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BEST PROXIMITY POINT OF ZAMFIRESCU CONTRACTIONS OF PEROV TYPE ON REGULAR CONE METRIC SPACES

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Abstract. T. Zamfirescu, [Fixed point theorems in metric spaces, Arch. Math. (Basel), 23 (1972), 292-298] obtained a very interesting fixed point theorem on complete metric spaces, by combining results of Banach, Kannan and Chatterjea. In this paper, we introduce the concept of Zamfirescu-Perov type cyclic contraction and obtain best proximity point theorems for such mapping in the frame work of regular cone metric spaces. Examples are given to support our results. Our results extend and generalize several comparable existing results in literature.
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1. Introduction

Let A and B be two nonempty subsets of a set X. A mapping $f : X \to X$ is said to be cyclic (with respect to A and B) if $f(A) \subseteq B$ and $f(B) \subseteq A$. The fixed point theory of cyclic contractive mappings is a recent development. Kirk *et al.* [18] in 2003 introduced a class of mappings which satisfy contraction condition for points x and y where $x \in A$ and $y \in B$ and hence extended Banach contraction principle. Petruşel [21] proved some results about periodic points of cyclic contraction maps and generalized the main result in [18].

Eldered and Veeramani [11] proved existence of best proximity points of cyclic contraction maps. Al-Thagafi and Shahzad [2] gave the convergence and existence of best proximity point of cyclic φ-contraction. Basha [4] proved best proximity point theorems for proximal cyclic contraction in the framework of complete metric space. There are many generalizations of the concept of metric spaces (see e.g., [26]).

Recently, Haghi *et al.* [13] defined the notion of a distance between two subsets in regular cone metric spaces and studied some conditions that guarantee the existence of best proximity points for cyclic contraction mappings in such spaces.

On the other hand, Perov [19, 20] generalized Banach contraction principle by replacing the contractive factor with a matrix with the spectral radius less than one. Cvetković and Rakočević [10] introduced Perov-type quasi-contractive mapping replacing contractive factor with bounded linear operator with spectral radius less than one and obtained some interesting fixed point results in the setup of cone metric spaces. Abbas *et al.* [1] obtained coincidence best proximity point results for proximal contractions of Perov type on regular cone metric spaces. Sultana [27] presented best proximity point result of quasi contraction mappings of Perov type in the frame work of regular cone metric spaces.

Zamfirescu [28] obtained a very interesting fixed point theorem on complete metric spaces, by combining results of Banach [3], Kannan [17] and Chatterjea [9].

The purpose of this paper is to introduce the concepts of Zamfirescu-Perov type cyclic contraction and proximal cyclic contraction mappings to obtain best proximity point theorems for such mappings in the frame work of regular cone metric spaces.

2. Preliminaries

In this section, we present some basic results and definitions concerning Zamfirescu theorem [28] (see also [5, 6, 7, 8, 24, 25]) and cone metric spaces.

Theorem 2.1. (Zamfirescu [28]) *Let (X, d) be a complete metric space and $T : X \to X$ a map for which there exist real numbers a, b and c satisfying $0 \leq a < 1, 0 \leq b, c < 1/2$ such that for each pair $x, y \in X$, at least one of the following is true:*

(1) $$d(Tx, Ty) \leq ad(x, y);$$

(2) $$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)];$$

(3) $$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)].$$

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \ n = 0, 1, 2, \ldots$$

converges to p, for any $x_0 \in X$.

An operator T which satisfies the contractive conditions in Theorem 2.1 is called a *Zamfirescu operator*. The class of Zamfirescu operators is one of the most studied classes of quasicontractive type operators. In this class all important fixed point iteration procedures, i.e., the Picard, Mann and Ishikawa iterations, are known to converge to the unique fixed point of T. The class of Zamfirescu operators is independent (see Rhoades [24]) of the class of strictly (strongly) pseudocontractive operators, extensively studied by several authors in the last years. For a recent survey and a comprehensive bibliography, we refer to the recent Berinde's monograph [7].

Let E be a real Banach space. A subset P of E is called a cone (a pointed closed cone) if

(i) P is nonempty, closed and $P \neq \{\theta\}$ (where θ is the zero element of E);
(ii) $a, b \in \mathbb{R}, a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

Partial ordering on E is defined with the help of a cone P as follows: $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ and $x \ll y$ stands for $y - x \in intP$, where $intP$ denotes the interior of P. If $intP$ is nonemtpy then P is called a solid cone. A cone P is normal if there is a number $K > 0$ such that for all $x, y \in P$,

$$\theta \preceq x \preceq y \quad \text{implies that} \quad \|x\| \leq K \|y\| . \tag{2.1}$$

The least positive number satisfying the above inequality is called a normal constant of P. A cone P is called regular if every bounded above increasing sequence in E is convergent, or equivalently a cone P is regular if every decreasing sequence which is bounded below is convergent. It is known that every regular cone is normal [23].

Definition 2.2. [14] Let X be a nonempty set. A mapping $d : X \times X \to E$ is said to be a cone metric on X if for any $x, y, z \in X$, the following conditions hold:

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$;
(d3) $d(x, y) \preceq d(x, z) + d(y, z)$.

