $x_{n} \rightarrow R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)} x$. From Lemma 2.3, we obtain that

$$
\begin{equation*}
\phi\left(x, R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)} x\right) \leq \phi\left(x, x^{*}\right) \tag{3.12}
\end{equation*}
$$

Also, for $x^{*} \in F_{J}(\Gamma) \cap \operatorname{GMEP}(f, A, \varphi) \subset C_{n+1}, x_{n+1}=R_{C_{n+1}} x$, and by Lemma 2.3, we have that $\phi\left(x, x_{n+1}\right) \leq \phi\left(x, R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)} x\right)$. Since $x_{n} \rightarrow x^{*}$ as $n \rightarrow \infty$, we get that $\phi\left(x, x^{*}\right) \leq \phi\left(x, R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)} x\right)$.
Using this and inequality (3.12) we get that $\phi\left(x, x^{*}\right)=\phi\left(x, R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)} x\right)$. By uniqueness of $R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)} x$, we conclude that $x^{*}=R_{F_{J}(\Gamma) \cap G M E P(f, A, \varphi)} x$. This proof is complete.

## 4. An example

Let $E=l_{p}, 1<p<\infty, \frac{1}{p}+\frac{1}{q}=1, C=\overline{B_{l_{p}}}(0,1)=\left\{u \in l_{p}:\|u\|_{l_{p}} \leq 1\right\}$.
Consider the following maps:
$\varphi: J C \rightarrow \mathbb{R}$ defined by $\varphi\left(u^{*}\right)=\left\|u^{*}\right\|, \forall u^{*} \in J C$;
$f: J C \times J C \rightarrow \mathbb{R}$ defined by $f\left(u^{*}, v^{*}\right)=\left\langle J^{-1} u^{*}, v^{*}-u^{*}\right\rangle, \forall v^{*} \in J C$;
$A: C \rightarrow l_{q}$ defined by $A u=J\left(u_{1}, u_{2}, u_{3}, \cdots\right), \forall u=\left(u_{1}, u_{2}, u_{3}, \cdots\right) \in C$;
$T: C \rightarrow l_{q}$ defined by $T u=J\left(0, u_{1}, u_{2}, u_{3}, \cdots\right), \forall u=\left(u_{1}, u_{2}, u_{3}, \cdots\right) \in C$;
$T_{n}: K \rightarrow l_{q}$ defined by $T_{n} u=J\left(\beta_{n} u+\left(1-\beta_{n}\right) J^{-1} o T u\right), \forall n \geq 1, u \in C, \alpha=\frac{1}{2}$, $\beta_{n} \subset(0,1),\left\{r_{n}\right\} \subset[1, \infty), \forall n \geq 1$ and $\Gamma=T$.

Proof. Then, (a) E, C, JC, $\varphi, f$ and $A$ satisfy all the conditions of Theorem 3.6. In particular, $f$ satisfies conditions $\left(A_{1}\right)$ to $\left(A_{4}\right)$ as follows: conditions $\left(A_{1}\right)$ and $\left(A_{4}\right)$ follow easily from direct computation; $\left(A_{2}\right)$ follows from the monotonicity of the normalized duality map $J^{-1}$, and condition $\left(A_{3}\right)$ follows from the continuity of $J^{-1}$. Furthermore, $0 \in G M E P(f, A, \varphi)$.
(b) $T_{n}$ is a generalized- $J$-nonexpansive map and satisfies the NST-condition with $\Gamma$. $F_{J}(T)=F_{J}\left(T_{n}\right)=F_{J}(\Gamma)=\{0\}, \forall n \geq 1$. Moreover, $F_{J}(\Gamma) \cap G M E P(f, A, \varphi)=\{0\}$.
Hence, by Theorem 3.6, the sequence $\left\{x_{n}\right\}$ generated by equation (3.7) converges strongly to an element of $F(\Gamma) \cap G M E P(f, A, \varphi)$. This completes the example.

Remark 5. Theorem 3.6 is applicable in $L_{p}, l_{p}$ or $W_{p}^{m}(\Omega)$ spaces, $1<p<\infty$, where $W_{p}^{m}(\Omega)$ denote the usual Sobolev space, since these spaces are uniformly convex and uniformly smooth. The analytical representations of the duality map in these spaces where $p^{-1}+q^{-1}=1$ (see e.g., Theorem 3.5, Alber and Ryazantseva [3]; page 36).

