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SETS WITH STRUCTURE, MAPPINGS AND FIXED POINT PROPERTY: FIXED POINT STRUCTURES

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Abstract. In the book, *Fixed Point Structure Theory* (I.A. Rus, *Fixed Point Structure Theory*, Cluj Univ. Press, Cluj-Napoca, 2006) there are studied fixed point structures on a set with structure. In this paper we introduce the notion of the set-mapping pair $(\mathcal{U}, M)$ (i.e., $\mathcal{U} :=$ a class of sets with the same type structure and for $X, Y \in \mathcal{U}$ a set of mappings, $M(X,Y)$, from X to Y is given) and the notion of fixed point structure (f.p.s.) on a such pair. After some examples of f.p.s. we study the preserving of the fixed point property by, $(\mathcal{U}, M)$-bijections, retractions, cartesian product and exponential. We give some fixed point results in terms of a f.p.s. and we consider some special f.p.s.: f.p.s. with common fixed point property, with coincidence point property and with coincidence producing mappings. Some open problems are also formulated.
Key Words and Phrases: Set with structure, set-mapping pair, category of sets with structure, fixed point structure, retraction, coretraction, coincidence point property, common fixed point property, coincidence producing mapping, cartesian product, exponential.
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1. Introduction

In the book [104] we have studied the following problem:

Let T be a fixed point theorem and f be a mapping which does not satisfy the conditions of T. In which conditions the mapping f has an invariant subset Y such that the restriction of f to Y, $f|_Y$, satisfies the conditions of T?

In order to formalize and study this problem, we introduced the notion *fixed point structure* on a set with structure.

The purpose of this paper is to study the following problem:

Let X be a set with structure and $M(X,X)$ be a nonempty set of mappings from X to X (morphisms or not). In which conditions for each $f \in M(X,X)$, the fixed point set of f, $F_f \neq \emptyset$?

There are various concepts of mathematical structure on an abstract set (see [77], [114], [2], [12], [18], [38], [22], [65], [75], [76], [82], [92], [93], [104], [105], [115], [118], [133], [47]). In this paper we do not work with a such abstract notion. By $\mathcal{U} :=$ a class of sets with structure, we understand that $\mathcal{U}$ is a class of monoids, or $\mathcal{U}$ is a class of ordered sets, or $\mathcal{U}$ is a class of topological spaces, or $\mathcal{U}$ is a class of metric spaces, ..., i.e., $\mathcal{U}$ is a class of sets with the same type of structure. If X is a set with structure and $Y \subset X$, we consider Y endowed with the structure induced by that of X. If $X, Y \in \mathcal{U}$, we denote by $\mathbb{M}(X, Y)$ the set of all mappings from the set X to the set Y. We also denote by the same symbol the set and the set with structure (i.e., $(X, \leq)$ by X, (X, d) by X, (X, τ) by X,...).

Instead of "Preliminaries" we indicate some references, as follow:

• *set with structure*: N. Bourbaki [22], P.R. Halmos [58], S. Mac Lane [76], [77], J. Dieudonné [40], [42], S. Vasilache [133];

• *category theory and graph theory*: S. Mac Lane [75], M. Barr and C. Wells [12], B. Plotkin [93], A. Blass [18];

• *fixed point, common fixed point and coincidence point theory*: A. Granas and J. Dugundji [57], R.F. Brown, M. Furi, L. Górniewicz and B. Jiang [27], A. Granas [56], A. McLennan [82], I.A. Rus, A. Petruşel and G. Petruşel [110], A. Buică [29], M.A. Şerban [120], R.F. Brown [26], H. Amann [2], L. Górniewicz [54], M. Florenzano [48], R. Precup [94], I.A. Rus [109], V. Berinde [50];

• *punctual problem in mathematics versus research program*: J. Dieudonné [41], Y.I. Manin [78], M. Atiyah [9], R. Thom [129].

Throughout this paper we follow the notations and terminology given in [104] and [109].

The structure of the paper is the following:

1. Introduction
2. Set-mapping pairs
3. Fixed point structures on a set-mapping pair
4. Invariance of $\mathcal{S}_{max}$ by some constructions in $(\mathcal{U}, M)$
5. Fixed point results in terms of a fixed point structure
6. Coincidence point property and fixed point property
7. Fixed point structures on a set-multivalued mapping pair

2. Set-mapping pairs

Let $\mathcal{U}$ be a class of nonempty sets with structure. We suppose that for each ordered pair (X, Y) with $X, Y \in \mathcal{U}$, a set of mappings from the set X to the set Y (morphisms or not), $M(X, Y)$ is given. We call the pair $(\mathcal{U}, M)$ a set-mapping pair. By $f \in M$ we understand that there exist $X, Y \in \mathcal{U}$ such that $f \in M(X, Y)$.

In the mathematical universe generated by the books which we refereed in Introduction, the following definitions and the definitions given in section 3 appear in a natural manner.

Definition 2.1. A pair $(\mathcal{U}, M)$ is with composition if $f, g \in M$ and $f \circ g$ is defined, then $f \circ g \in M$, i.e., M is a partial semigroup, since the composition is associative.