The pair (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space. Furthermore, the category of regular cone metric spaces is bigger then the category of metric spaces (Example 1.1. of [13]).

A sequence $\{x_n\}$ a sequence in a cone metric space (X, d) is called: Cauchy sequence if there is an N such that $d(x_n, x_m) \ll c$ for all $n, m > N$. Convergent if there exist an N and $x \in X$ such that $d(x_n, x) \ll c$ for all $n > N$. The limit of a convergent sequence is unique. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. If the cone is normal then a sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $d(x_n, x) \to \theta$ as $n \to \infty$. For further details of these properties, we refer to ([10, 12, 14, 16, 15, 22]). A subset A of X is closed if and only if every convergent sequence in A has its limit in A.

Throughout this paper (X, d) is a regular cone metric space, A and B nonempty subsets of X.

- If $c \gg \theta$, then there exists $\delta > 0$ such that $\| b \| < \delta$ implies $b \ll c$.
- For any given $c \gg \theta$ and $c_0 \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $\frac{c_0}{n_0} \ll c$.
- If $\{a_n\}, \{b_n\}$ are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \leq b_n$ for all $n \geq 1$, then $a \leq b$.

We write $\mathcal{B}(E)$ for the set of all bounded linear operators on E and $L(E)$ for the set of all linear operators on E. $\mathcal{B}(E)$ is a Banach algebra, and if $\mathcal{A} \in \mathcal{B}(E)$ let

$$r(\mathcal{A}) = \lim_{n\to\infty} \|\mathcal{A}^n\|^{\frac{1}{n}} = \inf_n \|\mathcal{A}^n\|^{\frac{1}{n}} \tag{2.2}$$

be the spectral radius of $\mathcal{A}$. We write $\mathcal{B}(E)^{-1}$ for the set of all invertible elements in $\mathcal{B}(E)$. Let us remark that if $r(\mathcal{A}) < 1$, then

(1) Series $\sum_{n=0}^{\infty} \mathcal{A}^n$ is absolutely convergent;
(2) $\mathcal{I} - \mathcal{A}$ is invertible in $\mathcal{B}(E)$ and

$$\sum_{n=0}^{\infty} \mathcal{A}^n = (\mathcal{I} - \mathcal{A})^{-1}. \tag{2.3}$$

Let E be a real Banach space, $P \subseteq E$ cone in E and $\mathcal{A} : E \to E$ a linear operator. The following conditions are equivalent: $\mathcal{A}$ is increasing, that is, $x \preceq y$ implies that $\mathcal{A}(x) \preceq \mathcal{A}(y)$ if and only if $\mathcal{A}$ is positive, that is, $\mathcal{A}(P) \subset P$.

Set $\triangle = \{p \in P : p \preceq d(x, y)$ for all $x \in A, y \in B\}$. Obviously this set is nonempty as $\theta \in \triangle$. We denote a unique vector $p \in \triangle$ by $dis(A, B) \equiv d(A, B)$ if for any q in $\triangle$, we have $q \preceq p$.

3. Main results

First we introduce the main definition for further work.

Definition 3.1. Let $f : A \cup B \to A \cup B$ with $f(A) \subseteq B$ and $f(B) \subseteq A$. A mapping f is called Zamfirescu cyclic contraction of Perov type if there exists $\mathcal{A} \in \mathcal{B}\ (E)$ such that $r(\mathcal{A}) < 1$ and

$$d(fx, fy) \preceq \mathcal{A}(u) + (\mathcal{I} - \mathcal{A})d(a, b), \tag{3.1}$$

where

$$u \in \left\{ d(x, y), \frac{1}{2}(d(x, fx) + d(y, fy)), \frac{1}{2}(d(x, fy) + d(y, fx)) \right\}, \tag{3.2}$$

for all $(a, b), (x, y) \in A \times B$.

Theorem 3.2. *Let A and B be nonempty subsets of a cone metric space X. Let $f : A \cup B \to A \cup B$ be a Zamfirescu cyclic contraction of Perov type. Then $dis(A, B)$ exists.*

Proof. Let x_0 be a given point in A. Define $x_{n+1} = f(x_n)$, and set $d_{n+1} = d(x_n, x_{n+1})$ for all $n \geq 1$.

Let us prove

$$d_{n+1} \leq d_n, \quad n = 1, 2, \ldots. \tag{3.3}$$

Concerning with (3.1) and (3.2), we have to consider the following three possibilities:
(1):

$$d_2 \preceq \mathcal{A}(d_1) + (\mathcal{I} - \mathcal{A})d(a, b); \tag{3.4}$$

and for $a = x_0$ and $b = f(x_0)$ we get $d_2 \preceq d_1$.
(2):

$$d_2 \preceq \mathcal{A}\left(\frac{1}{2}(d_1 + d_2)\right) + (\mathcal{I} - \mathcal{A})d(a, b) \tag{3.5}$$

and

$$2 \cdot d_2 \preceq \mathcal{A}(d_1) + \mathcal{A}(d_2) + 2(\mathcal{I} - \mathcal{A})d(a, b). \tag{3.6}$$

Now, for $a = x_0$ and $b = f(x_0)$ we get

$$(2 \cdot \mathcal{I} - \mathcal{A})d_2 \preceq (2\mathcal{I} - \mathcal{A})d_1. \tag{3.7}$$

