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A MODIFIED INERTIAL SUBGRADIENT EXTRAGRADIENT METHOD FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES AND COMMON FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a modified inertial subgradient extragradient method for solving a variational inequality problem with Lipschitz pseudomonotone mapping and a common fixed-point problem of a family of nonexpansive mappings. Under mild conditions, we obtain strong convergence theorems in a real Hilbert space. An application is also provided.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\text{Fix}(S)$ the set of fixed points of S. Let $A : H \to H$ be a mapping. Consider the classical variational inequality problem (VIP) of finding $x^* \in C$ such that $\langle Ax^*, x - x^* \rangle \geq 0, \forall x \in C$. The solution set of the VIP is denoted by $\text{VI}(C, A)$. Recently, much attention has been focused on solution methods for the VIP; see, e.g., [24, 17, 18, 8, 6, 23, 5, 1, 7, 3, 2] and references therein. One of effective methods to solve the VIP is the extragradient method, which was introduced by Korpelevich [13] in 1976. It generates a sequence $\{x_n\}$ in the following manner: $x_0 \in C$,

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n) \quad \forall n \geq 0, \end{cases} \tag{1.1}$$

where A is a L-Lipschitz continuous monotone mapping and $\tau \in (0, \frac{1}{L})$.

It deserves mentioning that there are two projections onto C for each iteration. In most cases, metric projections are not easy to calculate. In 2011, Censor, Gibali and Reich [4] first introduced the subgradient extragradient method, in which the second projection onto C was replaced by a projection onto a half-space:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ C_n = \{w \in H : \langle x_n - \tau A x_n - y_n, w - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n}(x_n - \tau A y_n) \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

where A is a L-Lipschitz continuous monotone mapping and $\tau \in (0, \frac{1}{L})$.

Combining the subgradient extragradient method and the Halpern's iteration method, Kraikaew and Saejung [14] proposed the Halpern subgradient extragradient method for solving the VIP in 2014. For any initial $x_0 \in H$, their iterative sequence $\{x_n\}$ was generated by

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ C_n = \{x \in H : \langle x_n - \tau A x_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{C_n}(x_n - \tau A y_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) z_n \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\tau \in (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$. They proved the strong convergence of $\{x_n\}$ to $P_{\mathrm{VI}(C,A)} x_0$.

In 2018, Thong and Hieu [19] first proposed the following inertial subgradient extragradient method. For any initial $x_0, x_1 \in H$, their iterative sequence $\{x_n\}$ was generated by

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau A w_n), \\ C_n = \{x \in H : \langle w_n - \tau A w_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{C_n}(w_n - \tau A y_n) \quad \forall n \geq 1, \end{cases} \tag{1.4}$$

with constant $\tau \in (0, \frac{1}{L})$. Under suitable conditions, they proved the weak convergence of $\{x_n\}$ to an element of $\mathrm{VI}(C, A)$.

Very recently, Thong et al. [20] introduced an inertial subgradient extragradient-type method for solving the VIP with pseudomonotone and Lipschitz continuous mapping in a real Hilbert space. Under appropriate conditions, they proved the strong convergence of $\{x_n\}$ to an element of $\mathrm{VI}(C, A)$.

In this paper, we introduce a modified inertial subgradient extragradient method for solving the VIP with a pseudomonotone and Lipschitz continuous mapping and a common fixed point problem (CFFP) of nonexpansive mappings in a real Hilbert space. Our proposed algorithm is based on the inertial subgradient extragradient method, hybrid steepest-descent method, and viscosity approximation method. Under mild conditions, we prove strong convergence of the proposed algorithm to a common solution of the VIP and CFPP. Our main result can also be applied to common solution problems of a fractional programming and a fixed-point problem.

This paper is organized as follows: In Section 2, we recall some definitions and preliminaries for the sequel use. Section 3 deals with the convergence analysis of the proposed algorithm. Finally, in Section 4, our main result is applied to a common solution problem of the fractional programming and the fixed-point problem.

2. Preliminaries

Let $\{x_n\}$ be a sequence in a Hilbert space H. We denote by $x_n \to x$ (respectively, $x_n \rightharpoonup x$) the strong (respectively, weak) convergence of $\{x_n\}$ to x.

A mapping $T : C \to H$ is said to be nonexpansive if $\|Tx-Ty\| \le \|x-y\|$, $\forall x, y \in C$. Recall that $T : C \to H$ is said to be

(i) L-Lipschitz continuous (or L-Lipschitzian) if $\exists L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \ \forall x, y \in C;$$

(ii) monotone if $\langle Tx - Ty, x - y\rangle \ge 0$, $\forall x, y \in C$;

(iii) pseudomonotone if $\langle Tx, y - x\rangle \ge 0 \Rightarrow \langle Ty, y - x\rangle \ge 0$, $\forall x, y \in C$;

(iv) α-strongly monotone if $\exists \alpha > 0$ such that

$$\langle Fx - Fy, x - y\rangle \ge \alpha\|x - y\|^2, \ \forall x, y \in C;$$

(v) sequentially weakly continuous if $\forall \{x_n\} \subset C$, the relation holds:

$$x_n \rightharpoonup x \Rightarrow Tx_n \rightharpoonup Tx.$$

It is easy to see that every monotone operator is pseudomonotone but the converse is not true. For each $x \in H$, we know that there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$, $\forall y \in C$. P_C is called a metric projection of H onto C.

