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UNIFIED RELATION-THEORETIC FIXED POINT RESULTS VIA $F_{\mathcal{R}}$-SUZUKI-CONTRACTIONS WITH AN APPLICATION

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Abstract. In this paper, we introduce the notion of $F_{\mathcal{R}}$*-Suzuki-contraction* where $\mathcal{R}$ stands for an arbitrary binary relation and utilize the same to establish some existence and uniqueness fixed point results on metric spaces (not necessarily complete) equipped with arbitrary relation. Our results generalize, extend and unify several results of the existing literature. We also provide some examples to demonstrate the generality of our results. As an application of our main results, the existence and uniqueness of solution of a family of nonlinear matrix equations is discussed.
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1. Introduction

Throughout this paper, respectively, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the set of all real numbers, the set of all positive real numbers, the set of all positive integers and the set of whole numbers.

The Banach contraction principle was originated in the Ph.D. thesis of Banach in 1920. This work was later published in the form of a research article [8] in 1922 which has already earned around 2000 Google citations. This work has been extended and generalized in the different directions. Historically speaking, in 1986 the idea of order-theoretic fixed points was initiated by Turinici [23]. In 2004, Ran and Reurings [16] formulated a relatively more natural order-theoretic version of classical Banach contraction principle. Recently, Samet and Turinici [20] established fixed point theorem for nonlinear contraction under symmetric closure of an arbitrary relation. Most recently, Alam and Imdad [6, 7] employed an amorphous relation to prove a relation-theoretic analogue of Banach contraction principle which in turn unify a host of well known relevant order-theoretic fixed point theorems. For the work of this kind one can be referred [1, 2, 3, 4, 5, 6, 7, 12, 16, 17, 18, 19, 20, 21, 23] and references cited therein.

In 2009, Suzuki [22] defined yet another new contraction, often referred as Suzuki contraction $\big($a self-mapping f defined on a metric space (X, d) is said to be a Suzuki contraction if $\forall x, y \in X$ with $x \neq y$ and $\frac{1}{2}d(x, fx) < d(x, y) \implies d(fx, fy) < d(x, y)\big)$ and utilize the same to prove fixed point result which is another noted generalization of the Banach contraction principle. In 2012, Wardowski [24] generalized the Banach contraction principle by introducing the concept of F-contraction:

Definition 1.1. [24] Let (X, d) be a metric space. A self-mapping f on X is called an F-contraction if there exists $\tau \in \mathbb{R}^+$ such that

$$\forall x, y \in X \text{ with } d(fx, fy) > 0 \implies \tau + F(d(fx, fy)) \leq F(d(x, y)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following:

(F_1) F is strictly increasing, *i.e.*, for $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta \implies F(\alpha) < F(\beta)$;

(F_2) for each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3) there exists $r \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^r F(\alpha) = 0$.

We denote by $\mathcal{F}$, the family of all such mappings $F : \mathbb{R}^+ \to \mathbb{R}$. Some examples of such mappings $F \in \mathcal{F}$ are

$$F(t) = \ln t,\ F(t) = t + \ln t,\ F(t) = ln(t + t^2),\ F(t) = -1/\sqrt{t}.$$

Recently, Piri and Kumam [15] extended the results of Wardowski [24] by defining F-Suzuki-contraction:

Definition 1.2. [15] Let (X, d) be a metric space. A self-mapping f on X is called an F-Suzuki-contraction if there exists $\tau \in \mathbb{R}^+$ such that

$$\forall x, y \in X \text{ with } fx \neq fy \text{ and } \frac{1}{2}d(x, fx) < d(x, y) \implies \tau + F(d(fx, fy)) \leq F(d(x, y)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the conditions (F_1), (F_2) together with:

(F_3') F is continuous.

We denote by $\mathfrak{F}$, the family of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the conditions $(F_1), (F_2)$ and (F_3'). Examples of mappings $\mathbf{F} \in \mathfrak{F}$ are

$$\mathbf{F}(t) = -\frac{1}{t},\ \mathbf{F}(t) = ln(t + t^2),\ \mathbf{F}(t) = \frac{1}{1 - e^t} \text{ etc.}$$

Notice that the family $\mathcal{F}$ and $\mathfrak{F}$ are incomparable. For example $\mathbf{F}(t) = -\frac{1}{t}$ is a member of $\mathfrak{F}$ but not a member of $\mathcal{F}$ as it does not satisfy the condition (F_3). Also, for $s > 1, \alpha \in (0, \frac{1}{s})$ $F(t) = \frac{-1}{(t+[t])^\alpha}$ (where $[t]$ stands for the integral part of t), satisfies the condition (F_3) for any $k \in (\frac{1}{s}, 1)$ while it does not satisfy (F_3').

We denote by $\mathbb{F}$, the family of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the conditions $(F_1), (F_2)$ and $\big((F_3)$ or $(F_3')\big)$. Here it can be pointed out that $\mathcal{F} \subset \mathbb{F}$ and $\mathfrak{F} \subset \mathbb{F}$.

Most recently, Sawangsup et al. [21] improved the notion of F-contraction (due to Wardowski) by introducing the notion of $F_{\mathcal{R}}$-contraction.

Definition 1.3. [21] Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$. Then a mapping $f : X \to X$ is called an $F_{\mathcal{R}}$-contraction if there exist $F \in \mathcal{F}$ and

$\tau \in \mathbb{R}^+$ such that

$$\forall x, y \in X \text{ with } (x,y) \in \mathcal{R},\ d(fx, fy) > 0 \implies \tau + F(d(fx, fy)) \leq F(d(x,y)).$$

In [21], some fixed point results are also proved for $F_{\mathcal{R}}$-contraction. For further details on F-contraction, one can consult [11, 15, 21, 22, 24, 25] and references cited therein.

In this paper, firstly, we define the notion of *$F_{\mathcal{R}}$-Suzuki-contraction* where $\mathcal{R}$ is an arbitrary binary relation (not necessarily partial order) and give some examples to demonstrate the genuineness of our newly introduced contraction over F-contraction, $F_{\mathcal{R}}$-contraction, F-Suzuki-contraction. Secondly, we prove some existence and uniqueness fixed point results for *$F_{\mathcal{R}}$-Suzuki-contraction* on metric spaces (not necessarily complete) employing an arbitrary binary relation which in turn unify several well known results of the existing literature. We also give some examples to demonstrate the generality of our main results. Finally, as an application of our main results, the existence and uniqueness of solution of a family of nonlinear matrix equations is discussed.