Taking $(2\cdot\mathcal{I}-\mathcal{A})^{-1}$, on the both sides, we get $d_2 \preceq d_1$.
(3):

$$d_2 \preceq \mathcal{A}\left(\frac{1}{2}\left(d(x_0,x_2)+d(x_1,x_1)\right)\right)+(\mathcal{I}-\mathcal{A})d(a,b) \tag{3.8}$$

and

$$d_2 \preceq \mathcal{A}\left(\frac{1}{2}(d_1+d_2)\right)+(\mathcal{I}-\mathcal{A})d(a,b), \tag{3.9}$$

and (3.9) implies $d_2 \preceq d_1$. Now, using the method of mathematical induction we prove (3.3). By the regularity of P, there exists $p\in P$ such that $\lim_{n\to\infty} d_n = p$.
Now, there exist, at most, two subsequences d_{n_k}, $k=1,2$, of d_n, such that

$$d_{n_1} \preceq \mathcal{A}(d_{n_1-1})+(\mathcal{I}-\mathcal{A})d(a,b) \tag{3.10}$$

and

$$d_{n_2} \preceq \mathcal{A}\left(\frac{1}{2}(d_{n_1-1}+d_{n_2})\right)+(\mathcal{I}-\mathcal{A})d(a,b). \tag{3.11}$$

Since, $d_{n_1}, d_{n_2} \to p$, $n\to\infty$, and $\mathcal{A}$ is a continuous mapping, (3.10) (and (3.11)) implies

$$p \preceq \mathcal{A}(p)+(\mathcal{I}-\mathcal{A})d(a,b),$$

and

$$(\mathcal{I}-\mathcal{A})(p) \preceq (\mathcal{I}-\mathcal{A})d(a,b).$$

Thus,

$$(\mathcal{I}-\mathcal{A})^{-1}(\mathcal{I}-\mathcal{A})(p) \preceq (\mathcal{I}-\mathcal{A})^{-1}(\mathcal{I}-\mathcal{A})d(a,b),$$

and $p \preceq d(a,b)$ holds for any $(a,b)\in A\times B$. Now, if $q\in\triangle$, then $q\preceq d_n$ for all $n\geq 1$. Hence, $q\preceq p$. Therefore, $dis(A,B)=p$. □

Theorem 3.3. *Let A and B be nonempty and closed subsets of a cone metric space X. Let $f: A\cup B\to A\cup B$ be a Zamfirescu cyclic contraction of Perov type and x_0 a given point in A. Define $x_n = fx_{n-1}$ for all $n\geq 1$. If $\{x_{2n}\}$ has a convergent subsequence in A, then there exists $x\in A$ such that $d(x,fx)=p$.*

Proof. As $\{x_{2n_k}\}$ is convergent subsequence of $\{x_{2n}\}$ in A, choose $x\in A$ such that $\lim_{n\to\infty} x_{2n_k} = x$. Note that

$$p = dis(A,B) \preceq d(x,x_{2n_k-1}) \preceq d(x,x_{2n_k})+d(x_{2n_k},x_{2n_k-1}) \tag{3.12}$$

holds for all $k\geq 1$. If $d_{n+1}=d(x_n,x_{n+1})$, then following the arguments similar to those in the proof of Theorem 3.2, we obtain that $\lim_{n\to\infty} d_n = p$. From (3.12), it follows that

$$\lim_{k\to\infty} d(x,x_{2n_k-1}) = p. \tag{3.13}$$

Also,

$$p = dis(A,B) \preceq d(x_{2n_k},fx) = d(fx_{2n_k-1},fx), \tag{3.14}$$

holds for all $k\geq 1$.
Now, in relation to (3.1) and (3.2), we have to consider the following three possibilities:
(1):

$$d(fx_{2n_k-1},fx) \preceq \mathcal{A}(d(x_{2n_k-1},x))+(\mathcal{I}-\mathcal{A})d(a,b); \tag{3.15}$$

and for $a = x$ and $b = x_{2n_k-1}$, we get

$$d(fx_{2n_k-1}, fx) \preceq d(x_{2n_k-1}, x). \tag{3.16}$$

(2):

$$\begin{aligned} d(fx_{2n_k-1}, fx) &\preceq \mathcal{A}\left(\frac{1}{2}\left(d(x_{2n_k-1}, x_{2n_k}) + d(x, fx)\right)\right) + (\mathcal{I} - \mathcal{A})d(a,b) \\ &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{2n_k-1}, x_{2n_k}) + d(x, x_{2n_k}) + d(fx_{2n_k-1}, fx))\right) \\ &+ (\mathcal{I} - \mathcal{A})d(a,b), \end{aligned} \tag{3.17}$$

and so

$$\begin{aligned} (2\cdot\mathcal{I} - \mathcal{A})(d(fx_{2n_k-1}, fx)) \preceq \mathcal{A}(d(x_{2n_k-1}, x_{2n_k}) \\ + d(x, x_{2n_k})) + 2\cdot(\mathcal{I} - \mathcal{A})d(a,b). \end{aligned} \tag{3.18}$$

