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AN IMPROVEMENT OF ALTERNATING DIRECTION METHOD FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS WITH SEPARABLE STRUCTURE

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Abstract. In this paper, we propose a new modified alternating direction method for solving a class of variational inequalities with separable structure. The proposed method uses a new searching direction which differs from the others existing in the literature. The global convergence of the proposed method is studied under certain assumptions. Several special cases are also discussed. The results presented in this paper extend and improve some well-known results in the literature. We also reported some numerical results to illustrate the effectiveness and superiority of the proposed method.
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1. Introduction

We consider the following constrained convex programming problem with separate structure:

$$\min \{\theta_1(x) + \theta_2(y) : Ax + By = b, x \in \mathcal{X}, y \in \mathcal{Y}\}, \tag{1.1}$$

where $\theta_1 : \mathcal{R}^{n_1} \to \mathcal{R}$ and $\theta_2 : \mathcal{R}^{n_2} \to \mathcal{R}$ are convex functions, $A \in \mathcal{R}^{l\times n_1}$ and $B \in \mathcal{R}^{l\times n_2}$ are given matrices and $b \in \mathcal{R}^l$ is a given vector; $\mathcal{X} \subset \mathcal{R}^{n_1}, \mathcal{Y} \subset \mathcal{R}^{n_2}$ are closed convex sets.

A large number of problems can be modeled as problem (1.1). In practice, these classes of problems have very large size and due to their practical importance, they have received a great deal of attention from many researchers. Various methods have been suggested to find the solution of problem (1.1). A popular approach is the alternating direction method (ADM) which was proposed by Gabay and Mercier

[8, 9]. The ADM can reduce the scale of variational inequalities by decomposing the original problem into a series of subproblems with a lower scale. To make the ADM more efficient and practical, some strategies have been studied; For further details, we refer [6, 7, 11, 13, 15, 16] and the references therein.

Let $\partial(.)$ denote the subgradient operator of a convex function, and $f(x) \in \partial\theta_1(x)$ and $g(y) \in \partial\theta_2(y)$ are the subgradient of $\theta_1(x)$ and $\theta_2(y)$, respectively. By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^l$ to the linear constraint $Ax + By = b$, problem (1.1) can be written in terms of finding $w \in \mathcal{W}$ such that

$$(w' - w)^\top Q(w) \geq 0, \quad \forall\, w' \in \mathcal{W}, \tag{1.2}$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} \quad Q(w) = \begin{pmatrix} f(x) - A^\top\lambda \\ g(y) - B^\top\lambda \\ Ax + By - b \end{pmatrix}, \quad \mathcal{W} = \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l. \tag{1.3}$$

The problem (1.2)–(1.3) is referred as *structured variational inequality problem* (in short, SVIP).

The alternating direction method (ADM) for solving the structured problem (1.2)–(1.3) was proposed by Gabay and Mercier [8, 9]. They decomposed the original problem into a series of subproblems with lower scale. This method appears to be one of the most powerful methods. For ADM with logarithmic-quadratic proximal regularization, see, [1, 2, 3, 4, 5, 14, 15, 17]. To make the ADM more efficient and practical, some strategies have been studied, see, for example, [6, 7, 11, 12, 15, 16].

In [10] is proposed the following algorithm: For a given

$$w^k = (x^k, y^k, \lambda^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l,$$

the predictor $(\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ is obtained via solving the following variational inequalities:

$$(x' - x)^\top (f(x) - A^\top[\lambda^k - H(Ax + By^k - b)]) \geq 0, \tag{1.4}$$

$$(y' - y)^\top (g(y) - B^\top[\lambda^k - H(Ax^k + By - b)]) \geq 0, \tag{1.5}$$

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - b), \tag{1.6}$$

where $H \in \mathcal{R}^{l\times l}$ is symmetric positive definite and the new iterate

$$w^{k+1}(\alpha_k) = (x^{k+1}, y^{k+1}, \lambda^{k+1})$$

is given by:

$$w^{k+1}(\alpha_k) = w^k - \alpha_k G^{-1} M(w^k - \tilde{w}^k),$$

where

$$G = \begin{pmatrix} A^\top HA & 0 & 0 \\ 0 & B^\top HB & 0 \\ 0 & 0 & H^{-1} \end{pmatrix}.$$

Jiang and Yuan [12] proposed a new parallel descent-like method for solving a class of variational inequalities with separate structure by using the same predictor as in [10] and the new iterate $w^{k+1}(\alpha_k) = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ is given by:

$$w^{k+1}(\alpha_k) = P_{\mathcal{W}}[w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k)],$$

where

$$d(w^k, \tilde{w}^k) = \begin{pmatrix} f(\tilde{x}^k) - A^\top \tilde{\lambda}^k + A^\top H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ g(\tilde{y}^k) - B^\top \tilde{\lambda}^k + B^\top H(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ A\tilde{x}^k + B\tilde{y}^k - b \end{pmatrix}.$$

Very recently, Wang et al. [16] proposed a new parallel splitting descent method for solving a class of variational inequalities with separable structure by combining the descent directions used by He [10] and Jiang and Yuan [12].