2. Preliminaries

Recall that a binary relation $\mathcal{R}$ on a non-empty set X is a subset of $X \times X$. We say that "x relates to y under $\mathcal{R}$" if and only if $(x,y) \in \mathcal{R}$. In this presentation, we always employ a non-empty binary relation ($i.e., \mathcal{R} \neq \emptyset$). A binary relation $\mathcal{R}$ is said to be a transitive if for all $x, y, z \in X$, $(x,y) \in \mathcal{R}$ and $(y,z) \in \mathcal{R} \implies (x,z) \in \mathcal{R}$. Here, we recall some basic definitions which are needed in the proofs of our results.

Definition 2.1. [5, 18] For a given self-mapping $f : X \to X$, a binary relation $\mathcal{R}$ on X is said to be f-transitive if for any $x, y, z \in X$,

$$(fx, fy) \in \mathcal{R}, (fy, fz) \in \mathcal{R} \implies (fx, fz) \in \mathcal{R}.$$

Notice that, f-transitivity of $\mathcal{R}$ is equivalent to transitivity of $\mathcal{R}|_{fX}$.

Definition 2.2. [6] Let $\mathcal{R}$ be a binary relation defined on a non-empty set X. Then a sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}\ \ \forall\ n \in \mathbb{N}_0$.

Definition 2.3. [6] Let f be a self-mapping defined on a non-empty set X. Then a binary relation $\mathcal{R}$ on X is called f-closed if

$$\text{for all } x, y \in X,\ (x,y) \in \mathcal{R} \Rightarrow (fx, fy) \in \mathcal{R}.$$

Here it can be pointed out that this property is equivalent to say f is $\mathcal{R}$-nondecreasing (see [19]).

Definition 2.4. [19] A subset Y of a metric space (X, d) is called precomplete if every Cauchy sequence $\{x_n\} \subseteq Y$ is convergent to a point of X.

Definition 2.5. [7] Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$. A subspace $Y \subseteq X$ is called $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in Y converges to a point in Y.

Definition 2.6. Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$. A subspace $Y \subseteq X$ is said to be *$\mathcal{R}$-precomplete* if every $\mathcal{R}$-preserving Cauchy sequence in Y converges to a point in X.

Remark 2.7. • Every $\mathcal{R}$-complete metric space is *an $\mathcal{R}$-precomplete*

• Every precomplete metric space is *an $\mathcal{R}$-precomplete*. Particularly, under the universal relation the notion of *$\mathcal{R}$-precompleteness* coincides with usual precompleteness.

Definition 2.8. [7] Let (X,d) be a metric space equipped with a binary relation $\mathcal{R}$. Then a mapping $f: X \to X$ is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, we have $f(x_n) \xrightarrow{d} f(x)$. As usual, f is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.

Remark 2.9. Every continuous mapping is an $\mathcal{R}$-continuous. Particularly, under the universal relation the notion of $\mathcal{R}$-continuity coincides with usual continuity.

Definition 2.10. [20] Let (X,d) be a metric space equipped with a binary relation $\mathcal{R}$. Then $(X,d,\mathcal{R})$ is called regular if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in \mathcal{R},\ \forall k \in \mathbb{N}$.

Definition 2.11. Let (X,d) be a metric space equipped with a binary relation $\mathcal{R}$. Then $(X,d,\mathcal{R})$ is said to be strong-regular if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$, we have $(x_n, x) \in \mathcal{R},\ \forall n \in \mathbb{N}$.

Here it can be pointed out that '$\mathcal{R}$-nondecreasing-regularity of (X,d)' (see [17]) is equivalent to 'strong-regularity of $(X,d,\mathcal{R})$.'

Remark 2.12. Regularity of $(X,d,\mathcal{R})$ is a weaker notion than strong-regularity of $(X,d,\mathcal{R})$. For the sake of convenience, let $(X=[0,3],d)$ be a usual metric space equipped with a binary relation $\mathcal{R}=\{(0,1),(1,2),(2,2)\}$. Take any $\mathcal{R}$-preserving sequence $\{x_n\}$ in X. Then $x_n \to 2$ and for a subsequence $x_{n_k}=2, \forall k \in \mathbb{N}$, we have $(x_{n_k},2) \in \mathcal{R}$ while for an $\mathcal{R}$-preserving sequence $\{x_n\}$ define as: $x_1=0$, $x_2=1$, $x_i=2$, $\forall i \geq 3$, we have $x_n \to 2$ and $(x_n,2) \notin \mathcal{R}$ for $n=1$.

Definition 2.13. [13] Let (X,d) be a metric space equipped with a binary relation $\mathcal{R}$. For a pair of points x,y in X, there is a finite sequence $\{z_0,z_1,z_2,...,z_l\} \subset X$ such that $z_0=x, z_l=y$ and $(z_i,z_{i+1}) \in \mathcal{R}$ for each $i \in \{0,1,2,3,\cdots,l-1\}$, then this finite sequence is called a path of length l (where $l \in \mathbb{N}$) from x to y in $\mathcal{R}$.

Observe that, a path of length l involves $(l+1)$ elements of X and they need not be distinct in general.

In the subsequent discussion, we denote

- $Fix(f)$: as the collection of all fixed points of f;
- $X(f;\mathcal{R})$: the set of all points in X such that $(x,fx) \in \mathcal{R}$;
- $\Upsilon(x,y;\mathcal{R})$: the family of all paths from x to y in $\mathcal{R}$.

For the sake of completeness, we state the following theorems:

Theorem 2.14. [21, Theorems 3.2 and 3.6] *Let (X,d) be a metric space equipped with a binary relation $\mathcal{R}$ and f a self-mapping on X. Assume that the following conditions hold:*

(i) *$X(f;\mathcal{R})$ is non-empty,*
(ii) *$\mathcal{R}$ is f-closed,*
(iii) *X is complete,*
(iv) *either f is continuous or (X,d) is $\mathcal{R}$-nondecreasing-regular $\big[(X,d,\mathcal{R})$ is strong-regular$\big]$,*
(v) *f is an $F_{\mathcal{R}}$-contraction (where $F \in \mathcal{F}$).*

Then f has a fixed point. Moreover, for each $x_0 \in X(f;\mathcal{R})$, the Picard sequence $\{f^n(x_0)\}$ converges to a fixed point of f.