Now, for $a = x_{2n_k}$ and $b = x_{2n_k-1}$, we get

$$(2\cdot\mathcal{I} - \mathcal{A})(d(fx_{2n_k-1}, fx)) \preceq (2\cdot\mathcal{I} - \mathcal{A})(d(x_{2n_k-1}, x_{2n_k})) + \mathcal{A}(d(x, x_{2n_k})), \tag{3.19}$$

that is,

$$d(fx_{2n_k-1}, fx) \preceq d(x_{2n_k-1}, x_{2n_k}) + (2\cdot\mathcal{I} - \mathcal{A})^{-1}\mathcal{A}d(x, x_{2n_k}). \tag{3.20}$$

(3):

$$\begin{aligned} d(fx_{2n_k-1}, fx) &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{2n_k-1}, fx) + d(x, x_{2n_k}))\right) + (\mathcal{I} - \mathcal{A})d(a,b) \\ &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{2n_k-1}, x_{2n_k}) + d(x_{2n_k}, fx) + d(x, x_{2n_k}))\right) \\ &+ (\mathcal{I} - \mathcal{A})d(a,b), \end{aligned} \tag{3.21}$$

and, as in the case (b) we get (3.20).
Thus, by (3.13), (3.14), (3.16) and (3.20), we have $\lim\limits_{k\to\infty} d(x_{2n_k}, fx) = p$. Also, we have

$$p = dis(A,B) \preceq d(x, fx) \preceq d(x, x_{2n_k}) + d(x_{2n_k}, fx)$$

for all $k \geq 1$. Hence, $d(x, fx) = dis(A,B)$. □

If $p = dis(A,B)$, we set

$$\begin{aligned} A_0 &= \{x \in A \ : \ d(x,y) = p \text{ for some } \ y \in B\}, \\ B_0 &= \{y \in B \ : \ d(x,y) = p \text{ for some } \ x \in A\}. \end{aligned}$$

For the further work we shall need the following definitions:

Definition 3.4. A map $f : A \to B$ is said to be Zamfirescu proximal contraction of Perov first (second) kind if there exists a bounded linear operator $\mathcal{A} \in \mathcal{B}\ (E)$ with $r(\mathcal{A}) < 1$ such that for any x, y, u and v in A, $d(u, fx) = d(v, fy) = p$ implies that

$$d(u,v) \preceq \mathcal{A}(u_1); \quad (d(fu, fv) \preceq \mathcal{A}(u_2)),$$

where

$$u_1 \in \left\{ d(x,y), \frac{1}{2}(d(x,u)+d(y,v)), \frac{1}{2}(d(x,v)+d(y,u)) \right\}$$

and

$$u_2 \in \left\{ d(fx,fy), \frac{1}{2}(d(fx,fu)+d(fy,fv)), \frac{1}{2}(d(fx,fv)+d(fy,fu)) \right\}.$$

Definition 3.5. Let $f : A \to B$ and $g : B \to A$ with $f(A) \subseteq B$ and $g(B) \subseteq A$. The pair (f, g) is said to be Zamfirescu proximal cyclic contraction of Perov type pair if there exists $\mathcal{A} \in \mathcal{B}\ (E)$ with $r(\mathcal{A}) < 1$ such that for any $(u, v), (x, y) \in A \times B$, $d(u, fx) = d(v, gy) = p$ implies that

$$d(u,v) \preceq \mathcal{A}(u_3) + (\mathcal{I} - \mathcal{A})(p),$$

where

$$u_3 \in \left\{ d(x,y), \frac{1}{2}(d(x,fx)+d(y,gy)), \frac{1}{2}(d(x,gy)+d(y,fx)) \right\}. \tag{3.22}$$

Definition 3.6. A bijective mapping $h : A \to A$ is said to be weak isometry if for any $x, y \in A$, we have $d(x, y) \preceq d(hx, hy)$.

Definition 3.7. Let $h : A \to A$. A mapping $f : A \to B$ is said to preserve distance with respect to h if $d(fhx_1, fhx_2) = d(fx_1, fx_2)$ holds for all $x_1, x_2 \in A$.

Theorem 3.8. *Let A and B be nonempty subsets of a complete cone metric space X, $f : A \to B$ and $g : B \to A$. If (f, g) is Zamfirescu proximal cyclic contraction of Perov type pair with $f(A_0) \subseteq B_0$, $g(B_0) \subseteq A_0$ and $h : A \cup B \to A \cup B$ is a weak isometry such that $A_0 \subseteq h(A_0)$ and $B_0 \subseteq h(B_0)$. Then there exists unique $x \in A$ and $y \in B$ such that $d(hx, fx) = d(hy, gy) = d(x, y) = p$ provided that (f, g) is Zamfirescu proximal contraction of Perov first kind. Further, for any fixed element x_0 in A_0, the sequence $\{x_n\}$ satisfying $d(hx_{n+1}, fx_n) = p$ converges to x and for any fixed element $y_0 \in B_0$, sequence $\{y_n\}$ satisfying $d(hy_{n+1}, gy_n) = p$ converges to y.*