Definition 2.2. A pair $(\mathcal{U}, M)$ is with identity if it is with composition and for each $X \in \mathcal{U}$, $1_X \in M(X, X)$, i.e., M is a partial monoid.

Definition 2.3. A pair $(\mathcal{U}, M)$ is with restriction if $U, X, Y \in \mathcal{U}$, $U \subset X$ and $f \in M(X, Y)$ imply that $f|_U \in M(U, Y)$, where $f|_U$ is defined by $f|_U(u) = f(u)$, $\forall\, u \in U$.

Definition 2.4. A pair $(\mathcal{U}, M)$ is with contraction if $U, V, X, Y \in \mathcal{U}$, $U \subset X$, $V \subset Y$, $f \in M(X,Y)$ with $f(U) \subset V$ imply that $f|_{U,V} \in M(U,V)$, where $f|_{U,V}$ is defined by $f|_{U,V}(u) = f(u)$, $\forall\ u \in U$.

Definition 2.5. A bijective mapping $f \in \mathbb{M}(X,Y)$ is a $(\mathcal{U}, M)$-bijection if for $h \in M(Y,Y)$ and $g \in M(X,X)$ we have that $f^{-1} \circ h \circ f \in M(X,X)$ and $f \circ g \circ f^{-1} \in M(Y,Y)$.

Definition 2.6. Let $(\mathcal{U}, M)$ be a set-mapping pair. By definition, $Y \in \mathcal{U}$ is a retract of $X \in \mathcal{U}$ if there exists two mappings, $r \in \mathbb{M}(X,Y)$ and $s \in \mathbb{M}(Y,X)$ such that:

(a) $r \circ s = 1_Y$;

(b) for all $h \in M(Y,Y)$ and $g \in M(X,X)$ we have that $s \circ h \circ r \in M(X,X)$ and $r \circ g \circ s \in M(Y,Y)$.

We call r a retraction mapping and s a corectraction mapping, or, r is a retraction with respect to s, or (r,s) is a retraction-coretraction pair.

If $Y \subset X$, then, in general, we take s the inclusion mapping.

For retraction theory in various structured sets see: [21], [63], [12], [25], [30], [32], [35], [52], [44], [56], [57], [66], [75], [82], [97], [104], [107], [108], [110], [117], [118], [128], [102].

Example 2.1. $\mathcal{U} :=$ the class of all nonempty metric spaces and for $X, Y \in \mathcal{U}$,

$$M(X,Y) := \{f : X \to Y \mid f \text{ is a contraction}\}.$$

This pair is with composition and is not with identity. If $cardX > 1$, then $1_X \notin M(X,X)$. The pair $(\mathcal{U}, M)$ is with restriction and with contraction. If $f \in \mathbb{M}(X,Y)$ is an isometry, then f is a $(\mathcal{U}, M)$-bijection and $f \notin M(X,Y)$, in general.

Example 2.2. $\mathcal{U} :=$ the class of all nonempty metric spaces and

$$M(X,Y) := \{f : X \to Y \mid f \text{ is nonexpansive}\}.$$

This pair is with composition, $1_X \in M(X,X)$, $\forall\ X \in \mathcal{U}$, is with restriction and with contraction. Moreover, $(\mathcal{U}, M)$ is a category. The morphisms, $Hom = M$ and the isomorphisms are $(\mathcal{U}, M)$-bijections.

Example 2.3. $\mathcal{U} :=$ the class of all ordered sets, (i.e., posets, see [88]) and

$$M(X,Y) := \{f : X \to Y \mid f \text{ is increasing}\}.$$

In this case, $(\mathcal{U}, M)$ is a category. The isomorphisms of this category are $(\mathcal{U}, M)$-bijections.

Example 2.4. $\mathcal{U} :=$ the class of all normed spaces over $\mathbb{K} := \mathbb{R} \vee \mathbb{C}$ and

$$M(X,Y) := \{f : X \to Y \mid f \in C(X,Y) \text{ and } \overline{f(X)} \text{ is compact}\}.$$

This pair is with composition but not with identity. The isometries are $(\mathcal{U}, M)$-bijections and are not in M.

Example 2.5. Let X be a normed space,

$$\mathcal{U} := P_{cl,cv}(X) := \{Y \in P(X) \mid Y \text{ closed and convex}\}$$

and

$$M(Y,Z) := \{f : Y \to Z \mid f \text{ is nonexpansive}\}.$$

This pair is with composition, with restriction and with contraction.

Example 2.6. For (X, τ), a topological space, we take

$$\mathcal{U} := P_{cp}(X) := \{(Y, \tau_Y) \mid Y \in P(X),\ Y - \text{compact}\}$$

and for $Y, Z \in \mathcal{U}$,

$$M(Y, Z) := C(Y, Z) := \{f : Y \to Z \mid f \text{ is continuous}\}.$$

For other examples see section 3.

3. Fixed point structures on a set-mapping pair

Let $(\mathcal{U}, M)$ be a set-mapping pair and $\mathcal{S} \subset \mathcal{U}$, $\mathcal{S} \neq \emptyset$.

Definition 3.1. The triple $(\mathcal{U}, \mathcal{S}, M)$ is a fixed point structure (f.p.s.) on $(\mathcal{U}, M)$ if for each $X \in \mathcal{S}$ and $f \in M(X, X)$ we have that, $F_f \neq \emptyset$.