Lemma 2.1. *The following conclusions hold in a Hilbert space H:*

(i) $\langle x - P_C x, y - P_C x\rangle \le 0 \ \forall x \in H, y \in C$;

(ii) $\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2 \ \forall x \in H, y \in C$;

(iii) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \ \forall x, y \in H$;

(iv) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2 \ \forall x, y \in H, \ \forall \lambda, \mu \in [0, 1]$ *with* $\lambda + \mu = 1$.

Lemma 2.2. [9] *For all $x \in H$ and $\alpha \ge \beta > 0$ the inequalities hold:*

$$\frac{\|x - P_C(x - \alpha Ax)\|}{\alpha} \le \frac{\|x - P_C(x - \beta Ax)\|}{\beta}$$

and

$$\|x - P_C(x - \beta Ax)\| \le \|x - P_C(x - \alpha Ax)\|.$$

Lemma 2.3. [4] *Let $A : C \to H$ be pseudomonotone and continuous. Then $x^* \in C$ is a solution to the VIP $\langle Ax^*, x - x^*\rangle \geq 0\ \forall x \in C$, if and only if*

$$\langle Ax, x - x^*\rangle \geq 0\ \forall x \in C.$$

Lemma 2.4. [21] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the conditions: $a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\gamma_n\ \forall n \geq 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that*

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$, *and*

(ii) $\limsup_{n\to\infty} \gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\lambda_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. [10] *Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$, where I is the identity mapping of H.*

Lemma 2.6. [22] *Let $\lambda \in (0, 1]$, $T : C \to H$ be a nonexpansive mapping, and the mapping $T^\lambda : C \to H$ be defined by $T^\lambda x := Tx - \lambda\mu F(Tx)\ \forall x \in C$, where $F : H \to H$ is κ-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, i.e.,*

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|,\ \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

3. Convergence theorems

In this section, let the feasible set C be a nonempty closed convex subset of a real Hilbert space H, and always assume that the following hold:

$T_i : H \to H$ is nonexpansive for $i = 1, ..., N$;

$A : H \to H$ is L-Lipschitz continuous, pseudomonotone monotone on H, and sequentially weakly continuous on C, such that $\Omega = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \neq \emptyset$;

$f : H \to H$ is a contraction with constant $\delta \in [0, 1)$, and $F : H \to H$ is η-strongly monotone and κ-Lipschitzian such that

$$\delta < \tau := 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} \text{ for } \rho \in \left(0, \frac{2\eta}{\kappa^2}\right);$$

$\{\beta_n\}, \{\gamma_n\}, \{\tau_n\}$ are positive sequences such that $\beta_n + \gamma_n < 1$,

$$\sum_{n=1}^{\infty} \beta_n = \infty,\ \lim_{n\to\infty} \beta_n = 0,\ 0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$$

and $\tau_n = o(\beta_n)$.

In addition, we write $T_n := T_{n \bmod N}$ for integer $n \geq 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$, that is, if $n = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_n = T_N$ if $q = 0$ and $T_n = T_q$ if $0 < q < N$.

Algorithm 3.1. Initialization. Let $\lambda_1 > 0,\ \alpha > 0,\ \mu \in (0,1)$ and $x_0, x_1 \in H$ be arbitrary.
Iterative Steps. Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \overline{\alpha_n}$, where

$$\overline{\alpha_n} = \begin{cases} \min\{\alpha, \frac{\tau_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute $w_n = T_n x_n + \alpha_n (T_n x_n - T_n x_{n-1})$ and $y_n = P_C(w_n - \lambda_n A w_n)$.
Step 3. Construct the half-space $C_n := \{z \in H : \langle w_n - \lambda_n A w_n - y_n, z - y_n \rangle \leq 0\}$, and compute $z_n = P_{C_n}(w_n - \lambda_n A y_n)$.
Step 4. Calculate $x_{n+1} = \beta_n f(x_n) + \gamma_n x_n + ((1 - \gamma_n) I - \beta_n \rho F) z_n$, and update

$$\lambda_{n+1} = \begin{cases} \min\{\mu \frac{\|w_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle A w_n - A y_n, z_n - y_n \rangle}, \lambda_n\} & \text{if } \langle A w_n - A y_n, z_n - y_n \rangle > 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.2}$$

Let $n := n + 1$ and return to **Step 1**.