## 5. Application in Hilbert spaces

Theorem 5.1. Let $E=H$ be a real Hilbert space. Let $C$ be a nonempty closed and convex subset of $H$. Let $\varphi: C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function. Let $A: C \rightarrow H$ be a continuous and monotone map. Let $f: C \times C \rightarrow \mathbb{R}$ be $a$ bifunction satisfying conditions $\left(A_{1}\right)-\left(A_{4}\right)$. Let $T_{n}: C \rightarrow H$ be a countable family of nonexpansive maps and $\Gamma$ be a family of closed and generalized nonexpansive maps from $C$ to $H$ such that

$$
\bigcap_{n=1}^{\infty} F\left(T_{n}\right) \cap G M E P(f, A, \varphi)=F(\Gamma) \cap G M E P(f, A, \varphi) \neq \emptyset
$$

$\alpha \in(0,1)$ and $\left\{r_{n}\right\} \subset[a, \infty)$ for some $a>0$. Let $\left\{x_{n}\right\}$ generated by:

$$
\left\{\begin{array}{l}
x_{1}=x \in C, C_{1}=C  \tag{5.1}\\
y_{n}=\alpha x_{n}+(1-\alpha) T_{n} x_{n}, v_{n}=K_{r_{n}} y_{n} \\
C_{n+1}=\left\{v \in C_{n}:\left\|v_{n}-v\right\| \leq\left\|x_{n}-v\right\|\right\} \\
x_{n+1}=P_{C_{n+1}} x, n \geq 1
\end{array}\right.
$$

Assume that $\left\{T_{n}\right\}$ satisfies the NST-condition with $\Gamma$, then, $\left\{x_{n}\right\}$ converges strongly to $P_{F(\Gamma) \cap G M E P(f, A, \varphi)} x$.

Proof. In Hilbert space, $J$ is the identity map and $\phi(x, y)=\|x-y\|^{2} \forall x, y \in H$. The result follows from Theorem 3.6.

Remark 6. Theorem 3.6 is a complementary analogue of the theorem of Klin-eam et al. [14] in the sense that, in the theorem of Klin-eam et al. [14], $T_{n}$, maps a subset $C \subset E$ to the space $E$ while in Theorem 3.6, $T_{n}$, maps a subset $C \subset E$ to the dual space $E^{*}$. Also, in Theorem 3.6, generalized mixed equilibrium problem is studied which is not the case in the theorem of Klin-ea et al. [14]. Finally, the condition that $\underline{\lim }\left(1-\alpha_{n}\right)>0$ in the theorem of Klin-eam [14] is dispensed with in Theorem 3.6.

Remark 7. Theorem 5.1 improves significantly the result of Pen and Yao [24], Nakajo and Takahashi [21], Martinez-Yanes and Xu [17], Qin and Su [27] in the following sense:
(1) In theorem 5.1, the set of generalized mixed equilibrium problem is studied which is not considered in Nakajo and Takahashi [21], Martinez-Yanes and Xu [17], Qin and Su [27].
(2) Theorem 5.1 extends the result in Pen and Yao [24], Nakajo and Takahashi [21], Martinez-Yanes and Xu [17], Qin and Su [27] from a nonexpansive selfmap to a countable family of generalized nonexpansive non self-maps.
(3) The iteration process of Theorem 5.1 is more efficient than that considered in Pen and Yao [24], Martinez-Yanes and Xu [17] which requires more arithmetic at each stage to implement because of the extra $y_{n}$ and $z_{n}$ terms involved in the iteration process, respectively.
(4) The control parameter in the algorithm considered in Theorem 5.1 is one arbitrarily fixed constant $\alpha \in(0,1)$, which is to be computed once and then used at each step of the iteration process, whereas the parameters in the algorithm studied in Pen and Yao [24], Nakajo and Takahashi [21], MartinezYanes and $\mathrm{Xu}[17]$, Qin and $\mathrm{Su}[27]$ are $\alpha_{n} \in(0,1)$ and $\beta_{n} \in(0,1)$ which have to be computed at each step of the iteration process.
(5) Finally, the sequence of Krasnoselskii-type algorithm is known to converge as fast as a geometric progression whereas that of a Mann-type algorithm is known to converge like $\frac{1}{n}$.

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