Inspired by the above cited works and by the recent work going on in this direction, we propose a descent alternating direction method for SVIP. Each iteration of the above method contains a prediction and a correction, the predictor is obtained via solving two subvariational inequalities at each iteration and the new iteration by searching the optimal step size along the integrated descent direction from two descent directions. Global convergence of the proposed method is proved under certain assumptions. The proposed method is quite general and flexible and includes several known alternating direction methods for solving variational inequalities with separable structure.

2. Alternating direction method

The following lemma provides some basic properties of the projection.

Lemma 2.1. *Let G be a symmetry positive definite matrix and Ω be a nonempty closed convex subset of R^l, we denote by $P_{\Omega,G}(.)$ the projection under the G-norm, that is,*

$$P_{\Omega,G}(v) = argmin\{\|v-u\|_G \ : \ u \in \Omega\}.$$

Then, we have the following inequalities.

$$(z - P_{\Omega,G}[z])^\top G(P_{\Omega,G}[z] - v) \geq 0, \quad \forall\, z \in \mathcal{R}^l, v \in \Omega; \tag{2.1}$$

$$\|P_{\Omega,G}[u] - P_{\Omega,G}[v]\|_G \leq \|u - v\|_G, \quad \forall\, u, v \in \mathcal{R}^l; \tag{2.2}$$

$$\|u - P_{\Omega,G}[z]\|_G^2 \leq \|z - u\|_G^2 - \|z - P_{\Omega,G}[z]\|_G^2, \quad \forall\, z \in \mathcal{R}^l,\ u \in \Omega. \tag{2.3}$$

We make the following standard assumptions.

Assumption A. f is monotone on $\mathcal{X}$, that is, $(f(x) - f(y))^\top (x - y) \geq 0$ for all $x, y \in \mathcal{X}$ and g is monotone on $\mathcal{Y}$.

Assumption B. The solution set of SVIP, denoted by $\mathcal{W}^*$, is nonempty.

We propose the following alternating direction method for solving SVIP:

Algorithm 2.2.
Step 0. *The initial step:*
Given $\varepsilon > 0$, $\tau \in \left(\frac{\sqrt{2}}{2}, 1\right)$, $\beta_1 \geq 0, \beta_2 \geq 0$ $(\beta_1 + \beta_2 > 0)$ and

$$w^0 = (x^0, y^0, \lambda^0) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l.$$

Set $k = 0$.

Step 1. *Prediction step:*
Compute $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$ by solving the following variational inequalities:

$$(x' - x)^\top (f(x) - A^\top[\lambda^k - H(Ax + By^k - b)] + R(x - x^k)) \geq 0, \quad \forall x' \in \mathcal{X}, \tag{2.4}$$

$$(y' - y)^\top (g(y) - B^\top[\lambda^k - H(Ax^k + By - b)] + S(y - y^k)) \geq 0, \quad \forall y' \in \mathcal{Y}, \tag{2.5}$$

$$\tilde{\lambda}^k = \lambda^k - \tau H(A\tilde{x}^k + B\tilde{y}^k - b) \tag{2.6}$$

where $H \in \mathcal{R}^{m\times m}$, $R \in \mathcal{R}^{n_1\times n_1}$ and $S \in \mathcal{R}^{n_2\times n_2}$ are symmetric positive definite matrices.
Step 2. *Convergence verification:*
If $\max\{\|x^k - \tilde{x}^k\|_\infty, \|y^k - \tilde{y}^k\|_\infty, \|\lambda^k - \tilde{\lambda}^k\|_\infty\} < \varepsilon$, then stop.
Step 3. *Correction step:*
The new iterate $w^{k+1}(\alpha_k) = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ is given by:

$$w^{k+1}(\alpha_k) = P_{\mathcal{W}}[w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k)], \tag{2.7}$$

where

$$\alpha_k = \frac{\varphi_k}{(\beta_1 + \beta_2)\|w^k - \tilde{w}^k\|_G^2}, \tag{2.8}$$

$$\varphi_k = \|w^k - \tilde{w}^k\|_G^2 + \frac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right), \tag{2.9}$$

$$d(w^k, \tilde{w}^k) = \beta_1 D(w^k, \tilde{w}^k) + \beta_2 G(w^k - \tilde{w}^k),$$

