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PICARD SEQUENCES IN b-METRIC SPACES

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Abstract. The purpose of this article is to provide much simpler and shorter proofs of some important results in the framework of b-metric spaces. Namely, we show that the given contractive condition implies b-Cauchyness of the corresponding Picard sequence. The obtained results improve well-known comparable results in the literature.
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1. Introduction

Banach's fixed point theorem is the most celebrated result in Nonlinear analysis. It shows that in a complete metric space, each contractive mapping has a unique fixed point. This principle is used as a main tool for the existence of solution of many non-linear problems. A great number of generalizations of this famous result appear in the literature. On the one side, the usual contractive condition is replaced by a weakly contractive condition, while, on the other side, the action space is replaced by some generalization of standard metric space (b-metric space, partial metric space, G-metric space, etc.).

A fundamental role in the foundations of the constructions is played by the fixed point theorems in metric spaces. They have been intensively studied for quite some time. The Banach fixed point theorem provides a technique for solving variety problems in mathematical science and engineering.

It is worth notice that S. Banach proved his famous result in 1922 (see [8]). Other important year for fixed point theory is probably 1926 (see [12], [39]). The classical Denjoy-Wolff theorem asserts that all orbits of a fixed point of holomorphic mapping $f : \mathbb{D} \to \mathbb{D}$ on the open unit disc $\mathbb{D} \subset \mathbb{C}$ converge to a unique point $\eta \in \partial\mathbb{D}$. Namely, this was the beginning of a research as well as an application of fixed point theory in complex analysis.

Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric on X if it satisfies: $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ and $d(x, z) \leq s(d(x, y) + d(y, z))$, for all $x, y, z \in X$. In this case, the pair (X, d) is called a b-metric space. Obviously, if $s = 1$, then b-metric space is usual metric space. For more notions such as b-convergence, b-completeness, b-Cauchy sequence in the framework of b-metric spaces, the reader is referred to [1]-[7], [9]-[11], [13], [3], [16], [18]-[20], [22]-[27], [29]-[31], [34]-[38], [40].

In this section we give a detailed overview of the various contraction conditions related to b-metric spaces.

For the convenience of the reader we repeat the relevant material from [1], thus making our exposition self-contained.

We follow the notation used in [1] and denote by $\mathcal{S}$ the set of all mappings $\gamma : [0, \infty) \to [0, 1)$ with the property $\gamma(t_n) \to 1$, as $t_n \to 0$. This set is non-empty since the function $f : [0, \infty) \to [0, 1)$ given by $f(x) = \frac{1}{1+x}$ belongs to $\mathcal{S}$.

The following contractive condition is considered in [1, Definition 1.20].

Definition 1.1. [1] Let $(X, \preceq, d)$ be a partially ordered b-metric space. Two mappings $f, g : X \to X$ are said to satisfy generalized b-order rational contractive condition if there exists $\beta \in S$ such that

$$s\, d(fx, gy) \leq \beta(d(x, y))M(x, y) + LN(x, y) \tag{1.1}$$

for all $x, y \in X$ with $x \preceq y$, where $L \geq 0$,

$$M(x, y) = \max\left\{ d(x, y), \frac{d(x, fx)\, d(y, gy)}{1 + d(fx, gy)} \right\},$$

$$N(x, y) = \min\{d(x, y), d(x, fx), d(x, gy), d(y, fx), d(y, gy)\}.$$

Here and subsequently, Ψ denotes the class of all nondecreasing mappings $\psi : [0, \infty) \to [0, \infty)$ satisfying $\lim_{n\to\infty} \psi^n(t) = 0$ whenever $t > 0$. If $\psi \in \Psi$, then $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$, as is easy to check (for details see [21]).

Definition 1.2. [1] Let $(X, \preceq, d)$ be a partially ordered b-metric space. Suppose that $f, g : X \to X$ are two weakly increasing mappings and there exists $x_0 \in X$ such that $fx_0 \preceq gfx_0$. Assume that there exists $\psi \in \Psi$ such that

$$s\, d(fx, gy) \leq \psi(M(x, y)) \tag{1.2}$$

holds, for all $x, y \in X$ with $x \preceq y$, where

$$M(x, y) = \max\left\{ d(x, y), \frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)} \right\}.$$

Then f and g are said to satisfy generalized b-order ψ-rational contractive condition.