Remark 3.1. From (3.1), we get $\lim_{n \to \infty} \frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| = 0$. Indeed, we have

$$\alpha_n \|x_n - x_{n-1}\| \leq \tau_n \ \forall n \geq 1,$$

which together with $\lim_{n \to \infty} \frac{\tau_n}{\beta_n} = 0$ implies that

$$\frac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| \leq \frac{\tau_n}{\beta_n} \to 0 \text{ as } n \to \infty.$$

Lemma 3.1. *Let $\{\lambda_n\}$ be generated by (3.2). Then $\{\lambda_n\}$ is a nonincreasing sequence with $\lambda_n \geq \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$ $\forall n \geq 1$, and $\lim_{n \to \infty} \lambda_n \geq \lambda := \min\{\lambda_1, \frac{\mu}{L}\}$.*

Proof. First, from (3.2) it is clear that $\lambda_n \geq \lambda_{n+1}$ $\forall n \geq 1$. Also, observe that

$$\left.\begin{array}{l} \frac{1}{2}(\|w_n - y_n\|^2 + \|z_n - y_n\|^2) \geq \|w_n - y_n\| \|z_n - y_n\| \\ \langle A w_n - A y_n, z_n - y_n \rangle \leq L \|w_n - y_n\| \|z_n - y_n\| \end{array}\right\} \Rightarrow \lambda_{n+1} \geq \min\left\{\lambda_n, \frac{\mu}{L}\right\}.$$

Remark 3.2. In terms of Lemmas 2.2 and 3.1, we claim that if $w_n = y_n$ or $A y_n = 0$, then y_n is an element of $\mathrm{VI}(C, A)$. Indeed, if $w_n = y_n$ or $A y_n = 0$, then

$$0 = \|y_n - P_C(y_n - \lambda_n A y_n)\| \geq \|y_n - P_C(y_n - \lambda A y_n)\|.$$

Thus, the assertion is valid.

The following lemmas are quite helpful for the convergence analysis of our algorithm.

Lemma 3.2. *Let $\{w_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by Algorithm 3.1. Then*

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|z_n - y_n\|^2,\ \forall p \in \Omega. \tag{3.3}$$

Proof. By the definition of $\{\lambda_n\}$, we claim that

$$2\langle Aw_n - Ay_n, z_n - y_n\rangle \le \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|^2 + \frac{\mu}{\lambda_{n+1}}\|z_n - y_n\|^2\ \forall n \ge 1. \tag{3.4}$$

Indeed, if $\langle Aw_n - Ay_n, z_n - y_n\rangle \le 0$, then inequality (3.4) holds. From (3.2), we get (3.4). Observe that, for each $p \in \Omega \subset C \subset C_n$,

$$\begin{aligned}\|z_n - p\|^2 &= \|P_{C_n}(w_n - \lambda_n Ay_n) - P_{C_n}p\|^2 \le \langle z_n - p, w_n - \lambda_n Ay_n - p\rangle \\ &= \frac{1}{2}\|z_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 - \frac{1}{2}\|z_n - w_n\|^2 - \langle z_n - p, \lambda_n Ay_n\rangle,\end{aligned}$$

which hence yields

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle. \tag{3.5}$$

From $p \in \mathrm{VI}(C, A)$, we get $\langle Ap, x - p\rangle \ge 0\ \forall x \in C$. By the pseudomonotonicity of A on C we have $\langle Ax, x - p\rangle \ge 0\ \forall x \in C$. Putting $x := y_n \in C$ we get $\langle Ay_n, p - y_n\rangle \le 0$. Thus,

$$\langle Ay_n, p - z_n\rangle = \langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - z_n\rangle \le \langle Ay_n, y_n - z_n\rangle. \tag{3.6}$$

Substituting (3.6) for (3.5), we obtain

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle. \tag{3.7}$$

Since $y_n = P_{C_n}(w_n - \lambda_n Aw_n)$ and $z_n \in C_n$, we have

$$2\langle w_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle \le 2\lambda_n\langle Aw_n - Ay_n, z_n - y_n\rangle,$$

which together with (3.4), implies that

$$2\langle w_n - \lambda_n Ay_n - y_n, z_n - y_n\rangle \le \mu\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 + \mu\frac{\lambda_n}{\lambda_{n+1}}\|z_n - y_n\|^2. \tag{3.8}$$

Therefore, substituting (3.8) for (3.7), we infer that inequality (3.3) holds.