$$D(w^k, \tilde{w}^k) = \begin{pmatrix} f(\tilde{x}^k) - A^\top\tilde{\lambda}^k + A^\top H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \frac{1-\tau}{\tau}H^{-1}(\lambda^k - \tilde{\lambda}^k)] \\ g(\tilde{y}^k) - B^\top\tilde{\lambda}^k + B^\top H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \frac{1-\tau}{\tau}H^{-1}(\lambda^k - \tilde{\lambda}^k)] \\ A\tilde{x}^k + B\tilde{y}^k - b \end{pmatrix} \tag{2.10}$$

and

$$G = \begin{pmatrix} R + A^\top HA & 0 & 0 \\ 0 & S + B^\top HB & 0 \\ 0 & 0 & \frac{1}{\tau}H^{-1} \end{pmatrix}.$$

Set $k := k + 1$ and go to Step 1.

Remark 2.3. As special cases of our method, we can obtain some alternating direction methods.

- If $\tau = 1$ and $R = S = 0$, we obtain the method proposed by Wang et al. [16].
- If $\tau = 1, \beta_1 = 0, \beta_2 = 1$ and $R = S = 0$, we obtain the method proposed by He [10].
- If $\tau = 1, \beta_1 = 1, \beta_2 = 0$ and $R = S = 0$, we obtain the method proposed by Jiang and yuan [12].

Remark 2.4. It is easy to check that $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ is solution of SVIP if and only if

$$\begin{cases} x^k - \tilde{x}^k = 0, \\ y^k - \tilde{y}^k = 0, \\ \lambda^k - \tilde{\lambda}^k = 0. \end{cases}$$

Thus, it is reasonable to take the magnitude

$$\max\{\|x^k - \tilde{x}^k\|_\infty, \|y^k - \tilde{y}^k\|_\infty, \|\lambda^k - \tilde{\lambda}^k\|_\infty\} < \varepsilon,$$

as the stopping criterion.

In the next theorem, we show that α_k is lower bounded away from zero. It is useful to study the convergence analysis of the proposed method.

Theorem 2.5. *For given $w^k \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$, let $\tilde{w}^k$ be generated by (2.4)-(2.6). Then,*

$$\varphi_k \geq \frac{2\tau - \sqrt{2}}{2\tau}\|w^k - \tilde{w}^k\|_G^2, \tag{2.11}$$

and

$$\alpha_k \geq \frac{2\tau - \sqrt{2}}{2\tau}. \tag{2.12}$$

Proof. It follows from (2.9) that

$$\begin{aligned} \varphi_k &= \|w^k - \tilde{w}^k\|_G^2 + \frac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top (A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) \\ &= \|x^k - \tilde{x}^k\|_R^2 + \|Ax^k - A\tilde{x}^k\|_H^2 + \|y^k - \tilde{y}^k\|_S^2 + \|By^k - B\tilde{y}^k\|_H^2 \\ &\quad + \frac{1}{\tau}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 + \frac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top (A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)). \end{aligned} \tag{2.13}$$

By using the Cauchy-Schwarz inequality, we have

$$(\lambda^k - \tilde{\lambda}^k)^\top (A(x^k - \tilde{x}^k)) \geq -\frac{1}{2}\left(\sqrt{2}\|A(x^k - \tilde{x}^k)\|_H^2 + \frac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right), \tag{2.14}$$

and

$$(\lambda^k - \tilde{\lambda}^k)^\top (B(y^k - \tilde{y}^k)) \geq -\frac{1}{2}\left(\sqrt{2}\|B(y^k - \tilde{y}^k)\|_H^2 + \frac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13), we get

$$\begin{aligned} \varphi_k &\geq \frac{2\tau - \sqrt{2}}{2\tau}\left(\|Ax^k - A\tilde{x}^k\|_H^2 + \|By^k - B\tilde{y}^k\|_H^2\right) + \frac{2 - \sqrt{2}}{2\tau}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\ &\quad + \|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2 \\ &\geq \frac{2\tau - \sqrt{2}}{2\tau}\left(\|Ax^k - A\tilde{x}^k\|_H^2 + \|By^k - B\tilde{y}^k\|_H^2 + \frac{1}{\tau}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\right) \\ &\quad + \frac{2\tau - \sqrt{2}}{2\tau}\left(\|x^k - \tilde{x}^k\|_R^2 + \|y^k - \tilde{y}^k\|_S^2\right) \\ &\geq \frac{2\tau - \sqrt{2}}{2\tau}\|w^k - \tilde{w}^k\|_G^2. \end{aligned}$$

Therefore, it follows from (2.8) and (2.11) that

$$\alpha_k \geq \frac{2\tau - \sqrt{2}}{2\tau}, \tag{2.16}$$

and this completes the proof. □

3. Basic results

We establish some basic properties, which will be used to prove the sufficient and necessary conditions for the convergence of the proposed method.