Throughout the paper, Φ stands for the set of all functions $\varphi : [0, \infty) \to [0, \infty)$ with the following properties: φ is right continuous, nondecreasing, and $\varphi(t) < t$ for every $t > 0$.

To facilitate access to the main results, we repeat Theorem 2.2 and Corollary 2.4 from [5] without proofs.

Theorem 1.1. [5] *Let $(X, \preceq)$ be a partially ordered set and there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Suppose that $s > 1$ and $f : X \to X$ is an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Assume that there exists $\psi \in \Psi$ with the property that*

$$s\, d(fx, fy) \frac{1 + sd(x, y)}{1 + \frac{1}{2}d(x, fx)} \leq \psi(M(x, y)) + LN(x, y)$$

holds for all comparable elements $x, y \in X$, where $L \geq 0$,

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)\, d(y, fy)}{1 + d(fx, fy)}\right\}$$

and

$$N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\}.$$

If f is continuous, then f has a fixed point.

Corollary 1.1. [5] *Let $(X, \preceq)$ be a partially ordered set and there exists a b-metric d on X such that (X, d) is a b-complete b-metric space. Suppose that $f : X \to X$ is an increasing mapping with respect to $\preceq$ for which there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Assume that there exists $\beta \in \mathcal{S}$ such that*

$$\frac{1 + s\, d(x, y)}{1 + \frac{1}{2}d(x, fx)} \cdot d(fx, fy) \leq \beta(d(x, y))\, M(x, y) + LN(x, y)$$

holds for all comparable elements $x, y \in X$, where $L \geq 0$,

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, fx)\, d(y, fy)}{1 + d(fx, fy)}\right\}$$

and

$$N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx), d(y, fy)\}.$$

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Definition 1.3. Let (X, d, s) be a b-metric space. A mapping $f : X \to X$ is said to satisfy b-Hardy-Rogers contractive condition if the inequality

$$d(fx, fy) \leq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 [d(x, fy) + d(y, fx)]$$

holds, for all $x, y \in X$, where a_1, a_2, a_3, a_4 are nonnegative numbers with the property

$$sa_1 + sa_2 + a_3 + (s^2 + s)\, a_4 < 1.$$

The next definition is a generalization being analogous with that from [15].

Definition 1.4. Let (X, d, s) be a b-metric space. A mapping $f : X \to X$ is said to satisfy b-Hicks-Rhoades contractive condition if the inequality

$$d(fx, f^2x) \leq \mu d(x, fx),$$

holds, for some $\mu \in (0, 1)$, and $x \in O_f y = \{y, fy, f^2y, \ldots, f^n y, \ldots\}$, where $y \in X$.

Essential to the proofs of fixed point theorems for the previous contractive conditions are the following two lemmas.

Lemma 1.1. [19] *Let (X,d) be a b-metric space with $s \geq 1$, and $\{x_n\}$ a sequence in X such that* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. *If $\{x_n\}$ is not a b-Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences of positive integers $\{n(k)\}$ and $\{m(k)\}$ such that the following four sequences*

$$\{d(x_{m(k)}, x_{n(k)})\}, \{d(x_{m(k)}, x_{n(k)+1})\}, \{d(x_{m(k)+1}, x_{n(k)})\}, \{d(x_{m(k)+1}, x_{n(k)+1})\}$$

exist and satisfy

$$\varepsilon \leq \liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \leq \varepsilon s,$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq \limsup_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \leq \varepsilon s^2,$$
$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \leq \varepsilon s^2,$$
$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq \limsup_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) \leq \varepsilon s^3.$$

Lemma 1.2. *Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent with the limits x and y, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. *Moreover, for each $z \in X$, we have*

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s\, d(x,z).$$

Instead of using previous lemmas, in this paper, we will show that much more subtle and convenient is the next, recent result.