Theorem 2.15. [15, Theorem 2.2] *Let (X,d) be a complete metric space and $f : X \to X$ an F-Suzuki-contraction (where $F \in \mathfrak{F}$). Then f has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{f^n(x_0)\}$ converges to the fixed point of f.*

The main results of this paper are based on the following motivations and observations:

- a relation-theoretic analogue of F-Suzuki-contraction, termed as $F_{\mathcal{R}}$*-Suzuki-contraction* is introduced and some examples are given which demonstrate the utility of $F_{\mathcal{R}}$*-Suzuki-contraction* over F-contraction, F-Suzuki-contraction and $F_{\mathcal{R}}$-contraction;
- the $\mathcal{R}$-precompleteness of fX is used which is relatively weaker than precompleteness fX, completeness of X or fX, completeness of a subspace Y (with $fX \subseteq Y \subseteq X$);
- Theorem 2.14 is improved by using the weaker notion $\mathcal{R}$-continuity of f instead of continuity;
- Theorems 2.14 and 2.15 are unified by employing relatively more general contractivity condition besides unifying both the families $\mathcal{F}$ and $\mathfrak{F}$ by taking $\mathbb{F}$;
- some examples are adopted to demonstrate the realized improvement in the results of this paper;
- as an application of our main results, the existence and uniqueness of the solution of a family of nonlinear matrix equations is established.

3. Main results

In this section, inspired by notion of F-Suzuki-contraction, we define relation-theoretic version of F-Suzuki-contraction, so-called $F_{\mathcal{R}}$*-Suzuki-contraction*.

Definition 3.1. Let (X,d) be a metric space equipped with a binary relation $\mathcal{R}$. For a given self-mapping f on X, let

$$\mathcal{R}^* = \{(x,y) \in \mathcal{R} : fx \neq fy\}.$$

Then the mapping f is called an $F_{\mathcal{R}}$*-Suzuki-contraction* if there exist $F \in \mathbb{F}$ and $\tau \in \mathbb{R}^+$ such that $\forall x,y \in X$ with $(x,y) \in \mathcal{R}^*$ and

$$\frac{1}{2}d(x,fx) < d(x,y) \implies \tau + F(d(fx,fy)) \leq F(d(x,y)).$$

Remark 3.2. From Definitions 1.1, 1.2, 1.3 and 3.1, we have the following implications:

$$\begin{array}{ccc} F\text{-}contraction & \Longrightarrow & F\text{-}Suzuki\text{-}contraction \\ \Downarrow & & \Downarrow \\ F_{\mathcal{R}}\text{-}contraction & \Longrightarrow & F_{\mathcal{R}}\text{-}Suzuki\text{-}contraction. \end{array}$$

Converse implications not true in general, as the following examples show.

Example 3.3. Let $(X=(-1,5),d)$ be a usual metric space *i.e.*, the metric $d(x,y) = |x-y|$ for all $x,y \in X$ and a binary relation $\mathcal{R} = \{(0,1),(3,2),(4,4),(4,1),(4,2)\}$.

Define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} 3, & x = 0; \\ 2, & x = 1; \\ 4, & \text{otherwise}. \end{cases}$$

Take $\tau = \frac{1}{10}$ and $F(t) = -\frac{1}{\sqrt{t}}, \forall t \in \mathbb{R}^+$. Then $F \in \mathbb{F}$ and $\mathcal{R}^* = \{(0,1),(4,1)\}$. Since $\frac{1}{2}d(x,fx) < d(x,y)$ only for $(x,y) = (4,1)$ in $\mathcal{R}^*$ and

$$\tau + F(d(f4, f1)) = \frac{1}{10} - \frac{1}{\sqrt{2}} < -\frac{1}{\sqrt{3}} = F(d(4,1)).$$

This shows that f is an $F_{\mathcal{R}}$*-Suzuki-contraction* but not $F_{\mathcal{R}}$-contraction as

$$\tau + F(d(f0, f1)) = \tau + F(1) > F(1) = F(d(0,1)), \text{ for any } \tau \in \mathbb{R}^+ \text{ and } F \in \mathbb{F}.$$

Now, as $1, 0 \in X$ with $f1 = 2 \neq 3 = f0$ such that $\frac{1}{2}d(1, f1) < d(1,0)$ but

$$\tau + F(d(f1, f0)) = \tau + F(1) > F(1) = F(d(1,0)), \text{ for any } \tau \in \mathbb{R}^+ \text{ and } F \in \mathbb{F}.$$

Thus f is not F-Suzuki-contraction.

Example 3.4. Let $(X = [0,11], d)$ be a usual metric space equipped with a binary relation $\mathcal{R} = \{(1,1),(1,3),(3,6)\}$. Define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} x, & x \in [0,1]; \\ 1, & x \in [1,3]; \\ \frac{2x-3}{3}, & x \in [3,6]; \\ 3, & x \in [6,11]. \end{cases}$$

Take $\tau = \frac{1}{10}$ and $F(t) = -\frac{1}{\sqrt{t}}, \forall t \in \mathbb{R}^+$. Then $F \in \mathbb{F}$ and $\mathcal{R}^* = \{(3,6)\}$. For $x = 3$ and $y = 6$, we have

$$\tau + F(d(f3, f6)) = \frac{1}{10} - \frac{1}{\sqrt{2}} < -\frac{1}{\sqrt{3}} = F(d(3,6)).$$

This shows that f is an $F_{\mathcal{R}}$-contraction but not F-contraction as

$$\tau + F(d(f0, f\frac{1}{2})) = \tau + F(\frac{1}{2}) > F(\frac{1}{2}) = F(d(0, \frac{1}{2})), \text{ for all } \tau \in \mathbb{R}^+ \text{ and } F \in \mathbb{F}.$$

Example 3.5. Let $(X = (0,1], d)$ be a usual metric space. Define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} \frac{1}{2}, & x \in [0,1); \\ \frac{3}{8}, & x = 1, \end{cases}$$

and take $F(t) = ln(t), \forall t \in \mathbb{R}^+$, so $F \in \mathbb{F}$. Then f is an F-Suzuki-contraction (for all $0 < \tau < \frac{4}{3}$) but not F-contraction as for $x = \frac{9}{10}$ and $y = 1$, one can not find any $\tau > 0$ such taht $\tau + F(d(fx, fy)) \leq F(d(x,y))$.

Now, we present our main result involving the family $\mathcal{F}$.

Theorem 3.6. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ and f a self-mapping on X. Assume that the following conditions hold:*

(*i*) *$X(f; \mathcal{R})$ is non-empty,*

(ii) *$\mathcal{R}$ is f-closed,*
(iii) *fX is $\mathcal{R}$-precomplete,*
(iv) *either f is $\mathcal{R}$-continuous or $(X, d, \mathcal{R})$ is regular,*
(v) *f is an $F_{\mathcal{R}}$-Suzuki-contraction (where $F \in \mathcal{F}$).*

Then f has a fixed point. Moreover, for each $x_0 \in X(f;\mathcal{R})$, the Picard sequence $\{f^n(x_0)\}$ converges to a fixed point of f.