Lemma 3.1. *For given $w^k = (x^k, y^k, \lambda^k) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^l$, let $\tilde{w}^k$ be generated by (2.4)–(2.6). Then for any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, we have*

$$\begin{aligned}&(w^k - w^*)^\top G(w^k - \tilde{w}^k)\\ &\geq \|w^k - \tilde{w}^k\|_G^2 + \frac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&(w^{k+1}(\alpha_k) - \tilde{w}^k)^\top D(w^k, \tilde{w}^k)\\ &\geq (w^{k+1}(\alpha_k) - w^k)^\top G(w^k - \tilde{w}^k) + \|w^k - \tilde{w}^k\|_G^2.\end{aligned} \tag{3.2}$$

Proof. By setting $x' = x^*$ in (2.4), we get

$$\begin{aligned}&(x^* - \tilde{x}^k)^\top \Big\{ f(\tilde{x}^k) - A^\top \tilde{\lambda}^k - A^\top H A(x^k - \tilde{x}^k)\\ &+ A^\top H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \frac{1-\tau}{\tau} H^{-1}(\lambda^k - \tilde{\lambda}^k)] - R(x^k - \tilde{x}^k)\Big\} \geq 0.\end{aligned} \tag{3.3}$$

Substituting $y' = y^*$ in (2.5), we obtain

$$\begin{aligned}&(y^* - \tilde{y}^k)^\top \Big\{ g(\tilde{y}^k) - B^\top \tilde{\lambda}^k - B^\top H B(y^k - \tilde{y}^k)\\ &+ B^\top H[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \frac{1-\tau}{\tau} H^{-1}(\lambda^k - \tilde{\lambda}^k)] - S(y^k - \tilde{y}^k)\Big\} \geq 0.\end{aligned} \tag{3.4}$$

Since (x^*, y^*, λ^*) is a solution of SVIP, $\tilde{x}^k \in \mathcal{X}$ and $\tilde{y}^k \in \mathcal{Y}$, we have

$$(\tilde{x}^k - x^*)^\top (f(x^*) - A^\top \lambda^*) \geq 0,$$

$$(\tilde{y}^k - y^*)^\top (g(y^*) - B^\top \lambda^*) \geq 0$$

and

$$Ax^* + By^* - b = 0.$$

Using the monotonicity of f and g, we obtain

$$\begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{\lambda}^k - \lambda^* \end{pmatrix}^\top \begin{pmatrix} f(\tilde{x}^k) - A^\top \tilde{\lambda}^k \\ g(\tilde{y}^k) - B^\top \tilde{\lambda}^k \\ A\tilde{x}^k + B\tilde{y}^k - b \end{pmatrix} \geq \begin{pmatrix} \tilde{x}^k - x^* \\ \tilde{y}^k - y^* \\ \tilde{\lambda}^k - \lambda^* \end{pmatrix}^\top \begin{pmatrix} f(x^*) - A^\top \lambda^* \\ g(y^*) - B^\top \lambda^* \\ Ax^* + By^* - b \end{pmatrix} \geq 0. \tag{3.5}$$

Adding (3.3), (3.4) and (3.5), we get

$$
\begin{aligned}
& (w^* - \tilde{w}^k)^\top G(w^k - \tilde{w}^k) \\
= \;& (x^* - \tilde{x}^k)^\top (R(x^k - \tilde{x}^k) + A^\top HA(x^k - \tilde{x}^k)) + (y^* - \tilde{y}^k)^\top (S(y^k - \tilde{y}^k) \\
& + B^\top HB(y^k - \tilde{y}^k)) + (\lambda^* - \tilde{\lambda}^k)^\top \left(A\tilde{x}^k + B\tilde{y}^k - b\right) \\
\le \;& (x^* - \tilde{x}^k)^\top A^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \tfrac{1-\tau}{\tau} H^{-1}(\lambda^k - \tilde{\lambda}^k)\right) \\
& + (y^* - \tilde{y}^k)^\top B^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \tfrac{1-\tau}{\tau} H^{-1}(\lambda^k - \tilde{\lambda}^k)\right) \\
= \;& -(A\tilde{x}^k + B\tilde{y}^k - b)^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) \\
& - \tfrac{1-\tau}{\tau}(A\tilde{x}^k + B\tilde{y}^k - b)^\top (\lambda^k - \tilde{\lambda}^k) \\
= \;& -\tfrac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right) - \tfrac{1-\tau}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|^2_{H^{-1}} \\
\le \;& -\tfrac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right)
\end{aligned}
\tag{3.6}
$$