Lemma 1.3. [26, Lemma 2.2], [38, Lemma 6] *Let $\{x_n\}$ be a sequence in a b-metric space (X,d,s) such that*

$$d(x_n, x_{n+1}) \leq \mu d(x_{n-1}, x_n)$$

for some $\mu \in [0,1)$, and each $n \in \mathbb{N}$. Then $\{x_n\}$ is a b-Cauchy sequence in (X,d,s).

The next lemma can be compared with the previous.

Lemma 1.4. [27, Lemma 1.12] *Let (X,d) be a b-metric space and $\{x_n\}$ a sequence in X such that all x_n, $n \geq 0$, are different. Suppose that there exist real nonnegative numbers $\lambda \in [0,1)$ and c_1, c_2 such that the inequality*

$$d(x_m, x_n) \leq \lambda d(x_{m-1}, x_{n-1}) + c_1 \lambda^m + c_2 \lambda^n$$

holds, for all $m, n \in \mathbb{N}$. Then $\{x_n\}$ is a b-Cauchy sequence.

We will now show that the proofs of the most fixed point theorems in the context of b-metric spaces become simpler and shorter if they are based on Lemma 1.3.

2. Main results

Let (X, d) be a metric space and f a self-mapping on X satisfying some contractive condition. It is of interest to know whether the corresponding Picard sequence for f with initial point $x_0 \in X$ is a Cauchy sequence with respect to the contractive condition.

In order to obtain the existence of fixed points of some mapping f, in quite a few proofs, the basic idea is to consider the Picard sequence of f given by

$$\{x_{n+1}\}_{n\in\mathbb{N}\cup\{0\}} = \{fx_n\}_{n\in\mathbb{N}\cup\{0\}} = \{f^n x_0\}_{n\in\mathbb{N}\cup\{0\}}$$

where $f^0 x_0 = x_0$.

It is clear that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is not a Cauchy sequence, if $d(x_n, x_{n+1}) \nrightarrow 0$, as $n \to \infty$.

Most of all, the proof falls naturally into the next parts. It is firstly shown that the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies Jungck's contractive condition, i.e. for all $n \in \mathbb{N}$ (or $n \geq k$ for fixed $k \in \mathbb{N}$) and $\lambda \in [0, 1)$, holds the inequality

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n), \tag{2.1}$$

which implies

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \tag{2.2}$$

for all $n \in \mathbb{N}$ (or $n \geq k$ for fixed $k \in \mathbb{N}$) and $\lambda \in [0, 1)$. From this, it may be concluded that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a Cauchy sequence. Indeed, let $n, m \in \mathbb{N}$ and $n > m$. From (2.2), we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &\leq \left(\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-2} + \lambda^{n-1}\right) d(x_0, x_1) \\ &= \lambda^m \frac{1-\lambda^{n-m}}{1-\lambda} d(x_0, x_1) \leq \frac{\lambda^m}{1-\lambda} d(x_0, x_1) \to 0, \quad m \to \infty. \end{aligned}$$

Since $n > m$, we obtain $d(x_n, x_m) \to 0$, as $n, m \to \infty$.

If $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ does not satisfy the condition (2.1), then it is enough to assume that the corresponding Picard sequence of f is not a Cauchy sequence and according to the following lemma, a contradiction is obtained.

Lemma 2.1. [2] *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$, as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences*

$$\{d(x_{m_k}, x_{n_k})\}, \{d(x_{m_k}, x_{n_k+1})\}, \{d(x_{m_k-1}, x_{n_k})\},$$
$$\{d(x_{m_k-1}, x_{n_k+1})\}, \{d(x_{m_k+1}, x_{n_k+1})\}$$

tend to ε^+, as $k \to +\infty$.

The previous lemma is important and widely used for assaying that the Picard sequence of f under some contractive condition is a Cauchy sequence. There are a large number of illustrations. We present just two of them.