Proof. Since $X(f;\mathcal{R}) \neq \emptyset$, let $x_0 \in X(f;\mathcal{R})$. Define a sequence $\{x_n\}$, with the initial point x_0 by $x_n = f^n x_0$ $\forall n \in \mathbb{N}_0$. Since $(x_0, fx_0) \in \mathcal{R}$, using f-closedness property of $\mathcal{R}$, we have

$$(x_n, fx_n) \in \mathcal{R} \ \ \forall n \in \mathbb{N}_0.$$

Now, we prove that $\{x_n\}$ is a Cauchy sequence. If $d(x_{n_0}, fx_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then result follows immediately. Otherwise, for all $n \in \mathbb{N}_0$, $d(x_n, fx_n) > 0$ so that $fx_n \neq fx_{n+1}$ and hence $(x_n, x_{n+1}) \in \mathcal{R}^*$ and $\frac{1}{2}d(x_n, fx_n) < d(x_n, fx_n)$. Since f is an *$F_{\mathcal{R}}$-Suzuki-contraction*, we have (for all $n \in \mathbb{N}$, for some fixed F and τ)

$$F(d(x_n, fx_n)) = F(d(fx_{n-1}, f^2x_{n-1})) \leq F(d(x_{n-1}, fx_{n-1})) - \tau. \tag{3.1}$$

Repeating this process, we get

$$\begin{aligned} F(d(x_n, fx_n)) &\leq F(d(x_{n-1}, fx_{n-1})) - \tau \\ &\leq F(d(x_{n-2}, fx_{n-2})) - 2\tau \\ &\leq F(d(x_{n-3}, fx_{n-3})) - 3\tau \\ &\ \ \vdots \\ &\leq F(d(x_0, fx_0)) - n\tau. \end{aligned} \tag{3.2}$$

Hence $\lim\limits_{n\to\infty} F(d(x_n, fx_n)) = -\infty$, together with (F_2), we have

$$\lim_{n\to\infty} d(x_n, fx_n) = 0. \tag{3.3}$$

On using (F_3), one can find some $r \in (0, 1)$ such that

$$\lim_{n\to\infty} (d(x_n, x_{n+1}))^r F(d(x_n, x_{n+1})) = 0. \tag{3.4}$$

In view of (3.2), we get

$$(d(x_n, x_{n+1}))^r F(d(x_n, fx_n)) \leq (d(x_n, x_{n+1}))^r F(d(x_0, fx_0)) - n\tau(d(x_n, x_{n+1}))^r.$$

Taking limit as $n \to \infty$ and using (3.3) and (3.4), we obtain

$$\lim_{n\to\infty} n(d(x_n, x_{n+1}))^r = 0. \tag{3.5}$$

Therefore there exists $n_0 \in \mathbb{N}$ such that $n(d(x_n, x_{n+1}))^r \leq 1$ for all $n \geq n_0$ and hence

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n \geq n_0. \tag{3.6}$$

Using this fact and triangular inequality (for all $n, m \in \mathbb{N}_0$ with $m > n \geq n_0$), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &\leq \sum_{j \geq n} \frac{1}{j^{\frac{1}{r}}}. \end{aligned}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{r}}}$ is convergent which amounts to saying that the sequence $\{x_n\}$ is Cauchy in fX. Henceforth, $\{x_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence in fX. Since fX is $\mathcal{R}$-precomplete, there exists $x^* \in X$ such that $x_n \xrightarrow{d} x^*$.

Firstly, suppose f is $\mathcal{R}$-continuous, then

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f \lim_{n\to\infty} x_n = fx^*,$$

and hence x^* is a fixed point of f.

Alternatively, assume that $(X, d, \mathcal{R})$ is regular. Since $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and $x_n \xrightarrow{d} x^*$, there is a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $(x_{n(k)}, x^*) \in \mathcal{R}$ for all $k \in \mathbb{N}$.

Now, we assert that (for all $k \in \mathbb{N}$)

$$\frac{1}{2}d(x_{n(k)}, fx_{n(k)}) < d(x_{n(k)}, x^*) \text{ or } \frac{1}{2}d(fx_{n(k)}, f^2x_{n(k)}) < d(fx_{n(k)}, x^*). \tag{3.7}$$

Assume that there exists $l \in \mathbb{N}$ such that

$$\frac{1}{2}d(x_{n(l)}, fx_{n(l)} \geq d(x_{n(l)}, x^*) \text{ and } \frac{1}{2}d(fx_{n(l)}, f^2x_{n(l)}) \geq d(fx_{n(l)}, x^*).$$

Therefore,

$$2d(x_{n(l)}, x^*) \leq d(x_{n(l)}, fx_{n(l)}) \leq d(x_{n(l)}, x^*) + d(x^*, fx_{n(l)}),$$

and hence

$$d(x_{n(l)}, x^*) \leq d(x^*, fx_{n(l)}) \leq \frac{1}{2}d(fx_{n(l)}, f^2x_{n(l)}).$$

Using (3.1) and (F_1), we obtain

$$d(fx_{n(l)}, f^2x_{n(l)}) < d(x_{n(l)}, fx_{n(l)}).$$

Now, we have

$$\begin{aligned} d(fx_{n(l)}, f^2x_{n(l)}) < d(x_{n(l)}, fx_{n(l)}) &\leq d(x_{n(l)}, x^*) + d(x^*, fx_{n(l)}) \\ &\leq \frac{1}{2}d(fx_{n(l)}, f^2x_{n(l)}) + \frac{1}{2}d(fx_{n(l)}, f^2x_{n(l)}) \\ &= d(fx_{n(l)}, f^2x_{n(l)}) \end{aligned}$$

which is a contradiction and hence (3.7) holds.

Now, we distinguish two cases depending on $K = \{k \in \mathbb{N} : fx_{n(k)} = fx^*\}$. If K is finite, then there exists $k_0 \in \mathbb{N}$ such that $fx_{n(k)} \neq fx^*$ for all $k > k_0$. It follows from (3.7), (for all $k > k_0$) either

$$\tau + F(d(fx_{n(k)}, fx^*)) \leq F(d(x_{n(k)}, x^*))$$

or,

$$\tau + F(d(f^2x_{n(k)}, fx^*)) \leq F(d(fx_{n(k)}, x^*)) = F(d(x_{n(k)+1}, x^*))$$

holds.

If the first inequality holds for infinite values of $k \in \mathbb{N}$, then passing $k \to \infty$; using $\lim_{k\to\infty} d(x_{n(k)}, x^*) = 0$ and (F_2), we get

$$\lim_{k\to\infty} F(d(fx_{n(k)}, fx^*)) = -\infty.$$

It follows from (F_2) that $\lim_{k\to\infty} d(fx_{n(k)}, fx^*) = 0$. Therefore

$$d(x^*, fx^*) = \lim_{k\to\infty} d(x_{n(k)+1}, fx^*) = \lim_{k\to\infty} d(fx_{n(k)}, fx^*) = 0.$$

Hence x^* is a fixed point of f.