Proof. Let $x_0 \in A_0$. As $f(A_0) \subseteq B_0$ and $A_0 \subseteq h(A_0)$, we choose $x_1 \in A_0$ such that $d(hx_1, fx_0) = p$. Also, from $f(A_0) \subseteq B_0$ and $A_0 \subseteq h(A_0)$, there exists an element $x_2 \in A_0$ such that $d(hx_2, fx_1) = p$. Continuing this way, we can obtain a sequence $\{x_n\}$ in A_0 such that $d(hx_n, fx_{n-1}) = p$. Having chosen x_n in A_0, we find x_{n+1} of A_0 such that $d(hx_{n+1}, fx_n) = p$. Note that

$$d(x_n, x_{n+1}) \preceq \mathcal{A}(u_1),$$

where

$$u_1 \in \{d(x_{n-1}, x_n), \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})), \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))\}.$$

(a): If $u_1 = d(x_{n-1}, x_n)$, then

$$d(x_n, x_{n+1}) \preceq \mathcal{A}(d(x_{n-1}, x_n)). \tag{3.23}$$

(b): If $u_1 = \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_n))\right) + \mathcal{A}\left(\frac{1}{2}(d(x_n, x_{n+1}))\right) \end{aligned}$$

$$\Rightarrow (2 \cdot \mathcal{I} - \mathcal{A})\, d(x_n, x_{n+1}) \preceq \mathcal{A}\left(d(x_{n-1}, x_n)\right)$$

$$\Rightarrow d(x_n, x_{n+1}) \preceq (2 \cdot \mathcal{I} - \mathcal{A})^{-1} \mathcal{A}\left(d(x_{n-1}, x_n)\right). \tag{3.24}$$

(c): If $u_1 = \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_{n+1}))\right) \\ &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_n))\right) + \mathcal{A}\left(\frac{1}{2}(d(x_n, x_{n+1}))\right) \end{aligned}$$

$$\Rightarrow (2 \cdot \mathcal{I} - \mathcal{A})\left(d(x_n, x_{n+1})\right) \preceq \mathcal{A}\left(d(x_{n-1}, x_n)\right)$$

$$\Rightarrow d(x_n, x_{n+1}) \preceq (2 \cdot \mathcal{I} - \mathcal{A})^{-1} \mathcal{A}\left(d(x_{n-1}, x_n)\right). \tag{3.25}$$

From (3.23), (3.24) and (3.25), we obtain that $\{x_n\}$ is a Cauchy sequence and hence convergent to an element x in A.
Following similar arguments to those given above, we find y in B such that $y_n \to y$ as $n \to \infty$.
Since (f, g) is a Zamfirescu proximal cyclic contraction of Perov type pair and h is weak isometry, therefore

$$d(x_{n+1}, y_{n+1}) \preceq d(hx_{n+1}, hy_{n+1}) \preceq \mathcal{A}(u) + (\mathcal{I} - \mathcal{A})(p),$$

where

$$u \in \{d(x_n, y_n), \frac{1}{2}(d(x_n, fx_n) + d(y_n + gy_n)), \frac{1}{2}(d(x_n, gy_n) + d(y_n + fx_n))\}.$$

(a): If $u = d(x_n, y_n)$, then

$$d(x_{n+1}, y_{n+1}) \preceq d(hx_{n+1}, hy_{n+1}) \preceq \mathcal{A}(d(x_n, y_n)) + (\mathcal{I} - \mathcal{A})(p).$$

Taking limit as $n \to \infty$, we obtain that

$$d(x, y) \preceq \mathcal{A}(d(x, y)) + (\mathcal{I} - \mathcal{A})(p),$$

that is, $(\mathcal{I} - \mathcal{A})d(x, y) \preceq (\mathcal{I} - \mathcal{A})(p)$. Hence $d(x, y) = p$.

(b): If $u = \frac{1}{2}(d(x_n, fx_n) + d(y_n + gy_n))$, we have

$$\begin{aligned}
d(x_{n+1}, y_{n+1}) &\preceq d(hx_{n+1}, hy_{n+1}) \\
&\preceq \mathcal{A}\left(\frac{1}{2}(d(x_n, fx_n) + d(y_n + gy_n))\right) + (\mathcal{I} - \mathcal{A})(p) \\
&\preceq \mathcal{A}\left(\frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n+1}, fx_n) + d(y_n + y_{n+1}) + d(y_{n+1} + gy_n))\right) \\
&+ (\mathcal{I} - \mathcal{A})(p) \\
&= \mathcal{A}\left(\frac{1}{2}(d(x_n, x_{n+1}) + p + d(y_n + y_{n+1}) + p)\right)) + (\mathcal{I} - \mathcal{A})(p).
\end{aligned}$$

Taking limit as $n \to \infty$, we obtain that

$$\begin{aligned}
d(x, y) \preceq d(hx, hy) &\preceq \mathcal{A}\left(\frac{2p}{2}\right) + (\mathcal{I} - \mathcal{A})(p) \\
d(x, y) \preceq d(hx, hy) &\preceq \mathcal{A}(p) + (\mathcal{I} - \mathcal{A})(p).
\end{aligned}$$