Example 2.1. Let (X,d) be a metric space and $f:X\to X$ a mapping on X with the property

$$d\left(fx,fy\right)\le\lambda\max\left\{d\left(x,y\right),d\left(x,fx\right),d\left(y,fy\right),d\left(x,fy\right),d\left(fx,y\right)\right\},$$

for all $x,y\in X$ and $\lambda\in[0,\frac{1}{2})$.

Obviously, if $\lambda=0$, then every Picard sequence is a Cauchy sequence. Assume that $\lambda\in\left(0,\frac{1}{2}\right)$. Let $x_0\in X$ be an arbitrary. Define $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ by $x_{n+1}=fx_n$. We obtain tree cases:

1° $d(x_n,x_{n+1})\le\lambda d(x_n,x_{n-1})$;
2° $d(x_n,x_{n+1})\le\lambda d(x_n,x_{n+1})<d(x_n,x_{n+1})$, which is impossible;
3° $d(x_n,x_{n+1})\le\lambda d(x_{n-1},x_{n+1})\le\lambda(d(x_{n-1},x_n)+d(x_n,x_{n+1}))$, i.e.
$d(x_n,x_{n+1})\le\dfrac{\lambda}{1-\lambda}d(x_{n-1},x_n)$.

Since $\lambda\in\left(0,\frac{1}{2}\right)$, then (2.1) holds, and, consequently, $\{x_n\}$ is a Cauchy sequence.

Example 2.2. Let (X,d) be a metric space and $f:X\to X$ a mapping satisfying

$$d\left(fx,fy\right)\le d\left(x,y\right)-\varphi\left(d\left(x,y\right)\right)\tag{2.3}$$

for all $x,y\in X$, where $\varphi:[0,\infty)\to[0,\infty)$ is a function with some continuity properties and $\varphi\left(t\right)=0$ if and only if $t=0$.

Suppose that $x_n\ne x_{n+1}$ for all $n\in\mathbb{N}$. We have

$$\begin{aligned}d\left(x_n,x_{n+1}\right)&=d\left(fx_{n-1},fx_n\right)\le d\left(x_{n-1},x_n\right)-\varphi\left(d\left(x_{n-1},x_n\right)\right)\\&<d\left(x_{n-1},x_n\right).\end{aligned}$$

Consequently, there exists d^* such that $0\le d^*\le\lim_{n\to\infty}d\left(x_n,x_{n+1}\right).$

Assume that $d^*>0$. From the inequality

$$d\left(x_n,x_{n+1}\right)\le d\left(x_{n-1},x_n\right)-\varphi\left(d\left(x_{n-1},x_n\right)\right),$$

letting n tend to ∞, we obtain $d^*=0$, a contradiction.

If $\{x_{n+1}\}_{n\in\mathbb{N}\cup\{0\}}=\{fx_n\}_{n\in\mathbb{N}\cup\{0\}}=\{f^nx_0\}_{n\in\mathbb{N}\cup\{0\}}$ is not a Cauchy sequence, then substituting $x=x_{m_k}$ and $y=x_{n_k}$ into (2.3), we have

$$d\left(x_{m_k+1},x_{n_k+1}\right)\le d\left(x_{m_k},x_{n_k}\right)-\varphi\left(d\left(x_{m_k},x_{n_k}\right)\right).$$

Letting $k\to\infty$ and by the continuity of φ, we obtain $\varepsilon=0$, which contradicts $\varepsilon>0$.

Remark 2.1. The elegance of the previous proof could be seen as an example of the power of Lemma 2.1. Using this lemma, it should be pretty easy to obtain that every Picard sequence is a Cauchy. The proofs are now shorter than they were before.

Now, we will focus our attention to b-metric spaces and show that some contractive conditions imply that each Picard sequence of initial point $x_0\in X$ is actually b-Cauchy sequence in the given b-metric space (X,d,s). Our basic idea is to apply Lemma 1.3. We present our first result.