Lemma 3.3. *Let $\{w_n\}, \{x_n\}, \{y_n\}$ be the sequences generated by Algorithm 3.1. If $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$ and $w_n - y_n \to 0$ and $\exists \{w_{n_k}\} \subset \{w_n\}$ such that $w_{n_k} \rightharpoonup z \in H$, then $z \in \Omega$.*

Proof. From Algorithm 3.1, we get $w_n - x_n = T_n x_n - x_n + \alpha_n(T_n x_n - T_n x_{n-1})\ \forall n \ge 1$. Hence

$$\|T_n x_n - x_n\| \le \|w_n - x_n\| + \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|.$$

Utilizing Remark 3.1 and the assumption $w_n - x_n \to 0$, we have

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{3.9}$$

Also, from $y_n = P_C(w_n - \lambda_n A w_n)$, we have

$$\langle w_n - \lambda_n A w_n - y_n, x - y_n \rangle \leq 0 \ \forall x \in C.$$

Hence

$$\frac{1}{\lambda_n}\langle w_n - y_n, x - y_n \rangle + \langle A w_n, y_n - w_n \rangle \leq \langle A w_n, x - w_n \rangle \quad \forall x \in C. \tag{3.10}$$

Note that $\{w_{n_k}\}$ is bounded. According to the Lipschitz continuity of A, $\{A w_{n_k}\}$ is bounded. Note that $\lambda_n \geq \min\{\lambda_1, \frac{\mu}{L}\}$. So, from (3.10) we get

$$\liminf_{k\to\infty} \langle A w_{n_k}, x - w_{n_k} \rangle \geq 0, \ \forall x \in C.$$

Meantime, observe that

$$\langle A y_n, x - y_n \rangle = \langle A y_n - A w_n, x - w_n \rangle + \langle A w_n, x - w_n \rangle + \langle A y_n, w_n - y_n \rangle.$$

Since $w_n - y_n \to 0$, we obtain from L-Lipschitz continuity of A that $A w_n - A y_n \to 0$, which together with (3.10) yields

$$\liminf_{k\to\infty} \langle A y_{n_k}, x - y_{n_k} \rangle \geq 0 \ \forall x \in C.$$

Next we show that $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for $l = 1, ..., N$. Indeed, for $i = 1, ..., N$,

$$\|x_n - T_{n+i} x_n\| \quad \leq 2\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\|.$$

Hence from (3.9) and the assumption $x_n - x_{n+1} \to 0$, we get

$$\lim_{n\to\infty} \|x_n - T_{n+i} x_n\| = 0$$

for $i = 1, ..., N$. This immediately implies that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \quad \text{for } l = 1, ..., N. \tag{3.11}$$

We now take a sequence $\{\varepsilon_k\} \subset (0, 1)$ satisfying $\varepsilon_k \downarrow 0$ as $k \to \infty$. For all $k \geq 1$, we denote by m_k the smallest positive integer such that

$$\langle A y_{n_j}, x - y_{n_j} \rangle + \varepsilon_k \geq 0 \quad \forall j \geq m_k. \tag{3.12}$$

Since $\{\varepsilon_k\}$ is decreasing, it is clear that $\{m_k\}$ is increasing. Noticing that $\{y_{m_k}\} \subset C$ guarantees $A y_{m_k} \neq 0 \ \forall k \geq 1$, we set

$$u_{m_k} = \frac{A y_{m_k}}{\|A y_{m_k}\|^2}$$

and get $\langle A y_{m_k}, u_{m_k} \rangle = 1 \ \forall k \geq 1$. So, from (3.12), we get

$$\langle A y_{m_k}, x + \varepsilon_k u_{m_k} - y_{m_k} \rangle \geq 0 \ \forall k \geq 1.$$

Again from the pseudomonotonicity of A, we have

$$\langle A(x + \varepsilon_k u_{m_k}), x + \varepsilon_k u_{m_k} - y_{m_k} \rangle \geq 0 \ \forall k \geq 1.$$

This immediately leads to

$$\langle Ax, x - y_{m_k} \rangle \geq \langle Ax - A(x + \varepsilon_k u_{m_k}), x + \varepsilon_k u_{m_k} - y_{m_k} \rangle - \varepsilon_k \langle Ax, u_{m_k} \rangle \quad \forall k \geq 1. \tag{3.13}$$

We claim that

$$\lim_{k\to\infty} \varepsilon_k u_{m_k} = 0.$$

Indeed, from $w_{n_k} \rightharpoonup z$ and $w_n - y_n \to 0$, we obtain $y_{n_k} \rightharpoonup z$. So, $\{y_n\} \subset C$ guarantees $z \in C$. Again from the sequentially weak continuity of A, we know that $Ay_{n_k} \rightharpoonup Az$. Thus, we have $Az \neq 0$ (otherwise, z is a solution). Taking into account the sequentially weak lower semicontinuity of the norm $\|\cdot\|$, we get

$$0 < \|Az\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\|.$$

Note that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $\varepsilon_k \downarrow 0$ as $k \to \infty$. So it follows that

$$0 \leq \limsup_{k\to\infty} \|\varepsilon_k u_{m_k}\| = \limsup_{k\to\infty} \frac{\varepsilon_k}{\|Ay_{m_k}\|} \leq \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0.$$

Hence we get $\varepsilon_k u_{m_k} \to 0$.

Next we show that $z \in \Omega$. Indeed, from $w_n - x_n \to 0$ and $w_{n_k} \rightharpoonup z$, we get $x_{n_k} \rightharpoonup z$. From (3.11) we have $x_{n_k} - T_l x_{n_k} \to 0$ for $l = 1, ..., N$. Note that Lemma 2.5 guarantees the demiclosedness of $I - T_l$ at zero for $l = 1, ..., N$. Thus $z \in \mathrm{Fix}(T_l)$. Since l is an arbitrary element in the finite set $\{1, ..., N\}$, we get $z \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$.