Hence $d(x, y) = p$.
(c): If $u = \frac{1}{2}(d(x_n, gy_n) + d(y_n + fx_n))$, we have

$$\begin{aligned}
d(x_{n+1}, y_{n+1}) &\preceq d(hx_{n+1}, hy_{n+1}) \\
&\preceq \mathcal{A}\left(\frac{1}{2}(d(x_n, gy_n) + d(y_n + fx_n))\right) + (\mathcal{I} - \mathcal{A})(p) \\
&\preceq \mathcal{A}\left(\frac{1}{2}(d(x_n, y_{n+1}) + d(y_{n+1}, gy_n) + d(y_n + x_{n+1}) + d(x_{n+1} + fx_n))\right) \\
&+ (\mathcal{I} - \mathcal{A})(p) \\
&= \mathcal{A}\left(\frac{1}{2}(d(x_n, y_{n+1}) + p + d(y_n + x_{n+1}) + p)\right) + (\mathcal{I} - \mathcal{A})(p).
\end{aligned}$$

Taking limit as $n \to \infty$, we obtain that

$$\begin{aligned}
d(x, y) \preceq d(hx, hy) &\preceq \mathcal{A}\left(\frac{1}{2}(2d(x, y) + 2p)\right) + (\mathcal{I} - \mathcal{A})(p) \\
d(x, y) \preceq d(hx, hy) &\preceq \mathcal{A}(d(x, y) + p) + (\mathcal{I} - \mathcal{A})(p).
\end{aligned}$$

Hence $d(x, y) = p$.

Thus $x \in A_0$ and $y \in B_0$. As $f(A_0) \subseteq B_0$ and $g(B_0) \subseteq A_0$, there exists $u \in A$ and $v \in B$ such that $d(u, fx) = d(v, gy) = p$. By $d(u, fx) = d(hx_{n+1}, fx_n) = p$, we have $d(u, hx_{n+1}) \leq \mathcal{A}(d(x, x_n))$. On taking limit as $n \to \infty$, we obtain that $d(u, hx) \leq \mathcal{A}(d(x, x))$ and hence $u = hx$. Thus $d(hx, fx) = p$. Similarly, we have $v = hy$ and hence $d(hy, gy) = p$. For uniqueness; Let $x^* \in A$ and $y^* \in B$ be such that $d(hx^*, fx^*) = d(hy^*, gy^*) = p$. As h is weak isometry and (f, g) is Perov type proximal contraction pair, we have, $d(x, x^*) \preceq d(hx, hx^*) \preceq \mathcal{A}(d(x, x^*))$, that is, $(\mathcal{I} - \mathcal{A})(d(x, x^*)) \preceq 0$ and $d(x, x^*) \preceq (\mathcal{I} - \mathcal{A})^{-1}(0) = 0$. Hence $x = x^*$. Similarly, we have $y = y^*$. □

If in above theorem, we take $h = I_{A \cup B}$ (identity mapping on $A \cup B$), then we have the following best proximity point theorem.

Corollary 3.9. *Let A and B be nonempty subsets of a complete cone metric space X, $f : A \to B$ and $g : B \to A$. If (f, g) is Zamfirescu proximal cyclic contraction of Perov type pair with $f(A_0) \subseteq B_0$ and $g(B_0) \subseteq A_0$. Then there exists $x \in A$ and $y \in B$ such that $d(x, fx) = d(y, gy) = d(x, y) = p$ provided that f is Perov type proximal contraction of first type. Further, for any fixed element x_0 in A_0, the sequence $\{x_n\}$ satisfying $d(x_{n+1}, fx_n) = p$ converges to x and for any fixed element $y_0 \in B_0$, sequence $\{y_n\}$ satisfying $d(y_{n+1}, gy_n) = p$ converges to y.*

Corollary 3.10. *Let A and B be nonempty subsets of a complete cone metric space X, $f : A \to B$ and $g : B \to A$. If the pair (f, g) satisfies*

$$d(u, v) \preceq \mathcal{A}(d(x, y)) + (\mathcal{I} - \mathcal{A})(p)$$

with $f(A_0) \subseteq B_0$ and $g(B_0) \subseteq A_0$. Then there exists $x \in A$ and $y \in B$ such that $d(x, fx) = d(y, gy) = d(x, y) = p$ provided that f is Perov type proximal contraction of first type. Further, for any fixed element x_0 in A_0, the sequence $\{x_n\}$ satisfying $d(x_{n+1}, fx_n) = p$ converges to x and for any fixed element $y_0 \in B_0$, sequence $\{y_n\}$ satisfying $d(y_{n+1}, gy_n) = p$ converges to y.

The following main result is a best proximity point theorem for non-self-mappings which are Zamfirescu proximal cyclic contraction of Perov first kind as well as of the second kind.

Theorem 3.11. *Let A and B be nonempty subsets of a cone metric space X, $f : A \to B$ Zamfirescu proximal cyclic contraction of Perov first kind and second kind with $f(A_0) \subseteq B_0$. Then there exists $x \in A$ such that $d(x, fx) = p$ provided that f is distance preserving map. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(x_{n+1}, fx_n) = p$, converges to x.*

Proof. Following arguments similar to those in the proof of Theorem 3.8, and the fact that $f(A_0) \subseteq B_0$, we obtain a sequence $\{x_n\}$ in A_0 satisfying $d(x_{n+1}, fx_n) = p$ for all $n \in \mathbb{N}$. Since f is Perov type proximal contraction of first kind, we have

$$d(x_n, x_{n+1}) \preceq \mathcal{A}(u_1),$$

where

$$u_1 \in \{d(x_{n-1}, x_n), \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1})), \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))\}.$$