Theorem 2.1. *The contractive condition* (1.1) *implies that the corresponding Picard sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *is a b-Cauchy sequence.*

Proof. We can follow line by line the proof of Theorem 2.2 in [1]. Hence, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Further, it follows that

$$d(x_n, x_{n+1}) < \frac{1}{s^2} d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. From [26, Lemma 2.2], it may be concluded that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a b-Cauchy sequence.

Remark 2.2. We see at once that the condition (1.2) imposes a new better condition for the proof of Theorem 2.2 from [1]. Indeed, (1.1) implies the inequality

$$d(fx, gy) \leq \frac{1}{s^2} M(x, y) + L_1 N(x, y),$$

where $L_1 = \frac{L}{s}$, $s > 1$. For $s = 1$ we obtain the result in the standard metric space but with a new proof.

The proof of the next lemma is straightforward and will be omitted.

Lemma 2.2. *Let (X, d, s) be a b-metric space and f a self-mapping on X. The following conditions are equivalent.*

(A) $d(fx, fy) \leq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(y, fy) + a_4 [d(x, fy) + d(y, fx)]$ for all $x, y \in X$ and $a_1, a_2, a_3, a_4 \geq 0$ with the property $a_1 + a_2 + a_3 + 2sa_4 < 1$.

(B) $d(fx, fy) \leq b_1 d(x, y) + b_2 d(x, fx) + b_3 d(y, fy) + b_4 d(x, fy) + b_5 d(y, fx)$ for all $x, y \in X$ and $b_1, b_2, b_3, b_4, b_5 \geq 0$ with the property $b_1 + b_2 + b_3 + (b_4 + b_5) s < 1$.

Lemma 2.3. *Let (X, d, s) be a b-metric space and f a self-mapping on X satisfying the property (A). Then:*

a) for arbitrary $x_0 \in X$, the corresponding Picard sequence

$$\{x_{n+1}\}_{n\in\mathbb{N}\cup\{0\}} = \{fx_n\}_{n\in\mathbb{N}\cup\{0\}}$$

is a b-Cauchy sequence;

b) for all $x \in X$ and some $k \in (0, 1)$ holds the inequality

$$d\left(fx, f^2 x\right) \leq k d(x, fx). \tag{2.4}$$

Proof. a) Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Substituting $x = x_n$, $y = x_{n-1}$ and $x = x_{n-1}$, $y = x_n$, respectively into (A), we obtain

$$d(x_{n+1}, x_n) \leq (a_1 + a_3 + a_4 s) d(x_n, x_{n-1}) + (a_2 + a_4 s) d(x_{n+1}, x_n) \tag{2.5}$$

and

$$d(x_n, x_{n+1}) \leq (a_1 + a_2 + a_4 s) d(x_n, x_{n-1}) + (a_3 + a_4 s) d(x_{n+1}, x_n). \tag{2.6}$$

Summing (2.5) and (2.6), we have

$$d(x_{n+1}, x_n) \leq \frac{2a_1 + a_2 + a_3 + 2a_4 s}{2 - a_2 - a_3 - 2a_4 s} d(x_n, x_{n-1}).$$

Since $a_1 + a_2 + a_3 + 2sa_4 < 1$, the inequality

$$\frac{2a_1 + a_2 + a_3 + 2a_4 s}{2 - a_2 - a_3 - 2a_4 s} < 1$$

implies $2-\beta-\gamma-2\mu s>0$. According to [26, Lemma 2.2], the condition (A) gives that $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a b-Cauchy sequence.

$b)$ To deduce (2.4) from (A), replace x and y by x and fx and then by fx and x, respectively. We have

$$d\left(fx,f^2x\right)\leq\left(a_1+a_2+a_4s\right)d\left(x,fx\right)+\left(a_3+a_4s\right)d\left(fx,f^2x\right),$$

and

$$d\left(f^2x,fx\right)\leq\left(a_1+a_3+a_4s\right)d\left(x,fx\right)+\left(a_2+a_4s\right)d\left(fx,f^2x\right).$$

From the above inequalities, we conclude that $d\left(fx,f^2x\right)\leq kd\left(x,fx\right)$ for all $x\in X$ where

$$k=\frac{a_1+a_4s+\frac{a_2+a_3}{2}}{1-a_4s-\frac{a_2+a_3}{2}}.$$

Since $k<1$ if and only if $a_1+a_2+a_3+2sa_4<1$, the condition (2.4) is checked.