If the second inequality holds for infinite values of $k \in \mathbb{N}$, then passing $k \to \infty$; using $\lim_{k\to\infty} d(x_{n(k)}, x^*) = 0$ and (F_2), we get

$$\lim_{k\to\infty} F(d(f^2x_{n(k)}, fx^*)) = -\infty.$$

It follows from (F_2) that $\lim_{k\to\infty} d(f^2x_{n(k)}, fx^*) = 0$. Therefore

$$d(x^*, fx^*) = \lim_{k\to\infty} d(x_{n(k)+2}, fx^*) = \lim_{k\to\infty} d(f^2x_{n(k)}, fx^*) = 0.$$

Hence x^* is a fixed point of f.

Otherwise, if K is not finite, then there is a subsequence $\{x_{n(k(l))}\}$ of $\{x_{n(k)}\}$ such that

$$x_{n(k(l))+1} = fx_{n(k(l))} = fx^* \ \forall l \in \mathbb{N}.$$

As $x_{n(k)} \xrightarrow{d} x^*$, therefore $fx^* = x^*$. This completes the proof. □

Theorem 3.7. *In addition to the hypotheses of Theorem 3.6, suppose that $\mathcal{R}$ is an f-transitive relation on X and $\Upsilon(x, y; \mathcal{R}|_{fX})$ is non-empty for all $x, y \in fX$. Then f has a unique fixed point.*

Proof. In view of Theorem 3.6, $Fix(f)$ is a non-empty. If $Fix(f)$ is singleton, then there is nothing to prove. Otherwise, let x^* and y^* be two distinct fixed points of f. Then $fx^* = x^* \neq y^* = fy^*$. Since $\Upsilon(x, y; \mathcal{R}|_{fX})$ is non-empty for all $x, y \in fX$, there exists a path $\{fz_0, fz_1, \cdots, fz_l,\}$ of some length l in $\mathcal{R}|_{fX}$ such that $fz_0 = x^*$, $fz_l = y^*$ and $(fz_i, fz_{i+1}) \in \mathcal{R}|_{fX}$ for each $i = 0, 1, 2, \cdots, l-1$. Since $\mathcal{R}$ is f-transitive, we have

$$(x^*, fz_1) \in \mathcal{R}, (fz_1, fz_2) \in \mathcal{R}, \cdots, (fz_{l-1}, y^*) \in \mathcal{R} \implies (x^*, y^*) \in \mathcal{R}.$$

Also due to the fact $\frac{1}{2}d(x^*, fx^*) < d(x^*, y^*)$, we have

$$\tau + F(d(x^*, y^*)) = \tau + F(d(fx^*, fy^*)) \leq F(d(x^*, y^*))$$

which is a contradiction because $\tau > 0$. Thus f has a unique fixed point. □

Next, we prove a result involving the family $\mathfrak{F}$.

Theorem 3.8. *Let (X,d) be a metric space equipped with a binary relation $\mathcal{R}$ on X and f a self-mapping on X. Assume that the following conditions hold:*

(i) $X(f;\mathcal{R})$ is non-empty,
(ii) $\mathcal{R}$ is f-closed and f-transitive,
(iii) fX is $\mathcal{R}$-precomplete,
(iv) either f is $\mathcal{R}$-continuous or $(X,d,\mathcal{R})$ is regular,
(v) f is an $\mathbf{F}_{\mathcal{R}}$-Suzuki-contraction (where $\mathbf{F}\in\mathfrak{F}$).

Then f has a fixed point. Moreover, for each $x_0\in X(f;\mathcal{R})$, the Picard sequence $\{f^n(x_0)\}$ converges to a fixed point of f. In addition, if $\Upsilon(x,y;\mathcal{R}|_{fX})$ is non-empty for all $x,y\in fX$. Then f has a unique fixed point.

Proof. Owing to Theorem 3.6, we construct an $\mathcal{R}$-preserving sequence $\{x_n\}$ (where $x_{n+1}=fx_n$) such that $\lim\limits_{n\to\infty} d(x_n,x_{n+1})=0$. Now, we wish to show that $\{x_n\}$ is a Cauchy sequence. To accomplish this, let on contrary $\{x_n\}$ is not a Cauchy, then there exist $\varepsilon>0$ and $k_0\in\mathbb{N}$ with $m(k)>n(k)>k\geq k_0$ such that

$$d(x_{m(k)},x_{n(k)})\geq\varepsilon \text{ and } d(x_{m(k)-1},x_{n(k)})<\varepsilon.$$

Now, we can have

$$\begin{aligned}\varepsilon\leq d(x_{m(k)},x_{n(k)}) &\leq d(x_{m(k)},x_{m(k)-1})+d(x_{m(k)-1},x_{n(k)})\\ &< d(x_{m(k)},x_{m(k)-1})+\varepsilon.\end{aligned}$$

Taking $k\to\infty$ and using the fact $\lim\limits_{n\to\infty} d(x_n,x_{n+1})=0$, we obtain

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)})=\varepsilon, \tag{3.8}$$

and hence

$$\lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1})=\varepsilon, \tag{3.9}$$

From (3.3) and (3.8), one can choose a positive integer $N\in\mathbb{N}$ such that

$$\frac{1}{2}d(x_{m(k)},fx_{m(k)})<\frac{1}{2}\varepsilon<d(x_{m(k)},x_{n(k)}),\ \forall k\geq N$$

Since $\mathcal{R}$ is f-transitive and the sequence $\{x_n\}$ is $\mathcal{R}$-preserving, so $(x_{m(k)},x_{n(k)})\in\mathcal{R}$ and we have

$$\tau+\mathbf{F}(d(x_{m(k)+1},x_{n(k)+1}))=\tau+\mathbf{F}(d(fx_{m(k)},fx_{n(k)}))\leq\mathbf{F}(d(x_{m(k)},x_{n(k)})),\ \forall k\geq N.$$

Taking $k\to\infty$ and on using (3.8), (3.9) and (F_3'), we get

$$\tau+\mathbf{F}(\varepsilon)\leq\mathbf{F}(\varepsilon),$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in fX. Owing to (iii), there is $x^*\in X$ such that $x_n\to x^*$. Finally, proceeding on the lines of the proof of Theorems 3.6 and 3.7, one can complete the proof. □

4. Consequences and examples

In this section, we derive several results in the existing literature as consequences of our newly proved results presented in the earlier section.