Definition 3.2. Let $\mathcal{S}_{max}$:= the class of all $X \in \mathcal{U}$ such that, $f \in M(X, X) \Rightarrow F_f \neq \emptyset$. By definition, the triple $(\mathcal{U}, \mathcal{S}_{max}, M)$ is the maximal *f.p.s.* on $(\mathcal{U}, M)$.

Example 3.1. $\mathcal{U}$:= the class off all ordered sets and

$$M(X, Y) := \{f : X \to Y \mid f \text{ is increasing}\}.$$

If $\mathcal{S}$:= the class of all complete ordered sets, then the triple $(\mathcal{U}, \mathcal{S}, M)$ is, by Tarski's fixed point theorem, a f.p.s. In this case, $\mathcal{S} \neq \mathcal{S}_{max}$ (see [80], [81], [44]).

It is a problem to characterize the ordered set with fixed point property with respect to increasing mappings (see [2], [16], [80], [81], [97], [98], [117], [44], [112]).

Example 3.2. $\mathcal{U}$:= the class of all lattice-like ordered sets,

$$M(X, Y) := \{f : X \to Y \mid f \text{ is increasing}\}$$

and $\mathcal{S}$:= the class of all complete lattice-like ordered sets. By Tarski's fixed point theorem, $(\mathcal{U}, \mathcal{S}, M)$ is a f.p.s. By Davis' theorem, $\mathcal{S} = \mathcal{S}_{max}$.

Remark 3.1. In the Mathematical Logic and in the Mathematical Informatic, some objects are defined by "fixed point constructions" (see [12], [71], [74], [79], [80], [83], [92], [93], [118], [132], [134], [44]). In order to have uniqueness results by these constructions, it has been introduced the least (i.e., minimum) fixed point property with respect to increasing mappings. These studies open some new research directions in the fixed point theory. For example, let X be a set with structure and P be a property with respect to the elements of X. Let $f : X \to X$ be a mapping. *The problem is to find the conditions in which the mapping f has a unique fixed point with property P.*

Example 3.3. $\mathcal{U}$:= the class of all metric spaces,

$$M(X, Y) := \{f : X \to Y \mid f \text{ is a contraction}\}$$

and $\mathcal{S}$:= the class of all complete metric spaces. By contraction principle, $(\mathcal{U}, \mathcal{S}, M)$ is a f.p.s. In this case, $\mathcal{S} \neq \mathcal{S}_{max}$ (see [104], [107], [126], [127]).

It stills an open problem to characterize the metric spaces with fixed point property with respect to contractions.

Example 3.4. If $(X, \leq)$ is an ordered set, we denote by $\max X$, the maximal element set of X. Now, let $(X, \leq)$ be an ordered set. If we take, $\mathcal{U} := \{\ (Y, \leq) \mid Y \in P(X)\ \}$, $M(Y, Z) := \{\ f : Y \to Z \mid f \text{ is progressive, i.e., } x \leq f(x),\ \forall\ x \in Y\}$ and $\mathcal{S} := \{\ Y \mid Y \in \mathcal{U} \text{ with } \max Y \neq \emptyset\}$. Then by a folklore remark, the triple $(\mathcal{U}, \mathcal{S}, M)$ is a f.p.s. In this case, $\mathcal{S} = \mathcal{S}_{max}$.

Example 3.5. Let (X, d) be a complete metric space. We take

$$\mathcal{U} := \{(Y, d|_{Y \times Y}) \mid Y \in P(X)\},$$

$$M(Y, Z) := \{f : Y \to Z \mid f \text{ is a Caristi-Kirk mapping}\},$$

and

$$\mathcal{S} := P_{cl}(X) := \{Y \in P(X) \mid Y = \overline{Y}\}.$$

By Caristi-Kirk fixed point theorem, $(\mathcal{U}, \mathcal{S}, M)$ is a f.p.s. Moreover, $\mathcal{S} = \mathcal{S}_{max}$ (see [104], [107], [28], [126], [127]).

Example 3.6. $\mathcal{U} :=$ the class of all groups, $M := Hom :=$ the class of all group morphisms. Then the triple $(\mathcal{U}, \mathcal{U}, M)$ is a f.p.s. Each group has the fixed point property with respect to the group morphisms. A group endomorphism which has only the trivial fixed point is called *fixed point free endomorphism*. The existence of such endomorphism is one of the basic problems of the theory of finite groups (see [53], [130], [99] and the references therein).

Example 3.7. $\mathcal{U} :=$ the class of all topological spaces and $M(X, Y) := C(X, Y)$. A topological space $X \in \mathcal{S}_{max}$ of $(\mathcal{U}, M)$ is by definition a topological space with fixed point property. In Topology, $X \in \mathcal{S}_{max}$ is called fixed point space and $f \in C(X, Y)$, a map.

Examples of fixed point spaces are given by fixed point theorems of: Brouwer, Schauder, Tychonoff, Cauty, Lefschetz,...

It is a problem to characterize the fixed point spaces, i.e., the elements of $\mathcal{S}_{max}$ (see [21], [56], [57], [67], [84], [96], [85], [27], [26],...).