Remark 2.3. Note that Lemma 2.3 $b)$ gives more. Namely, in this case, f has the property (P), i.e., $F\left(f\right)=F\left(f^n\right).$ For more details see [17].

For the convenience of the reader we now repeat the relevant material from [28], thus making our exposition self-contained.

Let (X,d) be a metric space.

$CB(X)=\{A\mid A$ is a nonempty, closed and bounded subset of $X\}$

$D\left(a,B\right)=\inf\left\{d\left(a,b\right)\mid b\in B\subset X\right\},\quad a\in X$

$H(A,B)=\max\left\{\sup\left\{D\left(a,B\right)\mid a\in A\right\},\sup\left\{D\left(b,A\right)\mid b\in B\right\}\right\},\ A,B\in CB(X).$

It is easy to see that H is a metric on $CB\left(X\right)$, called the Hausdorff-Pompeu metric by d.

A set valued mapping $T:X\to CB\left(X\right)$ is said to be multi-valued contraction mapping if there exists a fixed real number λ, $0<\lambda<1$, such that

$$H\left(Tx,Ty\right)\leq\lambda d\left(x,y\right)$$

for all $x,y\in X$. A point $x\in X$ is called a fixed point of T if $x\in Tx$.

For further information about the multi-valued mappings in b-metric spaces, we refer the interested readers to the following papers: [4], [6], [18].

In the sequel we restrict our attention to [4, Theorem 2.1]. We will now show how to dispense with the assumption of Geraghty contraction.

Theorem 2.2. *Let (X,d) be a b-metric space and $S,T:X\to CB(X)$ two multivalued mappings. If there exist $\beta\in\mathcal{S}$ and $\psi\in\Psi$ such that the pair (S,T) satisfies the following inequality*

$$\psi(s^3H(Sx,Ty))\leq\beta(\psi(M(x,y)))\psi(M(x,y)),\tag{2.7}$$

for all $x,y\in X$ where

$$M(x,y)=\max\left\{d(x,y),D(x,Sx),D(y,Ty),\frac{D(x,Ty)+D(y,Sx)}{2s}\right\},$$

then the corresponding Picard sequence with the initial point x_0 is a b-Cauchy sequence.

Proof. It is clear that (2.7) implies

$$H\left(Sx,Ty\right) \leq \frac{1}{s^3}M\left(x,y\right), \tag{2.8}$$

for all $x, y \in X$. Following the lines of the proof in [4, Theorem 2.1], we construct a sequence $\{x_n\}_{\mathbb{N}\cup\{0\}}$ in X such that $x_{2i+1} \in Sx_{2i}$, $x_{2i+2} \in Tx_{2i+1}$, $i \in \mathbb{N} \cup \{0\}$. We obtain

$$d\left(x_{2i+1},x_{2i+2}\right) \leq \frac{1}{s^3}d\left(x_{2i},x_{2i+1}\right) \tag{2.9}$$

and according to Lemma 1.3, we conclude that $\{x_n\}_{\mathbb{N}\cup\{0\}}$ is a b-Cauchy sequence.

Remark 2.4. From (2.7) we have

$$0 < D\left(x_1,Tx_1\right) \leq H\left(Sx_0,Tx_1\right) \leq \frac{1}{s^3}M\left(x_0,x_1\right) = \frac{1}{s^3}d\left(x_0,x_1\right),$$

since $M\left(x_0,x_1\right) = d\left(x_0,x_1\right)$. Also, there exists $x_2 \in Tx_1$ such that

$$0 < d\left(x_1,x_2\right) \leq H\left(Sx_0,Tx_1\right) \leq \frac{1}{s^3}d\left(x_0,x_1\right).$$

From [4, Theorem 2.1], we obtain that the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ satisfies

$$d\left(x_n,x_{n+1}\right) \leq \mu d\left(x_{n-1},x_n\right),$$

where $\mu = \frac{1}{s^3}$ and $n \in \mathbb{N}$.