On the other hand, letting $k \to \infty$, we deduce that the right hand side of (3.13) tends to zero by the uniform continuity of A, the boundedness of $\{w_{m_k}\}, \{u_{m_k}\}$ and the limit $\lim_{k\to\infty} \varepsilon_k u_{m_k} = 0$. Thus, we get

$$\langle Ax, x - z\rangle = \liminf_{k\to\infty} \langle Ax, x - y_{m_k}\rangle \geq 0 \ \forall x \in C.$$

By Lemma 2.3, we have $z \in \mathrm{VI}(C, A)$. Therefore,

$$z \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) = \Omega.$$

This completes the proof.

Theorem 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then $x_n \to x^* \in \Omega \ \Leftrightarrow \ x_n - x_{n+1} \to 0$, where $x^* \in \Omega$ is a unique solution to the VIP:*

$$\langle (\rho F - f)x^*, p - x^*\rangle \geq 0 \ \forall p \in \Omega.$$

Proof. We show that $P_\Omega(f + I - \rho F)$ is a contraction. Indeed, for any $x, y \in H$, by Lemma 2.6, we have

$$\|P_\Omega(f + I - \rho F)x - P_\Omega(f + I - \rho F)y\| \leq [1 - (\tau - \delta)]\|x - y\|,$$

which implies that $P_\Omega(f + I - \rho F)$ is a contraction. Banach's Contraction Mapping Principle guarantees that $P_\Omega(f + I - \rho F)$ has a unique fixed point. Say $x^* \in H$, that is, $x^* = P_\Omega(f + I - \rho F)x^*$. Thus, there exists a unique solution

$$x^* \in \Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A)$$

to the VIP

$$\langle(\rho F - f)x^*, p - x^*\rangle \geq 0 \quad \forall p \in \Omega. \tag{3.14}$$

It is clear that the necessity of the theorem is valid. Next we show the sufficiency of the theorem. To the aim, we assume $x_n - x_{n+1} \to 0$ and divide the proof of the sufficiency into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, since

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1,$$

we may assume, without loss of generality, that $\{\gamma_n\} \subset [a,b] \subset (0,1)$. Take an arbitrary

$$p \in \Omega = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C,A).$$

Then $T_n p = p\ \forall n \geq 1$, and inequality (3.3) holds, i.e.,

$$\begin{aligned}\|z_n - p\|^2 \leq \|w_n - p\|^2 &- \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 \\ &- \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|z_n - y_n\|^2\ \forall p \in \Omega.\end{aligned} \tag{3.15}$$

Since

$$\lim_{n\to\infty}\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 0,$$

we may assume, without loss of generality, that

$$1 - \mu\frac{\lambda_n}{\lambda_{n+1}} > 0\ \forall n \geq 1.$$

Therefore, we have

$$\|z_n - p\| \leq \|w_n - p\| \quad \forall n \geq 1. \tag{3.16}$$

It follows that

$$\begin{aligned}\|w_n - p\| &\leq \|T_n x_n - p\| + \alpha_n\|T_n x_n - T_n x_{n-1}\| \\ &\leq \|x_n - p\| + \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|.\end{aligned} \tag{3.17}$$

According to Remark 3.1, we have

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0$$

as $n \to \infty$, it follows that there exists a constant $M_1 > 0$ such that

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq M_1 \quad \forall n \geq 1. \tag{3.18}$$

Combining (3.16), (3.17) and (3.18), we obtain

$$\|z_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| + \beta_n M_1 \quad \forall n \geq 1. \tag{3.19}$$