(a): If $u_1 = d(x_{n-1}, x_n)$, then

$$d(x_n, x_{n+1}) \preceq \mathcal{A}(d(x_{n-1}, x_n)). \tag{3.26}$$

(b): If $u_1 = \frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_n))\right) + \mathcal{A}\left(\frac{1}{2}(d(x_n, x_{n+1}))\right) \end{aligned}$$

$$\Rightarrow (2 \cdot \mathcal{I} - \mathcal{A})\,(d(x_n, x_{n+1})) \preceq \mathcal{A}\,(d(x_{n-1}, x_n))$$

$$\Rightarrow d(x_n, x_{n+1}) \preceq (2 \cdot \mathcal{I} - \mathcal{A})^{-1}\,\mathcal{A}\,(d(x_{n-1}, x_n))\,. \tag{3.27}$$

(c): If $u_1 = \frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_{n+1}))\right) \\ &\preceq \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(x_{n-1}, x_n))\right) + \mathcal{A}\left(\frac{1}{2}(d(x_n, x_{n+1}))\right) \end{aligned}$$

$$\Rightarrow (2 \cdot \mathcal{I} - \mathcal{A})\left(d(x_n, x_{n+1})\right) \preceq \mathcal{A}\left(d(x_{n-1}, x_n)\right)$$
$$\Rightarrow d(x_n, x_{n+1}) \preceq (2 \cdot \mathcal{I} - \mathcal{A})^{-1} \mathcal{A}\left(d(x_{n-1}, x_n)\right). \tag{3.28}$$

From (3.26), (3.27) and (3.28), we obtain that $\{x_n\}$ is a Cauchy sequence and hence convergent to an element x in A. As f is Perov type proximal contraction of second kind, we have

$$d(fx_n, fx_{n+1}) \preceq \mathcal{A}(u_2),$$

where

$$\begin{aligned} u_2 \in \{d(fx_{n-1}, fx_n), &\frac{1}{2}(d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})), \\ &\frac{1}{2}(d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n))\}. \end{aligned}$$

(a): If $u_1 = d(fx_{n-1}, fx_n)$, then

$$d(fx_n, fx_{n+1}) \preceq \mathcal{A}(d(fx_{n-1}, fx_n)). \tag{3.29}$$

(b): If $u_1 = \frac{1}{2}(d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}))$, then

$$\begin{aligned} d(fx_n, fx_{n+1}) &\preceq \mathcal{A}\left(\frac{1}{2}(d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(fx_{n-1}, fx_n))\right) + \mathcal{A}\left(\frac{1}{2}(d(fx_n, fx_{n+1}))\right) \end{aligned}$$

$$\Rightarrow (2 \cdot \mathcal{I} - \mathcal{A})\left(d(fx_n, fx_{n+1})\right) \preceq \mathcal{A}\left(d(fx_{n-1}, fx_n)\right)$$
$$\Rightarrow d(fx_n, fx_{n+1}) \preceq (2 \cdot \mathcal{I} - \mathcal{A})^{-1} \mathcal{A}\left(d(fx_{n-1}, fx_n)\right). \tag{3.30}$$

(c): If $u_1 = \frac{1}{2}(d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n))$, then

$$\begin{aligned} d(fx_n, fx_{n+1}) &\preceq \mathcal{A}\left(\frac{1}{2}(d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(fx_{n-1}, fx_{n+1}))\right) \\ &\preceq \mathcal{A}\left(\frac{1}{2}(d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1}))\right) \\ &= \mathcal{A}\left(\frac{1}{2}(d(fx_{n-1}, fx_n))\right) + \mathcal{A}\left(\frac{1}{2}(d(fx_n, fx_{n+1}))\right) \end{aligned}$$

$$\Rightarrow (2\cdot\mathcal{I}-\mathcal{A})\left(d(fx_n,fx_{n+1})\right) \preceq \mathcal{A}\left(d(fx_{n-1},fx_n)\right)$$
$$\Rightarrow d(fx_n,fx_{n+1}) \preceq (2\cdot\mathcal{I}-\mathcal{A})^{-1}\mathcal{A}\left(d(fx_{n-1},fx_n)\right). \qquad (3.31)$$

Again by (3.29), (3.30) and (3.31), $\{fx_n\}$ is a Cauchy sequence and hence convergent to an element y in B. Thus, $d(x,y)=\lim\limits_{n\to\infty} d(x_{n+1},fx_n)=p$. Now $x\in A_0$ and since $f(A_0)\subseteq B_0$, we have $d(z_1,fx)=p$ for some $z_1\in A_0$. As f is Perov type proximal contraction of first kind, $d(z_1,x_{n+1})\preceq \mathcal{A}d(x,x_n)$. Consequently, $\{x_n\}$ must converge to u which further implies that $u=x$. Hence $d(x,fx)=p$. The uniqueness and the remaining part of the proof follow using similar arguments to those in the proof of Theorem 3.8.

Remark 3.12. Let us remark that all the results in this paper hold for regular cone (solid or non solid).