It is clear that (2.7) implies (2.8). This can be a considerable improvement of [4] since we exclude the Geraghty condition and obtain much shorter proof.

Remark 2.5. Note that the proof of Theorem 2.1 in [4] is not correct. Namely, on page 11, line 4^+, $D\left(x^*,Tx^*\right)$ is not necessarily equal with $\lim_{n\to\infty} D\left(x_n,Tx^*\right)$.

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References

[1] M. Abbas, I.Z. Chema, A. Razani, *Existence of common fixed point for b-metric rational type contraction,* Filomat, **30(6)**(2016), 1413-1429.

[2] A. Aghajani, M. Abbas, J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces,* Math. Slovaca, **64**(2014), no. 4, 941-960.

[3] R. Allahyari, R. Arab, A.S. Haghighi, *A generalization on weak contractions in partially ordered b-metric spaces and its applications to quadratic integral equations,* J. Inequal. Appl. 2014, 2014:355.

[4] E. Amer, M. Arshad, W. Shatanawi, *Common fixed point results for generalized α_*-ψ-contraction multivalued mappings in b-metric spaces,* J. Fixed Point Theory Appl., 19 (4), 3069-3086.

[5] A.H. Ansari, A. Razani, N. Hussain, *Fixed and coincidence points for hybrid rational Geraghty contractive mappings in ordered b-metric spaces,* Int. J. Nonlinear Anal. Appl., **8**(2017), no. 1, 315-329.

[6] H. Aydi, M.F. Bota, E. Karapinar, S. Mitrović, *A fixed point theorem for set-valued quasi-contractions in b-metric spaces,* Fixed Point Th. Appl., 2012, 2012:88.

[7] I.A. Bakhtin, *The contraction principle in quasimetric spaces,* Funct. Anal., **30**(1989), 26–37.

[8] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,* Fundam. Math., **3**(1922), 133-181.

[9] S. Chandok, M. Jovanović, S. Radenović, *Ordered b-metric spaces and Geraghty type contractive mappings,* Vojnotehnički Glasnik, Military Techical Courier, **65**(2017), no. 2, 331-345.

[10] S. Czerwik, *Contraction mappings in b-metric spaces,* Acta Math. Inform., Univ. Ostrav., **1**(1993), 5-11.

[11] M. Demma, P. Vetro, *Picard sequence and fixed point results on b-metric spaces,* J. Function Spaces, Volume 2015, Article ID 189861, 6 pages.

[12] A. Denjoy, *Sur l'iteration des fonctions analytiques,* C.R. Acad. Sc. Paris, Serie 1, **182**(1926), 255–257.

[13] T. Došenović, M. Pavlović, S. Radenović, *On a various contractive conditions in b-metric spaces,* Vojnotehnički Glasnik, Military Techical Courier, **65**(2017), no. 4, 851-865.

[14] A.A. Harandi, *Fixed point theory for quasi-contraction maps in b-metric spaces,* Fixed Point Theory, **15**(2014), no. 2, 351-358.

[15] T.L. Hicks, B.E. Rhoades, *A Banach type fixed point theorem,* Math. Japonica, **24**(1979), 327-320.

[16] N. Hussain, Z.D. Mitrović, *On multi-valued weak quasi-contractions in b-metric spaces,* J. Nonlinear Sci. Appl., **10**(2017), 3815-3823.

[17] G.S. Jeong, B.E. Rhoades, *Maps for which* $F(T) = F(T^n)$, Fixed Point Theory Appl., **6**(2005), 87-131.

[18] J.M. Josepl, D.D. Roselin, M. Marudi, *Fixed point theorems on multi valued mappings in b-metric spaces*, Springer Plus, (2016), 5:217.

[19] M. Jovanović, *Contribution to the Theory of Abstract Metric Spaces,* Doctoral Dissertation, Belgrade, 2016.