Since $\beta_n + \gamma_n < 1 \ \forall n \geq 1$, we get

$$\frac{\beta_n}{1-\gamma_n} < 1 \ \forall n \geq 1.$$

So, from Lemma 2.6 and (3.19) it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \beta_n \|f(x_n) - p\| + \gamma_n \|x_n - p\| \\
&\quad + (1-\beta_n-\gamma_n) \left\| \left(\frac{1-\gamma_n}{1-\beta_n-\gamma_n} I - \frac{\beta_n}{1-\beta_n-\gamma_n} \rho F \right) z_n - p \right\| \\
&\leq \beta_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) + \gamma_n \|x_n - p\| \\
&\quad + (1-\beta_n-\gamma_n) \left\| \left(\frac{1-\gamma_n}{1-\beta_n-\gamma_n} I - \frac{\beta_n}{1-\beta_n-\gamma_n} \rho F \right) z_n - p \right\| \\
&\leq \beta_n (\delta \|x_n - p\| + \|f(p) - p\|) + \gamma_n \|x_n - p\| \\
&\quad + (1-\gamma_n) \left\| \left(I - \frac{\beta_n}{1-\gamma_n} \rho F \right) z_n - \left(I - \frac{\beta_n}{1-\gamma_n} \rho F \right) p + \frac{\beta_n}{1-\gamma_n} (I - \rho F) p \right\| \\
&\leq \beta_n (\delta \|x_n - p\| + \|f(p) - p\|) + \gamma_n \|x_n - p\| \\
&\quad + (1-\gamma_n-\beta_n\tau) \|z_n - p\| + \beta_n \|(I-\rho F)p\| \\
&\leq [1 - \beta_n(\tau-\delta)] \|x_n - p\| + \beta_n(\tau-\delta) \cdot \frac{M_1 + \|f(p) - p\| + \|(I-\rho F)p\|}{\tau - \delta} \\
&\leq \max \left\{ \|x_n - p\|, \frac{M_1 + \|f(p) - p\| + \|(I-\rho F)p\|}{\tau - \delta} \right\}.
\end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{M_1 + \|f(p) - p\| + \|(I-\rho F)p\|}{\tau - \delta}\} \ \forall n \geq 1.$$

Thus, $\{x_n\}$ is bounded, and so are the sequences $\{w_n\}$, $\{y_n\}$, $\{z_n\}$, $\{f(x_n)\}$, $\{Fz_n\}$, $\{T_n x_n\}$.

Step 2. We show that

$$(1-\beta_n\tau-\gamma_n) \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) [\|w_n - y_n\|^2 + \|z_n - y_n\|^2] \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n M_4,$$

for some $M_4 > 0$. Indeed, observe that

$$\begin{aligned}
x_{n+1} - p &= \beta_n (f(x_n) - p) + \gamma_n (x_n - p) + (1-\beta_n-\gamma_n) \\
&\quad \times \left\{ \frac{1-\gamma_n}{1-\beta_n-\gamma_n} \left[\left(I - \frac{\beta_n}{1-\gamma_n} \rho F \right) z_n - \left(I - \frac{\beta_n}{1-\gamma_n} \rho F \right) p \right] \right. \\
&\quad \left. + \frac{\beta_n}{1-\beta_n-\gamma_n} (I - \rho F) p \right\} \\
&= \beta_n (f(x_n) - f(p)) + \gamma_n (x_n - p) + (1-\gamma_n) \\
&\quad \times \left[\left(I - \frac{\beta_n}{1-\gamma_n} \rho F \right) z_n - \left(I - \frac{\beta_n}{1-\gamma_n} \rho F \right) p \right] + \beta_n (f - \rho F) p.
\end{aligned}$$

Then by Lemma 2.6 and the convexity of the function $h(t) = t^2 \ \forall t \in \mathbf{R}$, we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\delta\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \beta_n\tau - \gamma_n)\|z_n - p\|^2 \\ &\quad + 2\beta_n\langle (f - \rho F)p, x_{n+1} - p\rangle \\ &\le \beta_n\delta\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 \\ &\quad + (1 - \beta_n\tau - \gamma_n)\|z_n - p\|^2 + \beta_n M_2, \end{aligned} \tag{3.20}$$

where $\sup_{n\ge1} 2\|(f - \rho F)p\|\|x_n - p\| \le M_2$ for some $M_2 > 0$. Substituting (3.15) for (3.20), we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\delta\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \beta_n\tau - \gamma_n)[\|w_n - p\|^2 \\ &\quad - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2 \\ &\quad - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|z_n - y_n\|^2] + \beta_n M_2. \end{aligned} \tag{3.21}$$

Also, from (3.19) we have

$$\|w_n - p\|^2 \le \|x_n - p\|^2 + \beta_n M_3, \tag{3.22}$$

where $\sup_{n\ge1}(2M_1\|x_n - p\| + \beta_n M_1^2) \le M_3$ for some $M_3 > 0$. Combining (3.21) and (3.22), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\delta\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + (1 - \beta_n\tau - \gamma_n)[\|x_n - p\|^2 + \beta_n M_3] \\ &\quad - (1 - \beta_n\tau - \gamma_n)\left[\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|w_n - y_n\|^2\right. \\ &\quad \left.+ \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|z_n - y_n\|^2\right] + \beta_n M_2 \\ &\le \|x_n - p\|^2 - (1 - \beta_n\tau - \gamma_n)\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)[\|w_n - y_n\|^2 \\ &\quad + \|z_n - y_n\|^2] + \beta_n M_4, \end{aligned}$$

where $M_4 := M_2 + M_3$. This immediately implies that

$$(1-\beta_n\tau-\gamma_n)\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)[\|w_n-y_n\|^2+\|z_n-y_n\|^2] \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n M_4. \tag{3.23}$$

Step 3. We show that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le [1 - \beta_n(\tau - \delta)]\|x_n - p\|^2 \\ &\quad + \beta_n(\tau - \delta)\left[\frac{2}{\tau - \delta}\langle (f - \rho F)p, x_{n+1} - p\rangle + \frac{3M}{\tau - \delta}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n - x_{n-1}\|\right]\end{aligned}$$

for some $M > 0$. Indeed, we have

$$\|w_n - p\|^2 \le \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \alpha_n\|x_n - x_{n-1}\|]. \tag{3.24}$$