where the last equality follows from (2.6). It follows from (3.6) that

$$
(w^k - w^*)^\top G(w^k - \tilde{w}^k) \ge \|w^k - \tilde{w}^k\|^2_G + \frac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)\right),
$$

and the first assertion of this lemma is proved.
As in (3.3) and (3.4), we have

$$
\begin{aligned}
& (x^{k+1} - \tilde{x}^k)^\top \Big\{ R(x^k - \tilde{x}^k) - f(\tilde{x}^k) + A^\top \tilde{\lambda}^k + A^\top HA(x^k - \tilde{x}^k) \\
& \qquad - A^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \frac{1-\tau}{\tau} H^{-1}(\lambda^k - \tilde{\lambda}^k) \right) \Big\} \le 0,
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
& (y^{k+1} - \tilde{y}^k)^\top \Big\{ S(y^k - \tilde{y}^k) - g(\tilde{y}^k) + B^\top \tilde{\lambda}^k + B^\top HB(y^k - \tilde{y}^k) \\
& \qquad - B^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)) + \frac{1-\tau}{\tau} H^{-1}(\lambda^k - \tilde{\lambda}^k) \right) \Big\} \le 0
\end{aligned}
\tag{3.8}
$$

It follows from (3.7) and (3.8) that

$$
(w^{k+1}(\alpha_k) - \tilde{w}^k)^\top (G(w^k - \tilde{w}^k) - D(w^k, \tilde{w}^k)) \le 0.
$$

By simple manipulation, we obtain

$$
\begin{aligned}
(w^{k+1}(\alpha_k) - \tilde{w}^k)^\top D(w^k, \tilde{w}^k) \;&\ge\; (w^{k+1}(\alpha_k) - \tilde{w}^k)^\top G(w^k - \tilde{w}^k) \\
&=\; (w^{k+1}(\alpha_k) - w^k)^\top G(w^k - \tilde{w}^k) + \|w^k - \tilde{w}^k\|^2_G
\end{aligned}
$$

and the second assertion of this lemma is proved. □

The following theorem provides a unified framework for proving the convergence of the proposed algorithm.

Theorem 3.2. *Let* $w^* \in \mathcal{W}^*, w^{k+1}(\alpha_k)$ *be defined by (2.7) and*

$$
\Theta(\alpha_k) := \|w^k - w^*\|^2_G - \|w^{k+1}(\alpha_k) - w^*\|^2_G, \tag{3.9}
$$

then

$$
\begin{aligned}
\Theta(\alpha_k) \ge \;& \|w^k - w^{k+1}(\alpha_k) - \alpha_k(\beta_1 + \beta_2)(w^k - \tilde{w}^k)\|^2_G \\
& + 2\alpha_k(\beta_1 + \beta_2)\varphi_k - \alpha_k^2(\beta_1 + \beta_2)^2 \|w^k - \tilde{w}^k\|^2_G.
\end{aligned}
\tag{3.10}
$$

Proof. Since $w^* \in \mathcal{W}^*$ and $w^{k+1}(\alpha_k) = P_{\mathcal{W}}[w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k)]$, it follows from (2.3) that

$$\begin{aligned} \|w^{k+1}(\alpha_k) - w^*\|_G^2 \leq \|w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k) - w^*\|_G^2 \\ - \|w^k - \alpha_k G^{-1} d(w^k, \tilde{w}^k) - w^{k+1}(\alpha_k)\|_G^2. \end{aligned} \tag{3.11}$$

Using the definition of $\Theta(\alpha_k)$ and (3.11), we get

$$\begin{aligned} \Theta(\alpha_k) \geq \|w^k - w^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k (w^{k+1}(\alpha_k) - w^k)^\top d(w^k, \tilde{w}^k) \\ + 2\alpha_k (w^k - w^*)^\top d(w^k, \tilde{w}^k). \end{aligned} \tag{3.12}$$

It follows from (3.5) that

$$\begin{aligned} & (\tilde{w}^k - w^*)^\top D(w^k, \tilde{w}^k) \\ \geq \quad & (\tilde{w}^k - w^*)^\top \begin{pmatrix} A^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \frac{1-\tau}{\tau} H^{-1}(\lambda^k - \tilde{\lambda}^k) \right) \\ B^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \frac{1-\tau}{\tau} H^{-1}(\lambda^k - \tilde{\lambda}^k) \right) \\ 0 \end{pmatrix} \\ = \quad & (A\tilde{x}^k + B\tilde{y}^k - b)^\top H \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) \right) + \frac{1-\tau}{\tau^2} \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2 \\ \geq \quad & \frac{1}{\tau} (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) \right). \end{aligned}$$