4. Invariance of $\mathcal{S}_{max}$ by some constructions in $(\mathcal{U}, M)$

In what follows we propose to study the invariance of $\mathcal{S}_{max}$, in a $(\mathcal{U}, M)$ pair, by $(\mathcal{U}, M)$-bijections, retractions, cartesian product and exponential. Here are some results.

Theorem 4.1. *Let $(\mathcal{U}, M)$ be a set-mapping pair, $X \in \mathcal{S}_{max}$ and $Y \in \mathcal{U}$. If there exists a $(\mathcal{U}, M)$-bijection $f \in \mathbb{M}(X, X)$, then $Y \in \mathcal{S}_{max}$.*

Proof. Let $g \in M(Y, Y)$. Then $f^{-1} \circ g \circ f \in M(X, X)$. Since $X \in \mathcal{S}_{max}$, there exists $x^* \in X$ such that, $f^{-1} \circ g \circ f(x^*) = x^*$. From this we have that, $g(f(x^*)) = f(x^*)$, i.e., $F_g \neq \emptyset$. So, $Y \in \mathcal{S}_{max}$.

Theorem 4.2. *Let $(\mathcal{U}, M)$ be a set-mapping pair. If $Y \in \mathcal{U}$ is a retract of $X \in \mathcal{S}_{max}$, then $Y \in \mathcal{S}_{max}$.*

Proof. Let $r \in \mathbb{M}(X, Y)$ and $s \in \mathbb{M}(Y, X)$ such that, $r \circ s = 1_Y$. Let $f \in M(Y, Y)$. Then, $s \circ f \circ r \in M(X, X)$. Since, $X \in \mathcal{S}_{max}$, there exists $x^* \in X$ such that, $s \circ f \circ r(x^*) = x^*$. This implies that $r \circ s \circ f \circ r(x^*) = r(x^*)$. Since the composition is associative, it follows that $f(r(x^*)) = r(x^*)$, i.e., $F_f \neq \emptyset$. So, $Y \in \mathcal{S}_{max}$.

Definition 4.1. A class $\mathcal{U}$ of sets with structure is with cartesian product if for all $X, Y \in \mathcal{U}$, the cartesian product set, $X \times Y$, endowed with the usual structure (structures!) induced by those of X and Y, with respect to which, $X \times Y \in \mathcal{U}$. A set-mapping pair, $(\mathcal{U}, M)$ is with cartesian product if it is with composition, with restriction, $\mathcal{U}$ is with cartesian product and for each, $f = (f_1, f_2) \in M(X \times Y, X \times Y)$ we have that:

(1) $f_1 \in M(X \times Y, X)$ and $f_2 \in M(X \times Y, Y)$;
(2) $f_1(\cdot, y) \in M(X, X)$ and $f_1(x, \cdot) \in M(Y, X)$, $\forall\, x \in X$, $\forall\, y \in Y$;
(3) $f_2(x, \cdot) \in M(Y, Y)$ and $f_2(\cdot, y) \in M(X, Y)$, $\forall\, x \in X$, $\forall\, y \in Y$.

Remark 4.1. In $\mathcal{U}$ is with cartesian product and for $p_X : X \times Y \to X$, $(x, y) \mapsto x$ and $p_Y : X \times Y \to Y$, $(x, y) \mapsto y$, we have that $p_X \in M(X \times Y, X)$ and $p_Y \in M(X \times Y, Y)$, then $(\mathcal{U}, M)$ is with cartesian product.

Example 4.1. $\mathcal{U} :=$ the class of all metric spaces and

$$M(X, Y) := \{f : X \to Y \mid f \text{ is a contraction}\}.$$

If we take for (X, d) and (Y, ρ), on $X \times Y$ (by abuse of notations!) the metric, $\max(d, \rho)$, or $d + \rho$, or $(d^2 + \rho^2)^{\frac{1}{2}}$, then $(\mathcal{U}, M)$ is with cartesian product. In this case, p_X and p_Y are not in M.

Example 4.2. $\mathcal{U} :=$ the class of all ordered sets and

$$M(X, Y) := \{f : X \to Y \mid f \text{ is increasing}\}.$$

In this case, $p_X, p_Y \in M$. So, $(\mathcal{U}, M)$ is with cartesian product.

Example 4.3. $\mathcal{U} :=$ the class of all Hausdorff topological spaces and

$$M(X, Y) := \{f \in C(X, Y) \mid \overline{f(X)} \in P_{cp}(Y)\}.$$

If we take on $X \times Y$ the usual topology, then $(\mathcal{U}, M)$ is with cartesian product and, p_X and p_Y are not in M, in general.

Problem 4.1. Let $(\mathcal{U}, M)$ be with cartesian product. If $X, Y \in \mathcal{S}_{max}$, in which conditions, $X \times Y \in \mathcal{S}_{max}$?

Remark 4.2. Let $(\mathcal{U}, M)$ be with cartesian product. We suppose that for $X, Y \in \mathcal{U}$, $x_0 \in X$, $y_0 \in Y$, the mapping p_X and p_Y are in M and in addition, the mapping $s_X : X \to X \times Y$, $x \mapsto (x, y_0)$ and $s_Y : Y \to X \times Y$, $y \mapsto (x_0, y)$ are in M. If $X \times Y \in \mathcal{S}_{max}$, then X and Y are in $\mathcal{S}_{max}$. Indeed, we remark that X is a retract of $X \times Y$ by (p_X, s_X) and Y is a retract of $X \times Y$ by (p_Y, s_Y) retraction-coretraction pair.