Combining (3.19), (3.20) and (3.24), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\delta\|x_n-p\|^2+\gamma_n\|x_n-p\|^2+(1-\beta_n\tau-\gamma_n)\{\|x_n-p\|^2\\ &\quad+\alpha_n\|x_n-x_{n-1}\|[2\|x_n-p\|+\alpha_n\|x_n-x_{n-1}\|]\}\\ &\quad+2\beta_n\langle(f-\rho F)p,x_{n+1}-p\rangle\\ &\le[1-\beta_n(\tau-\delta)]\|x_n-p\|^2\\ &\quad+\alpha_n\|x_n-x_{n-1}\|[2\|x_n-p\|+\alpha_n\|x_n-x_{n-1}\|]\\ &\quad+2\beta_n\langle(f-\rho F)p,x_{n+1}-p\rangle\\ &\le[1-\beta_n(\tau-\delta)]\|x_n-p\|^2\\ &\quad+\beta_n(\tau-\delta)\left[\frac{2\langle(f-\rho F)p,x_{n+1}-p\rangle}{\tau-\delta}+\frac{3M}{\tau-\delta}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|\right],\end{aligned}\tag{3.25}$$

where $\sup_{n\ge1}\{\|x_n-p\|,\alpha_n\|x_n-x_{n-1}\|\}\le M$ for some $M>0$.

Step 4. We show that $\{x_n\}$ converges strongly to a unique solution $x^*\in\Omega$ to the VIP (3.14). Indeed, putting $p=x^*$, we deduce from (3.25) that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le[1-\beta_n(\tau-\delta)]\|x_n-x^*\|^2\\ &\quad+\beta_n(\tau-\delta)\left[\frac{2\langle(f-\rho F)x^*,x_{n+1}-x^*\rangle}{\tau-\delta}+\frac{3M}{\tau-\delta}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n-x_{n-1}\|\right].\end{aligned}\tag{3.26}$$

By Lemma 2.4, it suffices to show that

$$\limsup_{n\to\infty}\langle(f-\rho F)x^*,x_{n+1}-x^*\rangle\le0.$$

From (3.23), $x_n-x_{n+1}\to0$, $\beta_n\to0$ and $\{\gamma_n\}\subset[a,b]\subset(0,1)$, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}(1-\beta_n\tau-b)(1-\mu\frac{\lambda_n}{\lambda_{n+1}})[\|w_n-y_n\|^2+\|z_n-y_n\|^2]\\ &\le\limsup_{n\to\infty}[\|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_nM_4]\\ &\le\limsup_{n\to\infty}(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|=0.\end{aligned}$$

This immediately implies that

$$\lim_{n\to\infty}\|w_n-y_n\|=0\quad\text{and}\quad\lim_{n\to\infty}\|z_n-y_n\|=0.\tag{3.27}$$

Thus, we get

$$\lim_{n\to\infty}\|z_n-w_n\|=0.\tag{3.28}$$

Also, from Algorithm 3.1 we get

$$x_{n+1}-x_n=\beta_n(f(x_n)-\rho Fz_n)+(1-\gamma_n)(z_n-x_n),$$

which hence implies that

$$\|z_n-x_n\|\le\frac{1}{1-b}[\|x_{n+1}-x_n\|+\beta_n\|f(x_n)-\rho Fz_n\|].$$

From $x_n - x_{n+1} \to 0$, $\beta_n \to 0$ and the boundedness of $\{f(x_n)\}$ and $\{Fz_n\}$ we conclude that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.29}$$

From the boundedness of $\{x_n\}$, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (f-\rho F)x^*, x_n - x^*\rangle = \lim_{k\to\infty} \langle (f-\rho F)x^*, x_{n_k} - x^*\rangle. \tag{3.30}$$

Since H is reflexive and $\{x_n\}$ is bounded, we may assume, without loss of generality, that $x_{n_k} \rightharpoonup \tilde{x}$. Hence from (3.30) we get

$$\begin{aligned}\limsup_{n\to\infty} \langle (f-\rho F)x^*, x_n - x^*\rangle &= \lim_{k\to\infty} \langle (f-\rho F)x^*, x_{n_k} - x^*\rangle \\ &= \langle (f-\rho F)x^*, \tilde{x} - x^*\rangle. \end{aligned}\tag{3.31}$$