Example 3.13. Let $X=[0,1]$, $E=X\times X$ a unit square in $\mathbb{R}^2$ and

$$P=\{(x,y)\in E : x,y\geq 0\}\subset E.$$

Define $d: X\times X\to E$ by $d(x,y)=(|x-y|,\alpha|x-y|)$, where $\alpha\geq 1$. Then (X,d) is a cone metric space. Let $A=\left[0,\frac{1}{2}\right]$ and $B=\left[\frac{2}{3},1\right]$ and define $f:A\to B$ by

$$fx=\begin{cases} 1 & x\in\left[0,\frac{1}{5}\right] \\ \frac{2}{3} & x\in\left(\frac{1}{5},\frac{1}{2}\right]. \end{cases}$$

Clearly, f is discontinuous map and $f(A)\subseteq B$.
Define a linear bounded operator $\mathcal{A}: E\to E$ by

$$\mathcal{A}=\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{3}{7} & \frac{2}{5} \end{bmatrix}.$$

Clearly with $\|\mathcal{A}\|<1$ and $\mathcal{A}(P)\subset P$.
Note that, for $u=v=\dfrac{1}{2}$ and $x,y\in\left(\dfrac{1}{5},\dfrac{1}{2}\right]$, we have

$$d(u,fx)=\left(\left|\frac{1}{2}-\frac{2}{3}\right|,\alpha\left|\frac{1}{2}-\frac{2}{3}\right|\right)=d(v,fy)=\left(\frac{1}{6},\frac{1}{6}\alpha\right)=p \text{ (say)}.$$

Also, for $u_1=d(x,y)$, we obtain

$$\begin{aligned} d(u,v) &= (0,0) \\ &\preceq \left(\left(\frac{1}{3}+\frac{1\alpha}{4}\right)|x-y|,\left(\frac{7}{3}+\frac{2}{5}\alpha\right)|x-y|\right) \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{3}{7} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} |x-y| \\ \alpha|x-y| \end{bmatrix} \\ &= \mathcal{A}(d(x,y)) \\ &= \mathcal{A}(u_1) \end{aligned}$$

and for $u_2 = \dfrac{1}{2}(d(fx, fu) + d(fy, fv))$,

$$\begin{aligned} d(fu, fv) &= (0, 0) \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{3}{7} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \\ &= \mathcal{A}\left(\frac{1}{2}(d(fx, fu) + d(fy, fv))\right) \\ &= \mathcal{A}(u_2), \end{aligned}$$

where

$$u_1 \in \left\{ d(x, y), \frac{1}{2}(d(x, u) + d(y, v)), \frac{1}{2}(d(x, v) + d(y, u)) \right\}$$

and

$$u_2 \in \left\{ d(fx, fy), \frac{d(fx, fu) + d(fy, fv)}{2}, \frac{d(fx, fv) + d(fy, fu)}{2} \right\}.$$

Thus, f is Zamfirescu proximal cyclic contraction of Perov first (second) kind with $f(A) \subseteq B$. Furthermore, there exists $x = \dfrac{1}{2} \in A$ such that $d(x, fx) = \left(\dfrac{1}{6}, \dfrac{1}{6}\alpha\right) = p$. Moreover, for any sequence $\{x_n\}$ satisfying $d(x_{n+1}, fx_n) = p$, converges to $\dfrac{1}{2}$. □

Example 3.14. Let $X = [0, 1]$, and $E = C^1[0, 1]$ equipped with a norm

$$\|x\| = \|x\|_\infty + \|x'\|_\infty$$

and $P = \{x \in E : x(t) \geq 0 \text{ on } [0, 1]\}$. Let $A = \left[0, \frac{1}{2}\right]$ and $B = \left[\frac{2}{3}, 1\right]$. Define a cone metric $d : X \times X \to E$ by

$$d(x, y) = \frac{|x - y|}{t + 1}.$$

Then $d(A, B) = \frac{1}{6(t+1)}$. Define $f : A \to B$ by

$$fx = \begin{cases} 1 & x \in [0, \frac{1}{5}] \\ \frac{2}{3} & x \in (\frac{1}{5}, \frac{1}{2}] \end{cases} \tag{3.32}$$

and a bounded linear operator $\mathcal{A} : E \to E$ by

$$\mathcal{A}(f(t)) = \int_0^{\frac{1}{5}} \frac{1}{t + 3} f(s) ds.$$

Clearly, $||\mathcal{A}|| \leq \frac{1}{15}$ and $f(A) \subseteq B$. Now, for $u = v = \frac{1}{2}$ and $x, y \in \left(\frac{1}{5}, \frac{1}{2}\right]$, we have

$$d(u, fx) = \frac{1}{6(t + 1)} = d(v, fy).$$

Also,

$$d(u, v) \preceq \mathcal{A}(u_1)$$

and

$$d(fu, fv) \preceq \mathcal{A}(u_2)$$

where

$$u_1 \in \left\{ d(x,y), \frac{1}{2}(d(x,u)+d(y,v)), \frac{1}{2}(d(x,v)+d(y,u)) \right\}$$

and

$$u_2 \in \left\{ d(fx,fy), \frac{d(fx,fu)+d(fy,fv)}{2}, \frac{d(fx,fv)+d(fy,fu)}{2} \right\}.$$

That is, f is Zamfirescu proximal cyclic contraction of Perov first kind and second kind. Moreover, all the conditions of Theorem 3.11 are satisfied.

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