If we consider $\mathcal{R} = \{(x, y) \in X^2 \mid x \preceq y\}$, then Theorem 3.6 (or Theorem 3.8) reduces to the following corollary which appears to be new in the literature:

Corollary 4.1. *Let $(X, d, \preceq)$ be an ordered metric space and f a self-mapping on X. Assume that the following conditions hold:*

(*i*) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,*
(*ii*) *f is increasing,*
(*iii*) *fX is $\preceq$-precomplete,*
(*iv*) *either f is $\preceq$-continuous or $(X, d, \preceq)$ is regular,*
(*v*) *f is an $F_{\preceq}$-Suzuki-contraction (where $F \in \mathbb{F}$).*

Then f has a fixed point. Moreover, for each $x_0 \in X$ such that $x_0 \preceq fx_0$, the Picard sequence $\{f^n(x_0)\}$ converges to a fixed point of f.

On using Remark 3.2, we obtain the following result which remains an improved version of Theorems 3.2 and 3.6 due to Sawangsup et al. [21] owing to the involved relatively weaker notions in the considerations of completeness, regularity, continuity and contractivity condition.

Corollary 4.2. *Let (X, d) be a metric space equipped with a binary relation $\mathcal{R}$ on X and f a self-mapping on X. Assume that the following conditions hold:*

(*i*) *$X(f; \mathcal{R})$ is non-empty,*
(*ii*) *$\mathcal{R}$ is f-closed,*
(*iii*) *fX is $\mathcal{R}$-precomplete,*
(*iv*) *either f is $\mathcal{R}$-continuous or $(X, d, \mathcal{R})$ is regular,*
(*v*) *f is an $F_{\mathcal{R}}$-contraction (where $F \in \mathbb{F}$).*

Then f has a fixed point. Moreover, for each $x_0 \in X(f; \mathcal{R})$, the Picard sequence $\{f^n(x_0)\}$ converges to a fixed point of f.

On setting, $\mathcal{R} = X \times X$ (*i.e.*, universal relation), Theorems 3.7 and 3.8 reduce to the following lone corollary which is sharpened version of Theorem 2.2 of Piri and Kumam [15] due to the involvement of relatively weaker notions namely: precompleteness of fX and newly introduced contractivity condition (where $F \in \mathbb{F}$). Observe that Example 3.5 justifies our claim.

Corollary 4.3. *Let (X, d) be a metric space and $f : X \to X$ an F-Suzuki-contraction (where $F \in \mathbb{F}$) such that fX is precomplete. Then f has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard sequence $\{f^n(x_0)\}$ converges to the fixed point of f.*

On combining Corollary 4.3 and Remark 3.2, we deduce the following corollary which is also an improved version of Theorem 2.1 of Wardowski [24] due to the aforementioned reason described in the context of Corollary 4.3 .

Corollary 4.4. *Let (X, d) be a metric space and $f : X \to X$ an F-contraction (where $F \in \mathbb{F}$) such that fX is precomplete. Then f has a unique fixed point. Moreover, for each $x_0 \in X$, the Picard sequence $\{f^n(x_0)\}$ converges to the fixed point of f.*

Now, we furnish some examples to highlight the realized improvements accomplished via our newly proved results.

Example 4.5. Let $(X = [0,6), d)$ be a usual metric space equipped with a binary relation $\mathcal{R} = \{(0,1),(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$. Define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} 3, & 0 \leq x < 1; \\ 2, & x = 1; \\ 4, & 1 < x < 6. \end{cases}$$

Then f is not continuous while it is $\mathcal{R}$-continuous, $\mathcal{R}$ is f-closed, fX is $\mathcal{R}$-precomplete, $\mathcal{R}^* = \{(0,1),(4,1)\}$ and $X(f;\mathcal{R}) \neq \emptyset$ as $(4, f4) = (4,4) \in \mathcal{R}$. Take $\tau = \frac{1}{10}$ and $F(t) = -\frac{1}{\sqrt{t}}, \forall t \in \mathbb{R}^+$, so then $F \in \mathbb{F}$.

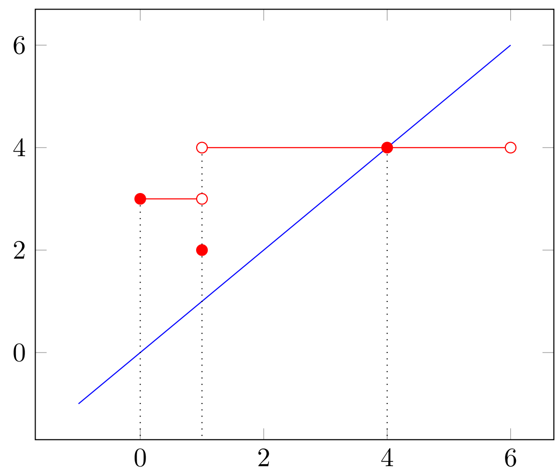

Fig. 1: Graph of $y = f(x)$ (red) and $y = x$ (blue) in Example 4.5.

Since $\frac{1}{2}d(x, fx) < d(x,y)$ only for $(x,y) = (4,1)$ in $\mathcal{R}^*$ and

$$\tau + F(d(f4, f1)) = \frac{1}{10} - \frac{1}{\sqrt{2}} < -\frac{1}{\sqrt{3}} = F(d(4,1)).$$

This shows that f is an $F_{\mathcal{R}}$*-Suzuki-contraction*. Thus all the conditions of Theorem 3.6 are satisfied, hence it has a fixed point. Moreover, $\mathcal{R}|_{fX}$ is transitive while $\mathcal{R}$ is not and for all $x, y \in fX$, we have $(x,y) \in \mathcal{R}$, so $\Upsilon(x, y; \mathcal{R}|_{fX})$ is nonempty for all $x, y \in fX$. Thus in view of Theorem 3.7, f has a unique fixed point. Observe that $x = 4$ is the only fixed point of f. Since $(0,1) \in \mathcal{R}$ and

$$\tau + F(d(f0, f1)) > F(d(0,1)), \text{ for all } \tau \in \mathbb{R}^+ \text{ and } F \in \mathbb{F},$$

which shows that f is not $F_{\mathcal{R}}$-contraction for any $F \in \mathbb{F}$. Also as $1, 0 \in X$ with $f1 = 2 \neq 3 = f0$ such that $\frac{1}{2}d(1, f1) < d(1,0)$ but

$$\tau + F(d(f1, f0)) > F(d(1,0)), \text{ for all } \tau \in \mathbb{R}^+ \text{ and } F \in \mathbb{F},$$

which shows that f is not F-Suzuki-contraction for any $F \in \mathbb{F}$. Hence Theorems 2.14 and 2.15 can not be applied to the present example, while our Theorems 3.6-3.8 are applicable. This shows that our results are genuine improvements over the corresponding results contained in Sawangsup et al. [21] and Piri and Kumam[15].