Thus,

$$\begin{aligned} (w^k - w^*)^\top D(w^k, \tilde{w}^k) \geq (w^k - \tilde{w}^k)^\top D(w^k, \tilde{w}^k) \\ + \frac{1}{\tau} (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) \right). \end{aligned} \tag{3.13}$$

Applying (3.1) and (3.13) to the last term on the right side of (3.12), we obtain

$$\begin{aligned} \Theta(\alpha_k) \quad \geq \quad & \|w^k - w^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k (w^{k+1}(\alpha_k) - w^k)^\top d(w^k, \tilde{w}^k) \\ & +2\alpha_k \{ \beta_1 (w^k - \tilde{w}^k)^\top D(w^k, \tilde{w}^k) \\ & + \tfrac{(\beta_1 + \beta_2)}{\tau} (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) \right) \\ & + \beta_2 \|w^k - \tilde{w}^k\|_G^2 \} \\ = \quad & \|w^k - w^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k \beta_1 (w^{k+1}(\alpha_k) - \tilde{w}^k)^\top D(w^k, \tilde{w}^k) \\ & + 2\alpha_k \beta_2 (w^{k+1}(\alpha_k) - w^k)^\top G(w^k - \tilde{w}^k) \\ & + \tfrac{2\alpha_k(\beta_1 + \beta_2)}{\tau} (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) \right) \\ & + 2\alpha_k \beta_2 \|w^k - \tilde{w}^k\|_G^2. \end{aligned} \tag{3.14}$$

Applying (3.2) to the second term in the right side of (3.14) and using the notation of φ_k in (2.9), we get

$$\begin{aligned} \Theta(\alpha_k) \geq & \|w^k - w^{k+1}(\alpha_k)\|_G^2 + 2\alpha_k(\beta_1 + \beta_2)(w^{k+1}(\alpha_k) - w^k)^\top G(w^k - \tilde{w}^k) \\ & + 2\alpha_k(\beta_1 + \beta_2) \left[\|w^k - \tilde{w}^k\|_G^2 + + \frac{1}{\tau} (\lambda^k - \tilde{\lambda}^k)^\top \left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) \right) \right] \\ & = \|w^k - w^{k+1}(\alpha_k) - \alpha_k(\beta_1 + \beta_2)(w^k - \tilde{w}^k)\|_G^2 - \alpha_k^2 (\beta_1 + \beta_2)^2 \|w^k - \tilde{w}^k\|_G^2 \\ & + 2\alpha_k(\beta_1 + \beta_2)\varphi_k \end{aligned}$$

and the theorem is proved. □

From the computational point of view, a relaxation factor $\gamma \in (0, 2)$ is preferable in the correction. We are now in a position to prove the contractive property of the iterative sequence.

Theorem 3.3. *Let $w^* \in \mathcal{W}^*$ be a solution of SVIP and let $w^{k+1}(\gamma\alpha_k)$ be generated by (2.7). Then w^k and $\tilde{w}^k$ are bounded, and*

$$\|w^{k+1}(\gamma\alpha_k) - w^*\|_G^2 \leq \|w^k - w^*\|_G^2 - c\|w^k - \tilde{w}^k\|_G^2, \tag{3.15}$$

where

$$c := \frac{\gamma(2-\gamma)(2\tau - \sqrt{2})^2}{4\tau^2} > 0.$$

Proof. It follows from (3.10), (2.11) and (2.12) that

$$\begin{aligned} & \|w^{k+1}(\gamma\alpha_k) - w^*\|_G^2 \\ \leq\ & \|w^k - w^*\|_G^2 - 2\gamma\alpha_k(\beta_1 + \beta_2)\varphi_k + \gamma^2\alpha_k^2(\beta_1 + \beta_2)^2\|w^k - \tilde{w}^k\|_G^2 \\ =\ & \|w^k - w^*\|_G^2 - \gamma(2-\gamma)(\beta_1 + \beta_2)\alpha_k\varphi_k \\ \leq\ & \|w^k - w^*\|_G^2 - \tfrac{\gamma(2-\gamma)(2\tau-\sqrt{2})^2}{4\tau^2}\|w^k - \tilde{w}^k\|_G^2. \end{aligned}$$

Since $\gamma \in (0, 2)$, we have

$$\|w^{k+1}(\alpha_k) - w^*\|_G \leq \|w^k - w^*\|_G \leq \cdots \leq \|w^0 - w^*\|_G,$$

and thus, $\{w^k\}$ is a bounded sequence.

It follows from (3.15) that

$$\sum_{k=0}^{\infty} c\|w^k - \tilde{w}^k\|_G^2 < +\infty,$$

which implies that

$$\lim_{k\to\infty} \|w^k - \tilde{w}^k\|_G = 0. \tag{3.16}$$

Since $\{w^k\}$ is a bounded sequence, we conclude that $\{\tilde{w}^k\}$ is also bounded. □

Now, we are ready to prove the convergence of the proposed method.