Remark 4.3. Let $(\mathcal{U}, M)$ be with cartesian product and $(X, \mathcal{S}, M)$ be a f.p.s. on $(\mathcal{U}, M)$. For $X, Y \in \mathcal{S}$, $f \in M(X \times Y, X \times Y)$, $f = (f_1, f_2)$, we consider the multivalued mappings:

$$P : Y \multimap X,\ y \mapsto F_{f_1(\cdot, y)} \text{ and } Q : X \multimap Y,\ x \mapsto F_{f_2(x, \cdot)}.$$

We suppose that a selection p of P and a selection q of Q exist such that, p and q are in M. It is clear that:

$$f_1(p(y), y) = p(y),\ \forall\, y \in Y$$

$$f_2(x, q(x)) = q(x),\ \forall\, x \in X.$$

Since $(\mathcal{U}, M)$ is with composition, the mapping, $q \circ p \in M(Y, Y)$. From $Y \in \mathcal{S}$, there exists $\overline{y} \in Y$ such that, $q(p(\overline{y})) = \overline{y}$. Let us denote, $\overline{x} := p(\overline{y})$. We remark that, $(\overline{x}, \overline{y}) \in F_f$. So, $X \times Y \in \mathcal{S}_{max}$.

From the above considerations it follows that the *cartesian product problem* is a *selection problem* for multivalued mappings, i.e., a difficult one (see, [57], [20], [89], [10], [90]).

For the history of Problem 4.1. see I. Rival [97] and, D. Duffus and I. Rival [44] in the case of ordered sets, R.F. Brown [24] in the case of topological spaces and M.A. Şerban [120] and [121] in the case of ordered sets and of metric spaces.

A general remark on Problem 4.1. is that we have more counterexamples than results. These counterexamples give rise to:

Problem 4.2. Let $(\mathcal{U}, \mathcal{S}, M)$ be a fixed point structure on $(\mathcal{U}, M)$. We suppose that $(\mathcal{U}, M)$ is with cartesian product. In which conditions, we have that:

$$X, Y \in \mathcal{S} \quad \Rightarrow \quad X \times Y \in \mathcal{S}\ ?$$

Example 4.4. Let $\mathcal{U}$:= the class of all ordered set,

$$M(X,Y) := \{f : X \to Y \mid f \text{ is increasing}\}$$

and $\mathcal{S}$:= the class of all complete ordered set. In this case, $X, Y \in \mathcal{S}$ imply that $X \times Y \in \mathcal{S}$. Indeed, if X and Y are complete ordered sets, then $X \times Y$ is also a complete ordered set.

Example 4.5. Let $\mathcal{U}$:= the class of all metric spaces,

$$M(X,Y) := \{f : X \to Y \mid f \text{ is a contraction}\}$$

and $\mathcal{S}$:= the class of all complete metric spaces. Then if X and Y are in $\mathcal{S}$, then $X \times Y \in \mathcal{S}$. Indeed, if X and Y are complete metric spaces, then $X \times Y$ is a complete metric space with max-metric, for example (see Example 4.1.).

Problem 4.3. Let $(\mathcal{U}, M)$ and $(\mathcal{U}, M_1)$ be two set-mapping pairs with $M_1 \subset M$, i.e., $X, Y \in \mathcal{U} \Rightarrow M_1(X,Y) \subset M(X,Y)$, for all $X, Y \in \mathcal{U}$. In which conditions we have that:

$$X, Y \in \mathcal{S}_{max} \quad \Rightarrow \quad X \times Y \in \mathcal{S}_{1max}\ ?$$

Remark 4.4. In the notations in Remark 4.3 we have that: If $f \in M_1(X \times Y, X \times Y)$ implies that there exist $p, q \in M$, then $X, Y \in \mathcal{S}_{max}$ implies that $X \times Y \in \mathcal{S}_{1max}$.

Example 4.6. $\mathcal{U}$:= the class of all metric spaces,

$$M(X,Y) := \{f : X \to Y \mid f \text{ is a contraction}\}$$

and

$$M_1(X,Y) := \left\{f : X \to Y \mid f \text{ is } l\text{-Lipschitz with } l < \frac{1}{2}\right\}.$$

Then the pairs $(\mathcal{U}, M)$ and $(\mathcal{U}, M_1)$ are a solution of Problem 4.3. Indeed, let (X, d) and (Y, ρ) in $\mathcal{U}$. We take on $X \times Y$ a usual metric, for example,

$$(\max(d,\rho)((x_1,y_1),(x_2,y_2)) := \max(d(x_1,x_2), \rho(y_1,y_2)).$$

In this case, $P = \{p\}$ and $Q = \{q\}$. Let $f \in M_1(X \times Y, X \times Y)$. Then p and q are with Lipschitz constant $2l$. So, p and q are in M. The proof follows form Remark 4.4.