[20] M.A. Khamsi, N. Hussain, *KKM mappings in metric type spaces,* Nonlinear Anal., **73**(2010), 3123-3129.

[21] M.S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc., **30**(1984), no. 1, 1-9.

[22] M. Kir, H. Kiziltunc, *On some well known fixed point theorems in b-metric spaces,* Turkish J. Anal. Numb. Theory, **1**(2013), no. 1, 13-16.

[23] P. Kumam, W. Sintunavarat, S. Sedghi, N. Shobkolaei, *Common fixed point of two R-weakly commuting mappings in b-metric spaces*, J. Function Spaces, Volume 2015, Article ID 350840, 5 pages.

[24] A. Latif, V. Parvaneh, P. Salimi, A.E. Al-Mazrooei, *Various Suzuki type theorems in b-metric spaces,* J. Nonlinear Sci. Appl., **8**(2015), 363-377.

[25] L. Liu, F. Gu, *Common fixed point theorems for six self-maps in b-metric spaces with nonlinear contractive conditions,* J. Nonlinear Sci. Appl., **9**(2016), 5909-5930.

[26] R. Miculescu, A. Mihail, *New fixed point theorems for set-valued contractions in b-metric spaces,* J. Fixed Point Theory Appl., 19 (3), 2153-2163.

[27] Z.D. Mitrović, S. Radenović, *The Banach and Reich contractions in* $b_v(s)$*-metric spaces,* J. Fixed Point Theory Appl., 19 (4), 3087-3095.

[28] S.B. Nadler, *Multivalued contraction mappings,* Pacific J. Math., **30**(1969), 475-488.

[29] Lj. Paunović, P. Kaushik, S. Kumar, *Some applications with new admissibility contractions in b-metric spaces,* J. Nonlinear Sci. Appl., **10**(2017), 4162-4174.

[30] H. Piri, P. Kumam, *Fixed point theorems for generalized F-Suzuki-contraction mappings in complete b-metric spaces,* Fixed Point Theory Appl., (2016), 2016:90.

[31] A. Petruşel, G. Petruşel, J. C. Yao, *Fixed point and coincidence point theorems in b-metric spacs with applications,* Appl. Anal. Discrete Math., **11**(2017), 199-215.

[32] B.E. Rhoades, *A comparison of various definitions of contractive mappings,* Trans. Amer. Math. Soc., **226**(1977), 257-290.

[33] J.R. Roshan, V. Parvaneh, I. Altun, *Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations,* Appl. Math. Comput., **226**(2014), 725-737.
[34] J.R. Roshan, V. Parvaneh, N. Shobkolaei, S. Sedghi, W. Shatanawi, *Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces*, Fixed Point Theory Appl., 2013, 2013:159.
[35] J.R. Roshan, N. Shobkolaei, S. Sedghi, M. Abbas, *Common fixed point of four maps in b-metric spaces,* Hacett. J. Math. Stat., **43**(2014), no. 4, 613-624.
[36] M. Sarwar, N. Jamal, Y. Li, *Coincidence point results via generalized (ψ, ϕ)-weak contractions in partial b-metric spaces with application,* J. Nonlinear Sci. Appl., **10**(2017), 3719-3731.
[37] M. Sarwar, M.U. Rahman, *Fixed point theorems for Ćirić and generalized contractions in b-metric spaces,* Int. J. Anal. Appl., **7(2015)**, no. 1, 70-78.
[38] T. Suzuki, *Basic inequality on a b-metric space and its applications*, J. Ineq. Appl., (2017), 2017:256, https://doi.org/10.1186/s13660-017-1528-3.
[39] J. Wolff, *Sur une generalization d'un théorème de Schwartz,* C.R. Hebd. Seanc. Acad. Paris, **182**(1926), 918-920 and **183**(1926), 500-502.
[40] F. Zabihi, A. Razani, *Fixed point theorems for hybrid rational Geraghty contractive mappings in ordered b-metric spaces,* J. Appl. Math., Vol. 2014, Art. ID 929821, 9 pages.

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