Theorem 3.4. *The sequence $\{w^k\}$ generated by the proposed method converges to some w^∞ which is a solution of SVIP.*

Proof. It follows from (3.16) that

$$\lim_{k\to\infty} \|x^k - \tilde{x}^k\|_R = 0, \qquad \lim_{k\to\infty} \|y^k - \tilde{y}^k\|_S = 0 \tag{3.17}$$

and

$$\lim_{k\to\infty} \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}} = \lim_{k\to\infty} \|A\tilde{x}^k + B\tilde{y}^k - b\|_H = 0. \tag{3.18}$$

Moreover, (2.4) and (2.5) imply that

$$\begin{aligned} (x - \tilde{x}^k)^\top (f(\tilde{x}^k) - A^\top\tilde{\lambda}^k) &\geq (x^k - \tilde{x}^k)^\top R(x - \tilde{x}^k) + (x - \tilde{x}^k)^\top A^\top HA(x^k - \tilde{x}^k) \\ &\quad - (x - \tilde{x}^k)^\top A^\top H\left(A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k) + \frac{1-\tau}{\tau}H^{-1}(\lambda^k - \tilde{\lambda}^k)\right), \end{aligned} \tag{3.19}$$

and

$$(y-\tilde{y}^k)^\top(g(\tilde{y}^k)-B^\top\tilde{\lambda}^k) \geq (y^k-\tilde{y}^k)^\top S(y-\tilde{y}^k) + (y-\tilde{y}^k)^\top B^\top HB(y^k-\tilde{y}^k) \\ -(y-\tilde{y}^k)^\top B^\top H\left(A(x^k-\tilde{x}^k)+B(y^k-\tilde{y}^k)+\frac{1-\tau}{\tau}H^{-1}(\lambda^k-\tilde{\lambda}^k)\right). \quad (3.20)$$

We deduce from (3.17) and (3.18) that

$$\begin{cases} \lim_{k\to\infty}(x-\tilde{x}^k)^\top\{f(\tilde{x}^k)-A^\top\tilde{\lambda}^k\} \geq 0, & \forall x\in\mathcal{X}, \\ \lim_{k\to\infty}(y-\tilde{y}^k)^\top\{g(\tilde{y}^k)-B^\top\tilde{\lambda}^k\} \geq 0, & \forall y\in\mathcal{Y}. \end{cases} \quad (3.21)$$

Since $\{w^k\}$ is bounded, it has at least one cluster point. Let w^∞ be a cluster point of $\{w^k\}$ and the subsequence $\{w^{k_j}\}$ converges to w^∞. Since $\mathcal{W}$ is a closed set, we have $w^\infty \in \mathcal{W}$. It follows from (3.18) and (3.21) that

$$\begin{cases} \lim_{j\to\infty}(x-x^{k_j})^\top\{f(x^{k_j})-A^\top\lambda^{k_j}\} \geq 0, & \forall x\in\mathcal{X}, \\ \lim_{j\to\infty}(y-y^{k_j})^\top\{g(y^{k_j})-B^\top\lambda^{k_j}\} \geq 0, & \forall y\in\mathcal{Y}, \\ \lim_{j\to\infty}(Ax^{k_j}+By^{k_j}-b)=0, \end{cases}$$

and consequently,

$$\begin{cases} (x-x^\infty)^\top\{f(x^\infty)-A^\top\lambda^\infty\} \geq 0, & \forall x\in\mathcal{X}, \\ (y-y^\infty)^\top\{g(y^\infty)-B^\top\lambda^\infty\} \geq 0, & \forall y\in\mathcal{Y}, \\ Ax^\infty+By^\infty-b=0, \end{cases}$$

which means that w^∞ is a solution of SVIP.
Now we prove that the sequence $\{w^k\}$ converges to w^∞. Since

$$\lim_{k\to\infty}\|w^k-\tilde{w}^k\|_G=0, \quad \text{and} \quad \{\tilde{w}^{k_j}\}\to w^\infty,$$

for any $\varepsilon>0$, there exists $l>0$ such that

$$\|\tilde{w}^{k_l}-w^\infty\|_G<\frac{\varepsilon}{2} \quad \text{and} \quad \|w^{k_l}-\tilde{w}^{k_l}\|_G<\frac{\varepsilon}{2}.$$

Therefore, for any $k\geq k_l$, it follows from (3.15) and (3) that

$$\|w^k-w^\infty\|_G \leq \|w^{k_l}-w^\infty\|_G \leq \|w^{k_l}-\tilde{w}^{k_l}\|_G+\|\tilde{w}^{k_l}-w^\infty\|_G<\varepsilon.$$