Example 4.7. Let $(\mathcal{U}, M)$ be with cartesian product and $(\mathcal{U}, \mathcal{S}, M)$ be a f.p.s. on $(\mathcal{U}, M)$. For $X, Y \in \mathcal{U}$, let

$$M^t(X \times Y, X \times Y) := \{f \in M(X \times Y, X \times Y) \mid f \text{ is a triangular mapping, i.e.,}$$

$$f(x,y) = (f_1(x), f_2(x,y)),\ \forall\ x \in X,\ \forall\ y \in Y\}.$$

If $X, Y \in \mathcal{S}$, then $X \times Y$ has the fixed point property with respect to $M^t(X \times Y, X \times Y)$.

Example 4.8. Let $(\mathcal{U}, M)$ be with cartesian product.
For $X \in \mathcal{U}$ and $f \in M(X \times X, X)$, by definition, a pair $(x, y) \in X \times X$ is called coupled fixed points (*intertwined fixed points* in H. Amann [2](1976)) for f if $x = f(x, y)$ and $y = f(y, x)$. Let us consider the mapping, $T_f : X \times X \to X \times X$, defined by

$$T_f(x, y) := (f(x, y), f(y, x)).$$

We observe that (x, y) are coupled fixed points for f if (x, y) is a fixed point of T_f. So, *the coupled fixed points problem is a particular problem of a fixed point problem of mappings on a cartesian product.*

Definition 4.2. Let $(\mathcal{U}, M)$ be a set-mapping pair with cartesian product and $(\mathcal{U}, \mathcal{S}, M)$ be a f.p.s. on $(\mathcal{U}, M)$. By definition, $X \in \mathcal{S}$ is with fixed point selection property with respect to $Y \in \mathcal{U}$, on the right, if for all $f \in M(X \times Y, X)$ the multivalued mapping, $P : Y \multimap X$ defined by, $y \multimap F_{f(\cdot,y)}$ has a selection $p \in M(Y, X)$. By definition, $Y \in S$ is with fixed point selection property with respect to $X \in \mathcal{U}$, on the left, if for all $f \in M(X \times Y, Y)$, the multivalued operator $Q : X \multimap Y$, defined by, $x \multimap F_{f(x,\cdot)}$ has a selection $q \in M(X, Y)$.

Remark 4.5. For the notions in Definition 4.2., in the case of topological structure, see [128].

Definition 4.3. Let $(\mathcal{U}, M)$ be with composition and with restriction. By definition, $(\mathcal{U}, M)$ is with exponential if for all $X, Y \in \mathcal{U}$, $Y^X := M(X, Y)$, endowed with the usual structure induced by those of X and Y, $M(X, Y) \in \mathcal{U}$.

The relevance of Definition 4.2 and 4.3 follow, for example, from the following results.

Theorem 4.3. *Let $(\mathcal{U}, M)$ be with cartesian product and with exponential. If for $X, Y \in \mathcal{U}$, $M(Y, X) \in \mathcal{S}_{max}$, then $X \in \mathcal{S}_{max}$ and X has the fixed point selection property with respect to Y, to the right.*

Proof. Let $T \in M(X \times Y, X)$. This mapping induces the mapping, $\tilde{T} : M(Y, X) \to M(Y, X)$, defined by, $h \mapsto T(h(\cdot), \cdot)$. Since $(\mathcal{U}, M)$ is with cartesian product, with exponential and $M(Y, X) \in \mathcal{S}_{max}$, hence there exists $h^* \in M(Y, X)$ such that, $T(h^*(y), y) = h^*(y)$, $\forall\ y \in Y$, i.e., X has the fixed point selection property with respect to Y, to the right.

From this result and Remark 4.3 we have,

Theorem 4.4. *Let $(\mathcal{U}, M)$ be with cartesian product and with exponential. If for X, Y in $\mathcal{U}$ we have that, $M(X, Y) \in \mathcal{S}_{max}$ and $M(Y, X) \in \mathcal{S}_{max}$, then $X \times Y \in \mathcal{S}_{max}$.*

Remark 4.6. In X and Y are sets with structure, for the standard structure of $X \times Y$ and Y^X induced by those of X and Y see [2], [12], [13], [22], [63], [65], [76], [82], [118], [128], [44], [27], [130], [37], [125], [112].

For the fixed point theory on cartesian product see [120], [24], [51], [57], [70], [97], [111], [128], [131], [44], [27], [45], [89], [90], [119], [113], [46], [125], [5], [121], [84], [100], [101].

The following Lawvere result has close connections with the above results (see [73], [123], [17], [12]).

Lawvere's Theorem. Let $\mathscr{C}$ be a category of sets with structure. We suppose that $\mathscr{C}$ is with finite products and powers. Let X and Y be two objects of $\mathscr{C}$. If there exists a surjective morphism, $f : X \to Hom(X, Y)$, then Y has the fixed point property, i.e., $h \in Hom(Y, Y) \Rightarrow F_h \neq \emptyset$.

In what follows, we give a variant of Lawvere's theorem in a $(\mathcal{U}, M)$ set-mapping pair. To do this, we need some preliminaries.

If X is a nonempty set, we consider the diagonal mapping $\Delta : X \to X \times X$, $x \mapsto (x, x)$. A mapping $f \in \mathbb{M}(X, \mathbb{M}(X, Y))$ induces the mapping, $\tilde{f} : X \times X \to Y$, $(x, y) \mapsto f(x)(y)$.