Example 4.6. Consider $(X,d), f, F$ and τ as in Example 3.3. Define a binary relation $\mathcal{R} = \{(3,2),(4,4),(4,1),(4,2)\}$. Then f is an $F_{\mathcal{R}}$-contraction. It is easy to verify that all requirements of Corollary 4.2 are fulfilled. Hence in view of Corollary 4.2, f has a fixed point. Observe that $x = 4$ is a fixed point of f. But Theorem 2.14 can not be applied to the present example (as (X,d) is not complete), which shows that even our Corollary 4.2 is an improved version of Theorem 2.14.

5. Application to nonlinear matrix equations

As an application of our main results, we establish the existence and uniqueness of the solution of the nonlinear matrix equation

$$X = P + \sum_{i=1}^{m} A_i^* G(X) A_i, \tag{5.1}$$

where P is a Hermitian positive definite matrix, A_i^* stands conjugate transpose of arbitrary $n \times n$ matrix A_i and G an order-preserving continuous mapping from the set of all Hermitian matrices to the set of all positive definite matrices such that $G(0) = 0$.

In this connection, we need to recall some basic notions as under:

We denote respectively, by $\mathcal{M}(n), \mathcal{H}(n), \mathcal{P}(n)$ and $\mathcal{H}^+(n)$, the set of all $n \times n$ complex matrices, the set of all Hermitian matrices in $\mathcal{M}(n)$, the set of all positive definite matrices in $\mathcal{M}(n)$ and the set of all positive semidefinite matrices in $\mathcal{M}(n)$. Also, we denote the element of $\mathcal{P}(n)$ as $X \succ 0$. If $X \succeq 0$, then $X \in \mathcal{H}^+(n)$. Furthermore, $X \succ Y$ (resp. $X \succeq Y$) is equivalent to saying $X - Y \succ 0$ (resp. $X - Y \succeq 0$). The symbol $\|.\|$ stands for the spectral norm of a matrix A defined by $\|A\| = \sqrt{\lambda^+(A^*A)}$, where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A and A^* is the conjugate transpose of A. We use the metric d induced by the trace norm $\|\cdot\|_{tr}$, defined as

$$\|A\|_{tr} = \sum_{j=1}^{n} s_j(A),$$

where $s_j(A)$ $(1 \leq j \leq n)$ are the singular values of $A \in \mathcal{M}(n)$. The induced metric space $(\mathcal{H}(n), d)$ is complete (see [9, 10, 16] for more details).

The following lemmas are needed in the subsequent discussion.

Lemma 5.1. [16] *If $A \succeq 0$ and $B \succeq 0$ are $n \times n$ matrices, then*

$$0 \leq tr(AB) \leq \|A\| tr(B).$$

Lemma 5.2. [14] *If $A \in \mathcal{H}(n)$ such that $A \prec I_n$, then $\|A\| < 1$.*

Theorem 5.3. *Consider the problem described by (5.1). Assume that there exist two positive real numbers τ and η such that*

(H_1) *for every* $X, Y \in \mathcal{H}(n)$ *such that* $X \preceq Y$ *with*

$$\sum_{i=1}^{m} A_i^* G(X) A_i \neq \sum_{i=1}^{m} A_i^* G(Y) A_i,$$

and

$$\left| tr\left(X - P - \sum_{i=1}^{m} A_i^* G(X) A_i \right) \right| < 2|tr(Y - X)|,$$

we have

$$tr\big(G(Y) - G(X)\big) \leq \frac{1}{\eta}\left(\sqrt{e^{-\tau}[tr(Y-X) + (tr(Y-X))^2] + 1/4} - \frac{1}{2} \right);$$

(H_2) *there exists* P*, such that* $\sum_{i=1}^{m} A_i^* G(P) A_i \succ 0$;

(H_3) $\sum_{i=1}^{m} A_i A_i^* \prec \eta I_n$.

Then the matrix equation (5.1) has a unique solution. Moreover, the iteration

$$X_n = P + \sum_{i=1}^{m} A_i^* G(X_{n-1}) A_i$$

where $X_0 \in \mathcal{H}(n)$ *satisfies*

$$X_0 \preceq P + \sum_{i=1}^{m} A_i^* G(X_0) A_i,$$

converges in the sense of trace norm $\|.\|_{tr}$ *to the solution of the matrix equation (5.1).*

Proof. Define a mapping $\mathcal{T} : \mathcal{H}(n) \to \mathcal{H}(n)$ by

$$\mathcal{T}(X) = P + \sum_{i=1}^{m} A_i^* G(X) A_i, \quad \text{for all } X \in \mathcal{H}(n), \tag{5.2}$$

and a binary relation

$$\mathcal{R} = \{(X, Y) \in \mathcal{H}(n) \times \mathcal{H}(n) : X \preceq Y\}.$$

Then fixed point of the mapping $\mathcal{T}$ is a solution of the matrix equation (5.1). Notice that $\mathcal{T}$ is well defined, $\mathcal{R}$-continuous and $\mathcal{R}$ is $\mathcal{T}$-closed. Since

$$\sum_{i=1}^{m} A_i^* G(P) A_i \succ 0,$$

for some $P \in \mathcal{H}(n)$, we have $\big(P, \mathcal{T}(P)\big) \in \mathcal{R}$ and hence $\mathcal{H}(n)(\mathcal{T}; \mathcal{R}) \neq \emptyset$.
Now, let $(X, Y) \in \mathcal{R}^* = \{(X, Y) \in \mathcal{R} : \mathcal{T}(X) \neq \mathcal{T}(Y)\}$ such that

$$\frac{1}{2}\|X - \mathcal{T}(X)\|_{tr} < \|Y - X\|_{tr}.$$

Then

$$
\begin{aligned}
\|\mathcal{T}(Y)-\mathcal{T}(X)\|_{tr} &= tr\big(\mathcal{T}(Y)-\mathcal{T}(X)\big)\\
&= tr\Big(\sum_{i=1}^{m} A_i^*\big(G(Y)-G(X)\big)A_i\Big)\\
&= \sum_{i=1}^{m} tr\big(A_i^*\big(G(Y)-G(X)\big)A_i\big)\\
&= \sum_{i=1}^{m} tr\big(A_iA_i^*\big(G(Y)-G(X)\big)\big)\\
&= tr\Big(\Big(\sum_{i=1}^{m} A_iA_i^*\Big)\big(G(Y)-G(X)\big)\Big)\\
&\le \Big\|\sum_{i=1}^{m} A_iA_i^*\Big\|\|G(Y)-G(X)\|_{tr}\\
&\le \frac{\|\sum_{i=1}^{m} A_iA_i^*\|}{\eta}\Big(\sqrt{e^{-\tau}\big[\|Y-X\|_{tr}+(\|Y-X\|_{tr})^2\big]+\frac{1}{4}}-\frac{1}{2}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by using hypothesis (H}_1))\\
&< \sqrt{e^{-\tau}\big[\|Y-X\|_{tr}+(\|Y-X\|_{tr})^2\big]+\frac{1}{4}}-\frac{1}{2},
\end{aligned}
$$

so that

$$\|\mathcal{T}(Y)-\mathcal{T}(X)\|_{tr}+(\|\mathcal{T}(Y)-\mathcal{T}(X)\|_{tr})^2\le e^{-\tau}\big[\|Y-X\|_{tr}+(\|Y-X\|_{tr})^2\big].$$