This implies that the sequence $\{w^k\}$ converges to w^∞ which is a solution of SVIP. □

4. Computational results

Let H_L, H_U and C be given $n\times n$ symmetric matrices. In order to verify the theoretical assertions, we consider the following optimization problem with matrix variables:

$$\min\left\{\frac{1}{2}\|X-C\|_F^2 : X\in S_+^n\cap\mathcal{B}\right\}, \quad (4.1)$$

where

$$S_+^n=\left\{H\in\mathcal{R}^{n\times n}: H^\top=H,\ H\succeq 0\right\}$$

and

$$\mathcal{B} = \left\{ H \in \mathcal{R}^{n \times n} : H^\top = H,\ H_L \le H \le H_U \right\}.$$

The matrices H_L and H_U are given by:

$$(H_U)_{jj} = (H_L)_{jj} = 1,\ \text{and}\ (H_U)_{ij} = -(H_L)_{ij} = 0.1, \quad \forall i \neq j, i, j = 1, 2, \dots, n.$$

Note that the problem (4.1) is equivalent to the following minimization problem:

$$\begin{aligned} &\min \left\{ \tfrac{1}{2}\|X - C\|^2 + \tfrac{1}{2}\|Y - C\|^2 \right\} \\ &\text{such that} \quad X - Y = 0, \\ &X \in S^n_+,\ Y \in \mathcal{B}. \end{aligned} \tag{4.2}$$

By attaching a Lagrange multiplier $Z \in \mathcal{R}^{n \times n}$ to the linear constraint $X - Y = 0$, then the Lagrange function of (4.2) is

$$L(X, Y, Z) = \frac{1}{2}\|X - C\|^2 + \frac{1}{2}\|Y - C\|^2 - \langle Z, X - Y \rangle,$$

which is defined on $S^n_+ \times \mathcal{B} \times \mathcal{R}^{n \times n}$. If $(X^*, Y^*, Z^*) \in S^n_+ \times \mathcal{B} \times \mathcal{R}^{n \times n}$ is a KKT point of (4.2), then (4.2) can be converted to the following variational inequality problem: Find $u^* = (X^*, Y^*, Z^*) \in \mathcal{W} = S^n_+ \times \mathcal{B} \times \mathcal{R}^{n \times n}$ such that

$$\begin{cases} \langle X - X^*, (X^* - C) - Z^* \rangle \ge 0, \\ \langle Y - Y^*, (Y^* - C) + Z^* \rangle \ge 0, \quad \forall\, u = (X, Y, Z) \in \mathcal{W}, \\ X^* - Y^* = 0. \end{cases} \tag{4.3}$$

Problem (4.3) is a special case of (1.2)–(1.3) with matrix variables, where $A = I_{n \times n}$, $B = -I_{n \times n}$, $b = 0$, $f(X) = X - C$, $g(Y) = Y - C$ and $\mathcal{W} = S^n_+ \times \mathcal{B} \times \mathcal{R}^{n \times n}$.

For simplification, we take $R = rI_{n \times n}$, $S = sI_{n \times n}$ and $H = I_{n \times n}$, where $r > 0$ and $s > 0$ are scalars. In all the tests, we take $\gamma = 1.8$, $\tau = 0.87$, $\beta_1 = 0.01$, $\beta_2 = 0.01$, $C = \text{rand}(n)$ and $(X^0, Y^0, Z^0) = (I_{n \times n}, I_{n \times n}, 0_{n \times n})$ as the initial point, and $r = 0.5$, $s = 5$. The iteration is stopped as soon as

$$\max \left\{ \|X^k - \tilde{X}^k\|, \|Y^k - \tilde{Y}^k\|, \|Z^k - \tilde{Z}^k\| \right\} \le 10^{-6}.$$

All codes were written in Matlab. We compare the proposed method with those in [16], [10] and [12]. The numerical results for problem (4.1) with different dimensions are given in Table 1, which demonstrates that the proposed algorithm is effective and reliable in practice.

Table 1. Numerical results for the problem (4.1)

Dimension of the problem	The proposed method		The method in [16]		The method in [12]		The method in [10]	
	k	CPU(Sec.)	k	CPU(Sec.)	k	CPU(Sec.)	k	CPU(Sec.)
100	34	0.0.42	80	0.74	80	0.95	83	0.81
200	64	3.43	105	5.29	117	6.24	128	6.04
300	96	16.67	172	25.66	178	37.26	183	19.29
400	132	43.19	238	73.96	244	84.21	246	51.71
500	176	104.59	307	183.51	309	188.21	313	159.26
600	220	216.66	371	334.12	384	507.21	397	366.45

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