Theorem 4.5. *Let $(\mathcal{U}, M)$ be with cartesian product and with exponential, and X, Y in $\mathcal{U}$. We suppose that there exists $f \in M(X, M(X, Y))$ such that:*

1. *f is surjective;*
2. *$\tilde{f} \circ \Delta \in M(X, Y)$.*

Then, $Y \in \mathcal{S}_{max}$.

Proof. Let $h \in M(Y, Y)$. Then $h \circ \tilde{f} \circ \Delta \in M(X, Y)$. Since f is surjective, there exists $y \in X$ such that, $h \circ \tilde{f} \circ \Delta = f(y)$. This implies that, $h \circ \tilde{f} \circ \Delta(x) = f(y)(x)$, $\forall\, x \in X$, i.e., $h \circ f(x)(x) = f(y)(x)$, $\forall\, x \in X$. For $x = y$, we have, $h(f(y)(y)) = f(y)(y)$. So, $F_h \neq \emptyset$.

From the above results the following problem appears.

Problem 4.4. In which set-mapping pairs, $(\mathcal{U}, M)$, there exists $X \in \mathcal{U}$ for which there exists a bijection, $f \in M(X, M(X, X))$?

A solution X of this problem is by definition a reflexive set with structure. For more considerations on this notion see [73], [12], [123], [17].

5. Fixed point results in terms of a f.p.s.

In section 4 we have considered some constructions in a set-mapping pair which preserve fixed point in a set-mapping pair. In what follows we present the f.p.s. as a frame-work that allows us to give a fixed point result from old one, i.e., to translate results on a set with structure to another one.

Theorem 5.1. *Let $(\mathcal{U}, M)$ be a set-mapping pair with composition, $X \in \mathcal{S}_{max}$, $Y \in \mathcal{U}$ and $f \in M(Y, Y)$. If f factors through X, in $(\mathcal{U}, M)$, then, $F_f \neq \emptyset$.*

Proof. Since f factors through X, in $(\mathcal{U}, M)$, hence there exist $g \in M(Y, X)$ and $h \in M(X, Y)$, such that, $f = h \circ g$. Since $(\mathcal{U}, M)$ is with composition, $g \circ h \in M(X, X)$. Now, $X \in \mathcal{S}_{max}$ implies that there exists $x^* \in X$ such that, $g(h(x^*)) = x^*$. This implies that, $h \circ g \circ h(x^*) = h(x^*)$, i.e., $f(h(x^*)) = h(x^*)$. So, $F_f \neq \emptyset$.

Remark 5.1. For Theorem 5.1 in the case of topological structure see, [56] and [57].

Theorem 5.2. *Let $(\mathcal{U}, M)$ be a set-mapping pair, $X, Y \in \mathcal{U}$, $X \subset Y$ and $f \in \mathbb{M}(X, Y)$. We suppose that:*

1. *$X \in \mathcal{S}_{max}$;*
2. *$r : Y \to X$ is a set retraction, i.e., $r|_X = 1_X$;*
3. *$r \circ f \in M(X, X)$.*

Then, or $F_f \neq \emptyset$, or $r(f(X) \setminus X) \cap F_{r \circ f} \neq \emptyset$.

Proof. From (1) and (3), there exists $x^* \in X$ such that, $r(f(x^*)) = x^*$. If $f(x^*) \in X$, since r is a set retraction, $f(x^*) = x^*$. If $f(x^*) \in Y \setminus X$, then $x^* \in r(f(X) \setminus X)$, i.e., $r(f(X) \setminus X) \cap F_{r \circ f} \neq \emptyset$.

Remark 5.2. For the impact of set retraction in the fixed point theory of nonself mappings see [25], [108], [109]. Here are some examples.

Example 5.1. Let H be a Hilbert space, $\mathcal{U} := P_{cl}(H)$,

$$M(X,Y) := \{f : X \to Y \mid f \text{ is a contraction}\}.$$

Let $X := \overline{B}(0;R)$, $Y := H$, $f : \overline{B}(0;R) \to H$ be a contraction. Since r is nonexpansive, $r \circ f \in M(\overline{B}(0;R), \overline{B}(0;R))$. From Theorem 5.2, or $F_f \neq \emptyset$, or there exists $x^* \in \partial\overline{B}(0;R)$ and $\lambda > 1$ such that $f(x^*) = \lambda x^*$. We remark that $r \notin M$.
Example 5.2. Let $\mathbb{B}$ be a Banach space, $\mathcal{U} := P_{cl,cv}(\mathbb{B})$,

$$M(X,Y) := \{f : X \to Y \mid f \text{ is complete continuous}\}.$$

Let $f \in M(\overline{B}(0;R), \mathbb{B})$. We remark that by second Schauder's fixed point theorem, $\overline{B}(0;R) \in \mathcal{S}_{max}$. Let $r : \mathbb{B} \to \overline{B}(0;R)$ be the radial retraction. Since r is continuous, $r \circ f \in M(\overline{B}(0;R), \overline{B}(0;R))$. From Theorem 5.2, or $F_f \neq \emptyset$ or there exists $x^* \in \partial\overline{B}(0;R)$ and $\lambda > 1$ such that, $f(x^*) = \lambda x^*$.
Example 5.3. Let $(X, \leq)$ be an ordered set,