If we consider $F(t)=ln(t+t^2)$, for all $t\in\mathbb{R}^+$ (so, $F\in\mathbb{F}$), then

$$\tau+F\big(\|T(Y)-T(X)\|_{tr}\big)\le F\big(\|Y-X\|_{tr}\big)$$

which shows that $\mathcal{T}$ is an $F_{\mathcal{R}}$-*Suzuki-contraction.* Thus all the hypotheses of Theorem 3.6 are satisfied, therefore on using Theorem 3.6, there exists $\widehat{X}\in\mathcal{H}(n)$ such that $\mathcal{T}(\widehat{X})=\widehat{X}$, and hence the matrix equation (5.1) has a solution in $\mathcal{H}(n)$. Furthermore, due to existence of least upper bound and greatest lower bound for each $X,Y\in\mathcal{T}(\mathcal{H}(n))$, We have $\Upsilon(X,Y;\mathcal{R}|_{\mathcal{T}(\mathcal{H}(n))})\neq\emptyset$ for each $X,Y\in\mathcal{T}(\mathcal{H}(n))$. Hence on using Theorem 3.7 (or, Theorem 3.8), $\mathcal{T}$ has a unique fixed point, and hence we conclude that the matrix equation (5.1) has a unique solution in $\mathcal{H}(n)$. □

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References

[1] M. Ahmadullah, J. Ali, M. Imdad, *Unified relation-theoretic metrical fixed point theorems under an implicit contractive condition with an application,* Fixed Point Theory Appl., **2016(42)**(2016).

[2] M. Ahmadullah, M. Imdad, M. Arif, *Common fixed point theorems under an implicit contractive condition on metric spaces endowed with an arbitrary binary relation and an application*, Asian-European J. Math., DOI: 10.1142/S1793557120501466.

[3] M. Ahmadullah, M. Imdad, M. Arif, *Relation-theoretic metrical coincidence and common fixed point theorems under nonlinear contractions,* Appl. Gen. Topol., **19**(2018), no. 1, 65-84.
[4] M. Ahmadullah, M. Imdad, R. Gubran, *Relation-theoretic metrical fixed point theorems under nonlinear contractions,* Fixed Point Theory, **20**(2019), no. 1, 3-18.
[5] A. Alam, M. Imdad, *Nonlinear contractions in metric spaces under locally T-transitive binary relations*, Fixed Point Theory, **19**(2018), no. 1, 13-24.
[6] A. Alam, M. Imdad, *Relation-theoretic contraction principle,* J. Fixed Point Theory Appl., **17**(2015), no. 4, 693-702.
[7] A. Alam, M. Imdad, *Relation-theoretic metrical coincidence theorems,* Filomat, **31**(14)(2017), 4421-4439.
[8] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,* Fund. Math., **3**(1922), 133-181.
[9] M. Berzig, *Solving a class of matrix equations via the Bhaskar- Lakshmikantham coupled fixed point theorem*, Appl. Math. Lett., **25**(2012), 1638-1643.
[10] M. Berzig, B. Samet, *Solving systems of nonlinear matrix equations involving Lipschitzian mappings*, Fixed Point Theory Appl., **2011(89)**(2011).
[11] G. Durmaz, G. Minak, I. Altun, *Fixed points of ordered F-contractions,* Hacet. J. Math. Stat., **45**(2016), no. 1, 15-21.
[12] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136**(2008), 1359-1373.
[13] B. Kolman, R.C. Busby, S. Ross, *Discrete Mathematical Structures,* Third Edition, PHI Pvt. Ltd., New Delhi, 2000.
[14] J.H. Long, X.Y. Hu, L. Zhang, *On the Hermitian positive definite solution of the nonlinear matrix equation* $X + A^*X^{-1}A + B^*X^{-1}B = I$, Bull. Braz. Math. Soc., **39**(2015), no. 3, 317-386.
[15] H. Piri, P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces,* Fixed Point Theory Appl., **2014(210)**(2014).
[16] A.C.M. Ran, M.C.B. Reurings, *A fixed point theorem in partially ordeded sets and some applications to matrix equations,* Proc. Amer. Math. Soc., **132**(2004), no. 5, 1435-1443.
[17] A.F. Roldán-López-de-Hierro, *A unified version of Ran and Reuring's theorem and Nieto and Rodríguez-López's theorem and low-dimensional generalizations*, Appl. Math. Inf. Sci., **10**(2016), no. 2, 383-393.
[18] A.F. Roldán-López-de-Hierro, E. Karapinar, M. De-la-Sen, *Coincidence point theorems in quasi-metric spaces without assuming the mixed monotone property and consequences in G-metric spaces,* Fixed Point Theory Appl., **2014(184)** (2014).
[19] A. F. Roldán-López-de-Hierro, N. Shahzad, *Common fixed point theorems under* $(\mathcal{R}, \mathcal{S})$-*contractivity conditions*, Fixed Point Theory Appl., **2016(55)**(2016).
[20] B. Samet, M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications,* Commun. Math. Anal., **13**(2012), 82-97.
[21] K. Sawangsup, W. Sintunavarat, A.F. Roldán-López-de-Hierro, *Fixed point theorems for* $F_{\mathcal{R}}$-*contractions with applications to solution of nonlinear matrix equations*, J. Fixed Point Theory Appl., **19**(2017), 1711-1725.
[22] T. Suzuki, *A new type of fixed point theorem in metric spaces,* Nonlinear Anal., **71**(2009), 5313-5317.
[23] M. Turinici, *Fixed points for monotone iteratively local contractions,* Demonstratio Math. **19**(1986), no. 1, 171-180.
[24] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012(94)**(2012).
[25] D. Wardowski, N. Van Dung, *Fixed points of F-weak contractions on complete metric spaces,* Demonstratio Math., **1**(2014), 146-155.

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