Also, from (3.28) and (3.29) we have

$$\|w_n - x_n\| \le \|w_n - z_n\| + \|z_n - x_n\| \to 0 \quad (n\to\infty),$$

which together with $x_{n_k} \rightharpoonup \tilde{x}$, implies that $w_{n_k} \rightharpoonup \tilde{x}$.
Since $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - y_n \to 0$ and $w_{n_k} \rightharpoonup \tilde{x}$, by Lemma 3.3 we infer that $\tilde{x} \in \Omega$. Hence from (3.14) and (3.31) we get

$$\limsup_{n\to\infty} \langle (f-\rho F)x^*, x_n - x^*\rangle = \langle (f-\rho F)x^*, \tilde{x} - x^*\rangle \le 0, \tag{3.32}$$

which immediately leads to

$$\begin{aligned}&\limsup_{n\to\infty} \langle (f-\rho F)x^*, x_{n+1} - x^*\rangle \\ &\le \limsup_{n\to\infty} [\|(f-\rho F)x^*\| \|x_{n+1} - x_n\| + \langle (f-\rho F)x^*, x_n - x^*\rangle] \le 0.\end{aligned}\tag{3.33}$$

Note that $\{\beta_n(\tau-\delta)\} \subset [0,1]$, $\sum_{n=1}^{\infty} \beta_n(\tau-\delta) = \infty$, and

$$\limsup_{n\to\infty} \left[\frac{2\langle (f-\rho F)x^*, x_{n+1} - x^*\rangle}{\tau-\delta} + \frac{3M}{\tau-\delta}\cdot\frac{\alpha_n}{\beta_n}\cdot\|x_n - x_{n-1}\|\right] \le 0.$$

Consequently, applying Lemma 2.4 to (3.26), we have $\lim_{n\to 0} \|x_n - x^*\| = 0$.
This completes the proof.

4. An application

In this section, our main result is applied to find a common solution of the fractional programming and fixed-point problems. Since the exact solution of the problem is not known, we make use of $\|x_{n+1} - x_n\|$ to measure the error of the n-th iteration, which also serves as the role of checking whether or not the proposed algorithm converges to the solution.

The initial point x_0 is randomly chosen in $\mathbf{R}^m$. Take

$$f(x) = F(x) = \frac{1}{2}x,\ \mu = 0.3,\ \beta_n = \frac{1}{n+1},\ \alpha = 0.1,\ \gamma_n = \frac{1}{3},\ \rho = 2,$$

and

$$\alpha_n = \begin{cases} \min\left\{\frac{\beta_n^2}{\|x_n - x_{n-1}\|}, \alpha\right\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases}$$

Then we know that $\kappa = \eta = \frac{1}{2}$, and

$$\tau = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} = 1 - \sqrt{1 - 2\left(2 \cdot \frac{1}{2} - 2\left(\frac{1}{2}\right)^2\right)} = 1 \in (0, 1].$$

First, we set the operator $\Gamma(x) := Mx + q$, which comes from [11] and has been considered by many authors for applicable examples (see, for example [16]), where $M = BB^T + D + G$, and B is an $m \times m$ matrix, D is an $m \times m$ skew-symmetric matrix, G is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative (so M is positive semidefinite), q is a vector in $\mathbf{R}^m$. The feasible set $C \subset \mathbf{R}^m$ is a closed and convex subset defined by $C := \{x \in \mathbf{R}^m : Hx \leq d\}$, where H is an $l \times m$ matrix and d is a nonnegative vector. It is clear that Γ is β-monotone and L-Lipschitz-continuous with $\beta = \min\{\text{eig}(\Gamma)\}$ and $L = \max\{\text{eig}(\Gamma)\}$. Next we give the operator A.
Consider the following fractional programming problem:

$$\min g(x) = \frac{x^T Qx + a^T x + a_0}{b^T x + b_0},$$
$$\text{subject to } x \in X := \{x \in \mathbf{R}^4 : b^T x + b_0 > 0\},$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \ a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2, \ b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite in $\mathbf{R}^4$ and consequently g is pseudoconvex on $X = \{x \in \mathbf{R}^4 : b^T x + b_0 > 0\}$. Then

$$Ax := \nabla g(x) = \frac{(b^T x + b_0)(2Qx + a) - b(x^T Qx + a^T x + a_0)}{(b^T x + b_0)^2}.$$

It is known that A is pseudomonotone (see, e.g., [12, 15] for more details). Now, we give a nonexpansive mapping $T_1 : H \to C$ defined by $T_1 x = P_C x \ \forall x \in H$. Thus, Algorithm 3.1 can be rewritten as follows:

$$\begin{cases} w_n = T_1 x_n + \alpha_n (T_1 x_n - T_1 x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = P_{C_n}(w_n - \lambda_n A y_n), \\ x_{n+1} = \frac{1}{n+1} \cdot \frac{1}{2} x_n + \frac{1}{3} x_n + (\frac{n}{n+1} I - \frac{1}{3} I) z_n \quad \forall n \geq 1, \end{cases}$$

where for each $n \geq 1$, C_n and λ_n are chosen as in Algorithm 3.1. Therefore, utilizing Theorem 3.1, we know that $\{x_n\}$ converges to a common solution of the fractional programming problem and the fixed-point problem of T_1 provided $\|x_n - x_{n+1}\| \to 0$.

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