$$\mathcal{U} := \{(Y, \leq) \mid Y \in P(X)\},$$

$$M(Y,Z) := \{f : Y \to Z \mid f \text{ is increasing}\}$$

and $\mathcal{S} := \{\, Y \in \mathcal{U} \mid (Y, \leq) \text{ is a complete ordered set}\}$. Let $Y \in \mathcal{S}$ such that the least element of X, $\perp \in Y$. Let $f \in M(Y,Z)$ and the retraction $r \in M(Z,Y)$, defined by,

$$r(x) := \begin{cases} x, & \text{for } x \in Y; \\ \sup_Y([\perp, x] \cap Y), & \text{for } x \in Z \setminus Y. \end{cases}$$

From Theorem 5.2. we have that there exists $x^* \in Y$ such that if $f(x^*) \in Y$, then $f(x^*) = x^*$ and if $f(x^*) \in Z \setminus Y$, then $\sup_Y([\perp, f(x^*)] \cap Y) = x^*$.

The following problems are a source for new results from old ones.
Problem 5.1. Let $(\mathcal{U}, M)$ be a set-mapping pair with composition and restriction, and $(\mathcal{U}, \mathcal{S}, M)$ be a *f.p.s.* on $(\mathcal{U}, M)$. In which conditions on f, the following implication holds:

$$Y \in \mathcal{S},\ X \in \mathcal{U},\ Y \subset X,\ f \in \mathbb{M}(Y,X),\ Y \subset f(Y) \quad \Rightarrow \quad F_f \neq \emptyset\ ?$$

The following problems are particular cases of Problem 5.1.
Problem 5.1$_a$. (O.H. Hamilton [59]). Let $Y \subset \mathbb{R}^2$ be a continuum which does not separate $\mathbb{R}^2$ and $f : Y \to \mathbb{R}^2$ be an interior (i.e., open) continuous function such that, $Y \subset f(Y)$. Does $F_f \neq \emptyset$?
Problem 5.1$_b$. (S.B. Nadler [86]). Let $\overline{B}(0;R)$ be the Euclidean ball in $\mathbb{R}^n$, $n > 1$, Let $0 < R_1 < R_2$ and $f \in C(\overline{B}(0;R_1), \overline{B}(0,R_2))$ a surjective function. Does $F_f \neq \emptyset$?
Problem 5.1$_c$. (T.L. Hicks - L.M. Saliga [60]). Let (X,d) be a complete metric space, $Y \in P_{cl}(X)$ and $f : Y \to X$ be such that, $Y \subset f(Y)$. In which conditions on f, we have that, $F_f \neq \emptyset$?

A folklore result is the following:

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If there exists an interval $[a,b] \subset \mathbb{R}$ such that, $[a,b] \subset f([a,b])$, then, $F_f \neq \emptyset$.
For some results on the above problems see: [59], [4], [86], [60], [104], [107], [6], [36].

Problem 5.2. Let $(\mathcal{U}, M)$ be a set-mapping pair with composition and with contraction, and $(\mathcal{U}, \mathcal{S}, M)$ be a f.p.s. on $(\mathcal{U}, M)$. Let $X \in \mathcal{S}$ and $f, g \in M(X,X)$ such that, $f \circ g = g \circ f$. In which conditions we have that, $F_f \cap F_g \neq \emptyset$?
Definition 5.1. Let $(\mathcal{U}, \mathcal{S}, M)$ be as in Problem 5.2. An element $X \in \mathcal{S}$ is with common fixed point property if for all $f, g \in M(X,X)$ with, $f \circ g = g \circ f$, $F_f \cap F_g \neq \emptyset$, i.e, X is a solution of Problem 5.2.
Remark 5.3. Let $(\mathcal{U}, \mathcal{S}, M)$ be as in Problem 5.2, and $X \in \mathcal{S}$. If $f \in M(X,X)$ implies that F_f endowed with the structure induced by that of X is in $\mathcal{S}$, then X is with common fixed point property. Indeed, since $f \circ g = g \circ f$, we have that, $f(F_g) \subset F_g$ and $g(F_f) \subset F_f$.
Example 5.5. A fixed point structure $(\mathcal{U}, \mathcal{S}, M)$ is with uniqueness if $X \in \mathcal{S}$ and $f \in M(X,X)$ implies that, $card F_f = 1$. It is clear that each fixed point structure with uniqueness is with common fixed point property, i.e., each $x \in X$ is with common fixed point property.
Example 5.6. $\mathcal{U} :=$ the class of all lattice-like ordered sets,

$$M(X,Y) := \{f : X \to Y \mid f \text{is increasing}\}$$

and $\mathcal{S} :=$ the class of all complete lattice-like ordered sets. Since for $X \in \mathcal{S}$ and $f \in M(X,X)$ we have that, $F_f \in \mathcal{S}$, hence all $X \in \mathcal{S}$ is with common fixed point property, i.e., $(\mathcal{U}, \mathcal{S}, M)$ is with the common fixed point property.

For other examples and for counterexamples of sets with structure with fixed point property which are with common fixed point property see [104], [107], [57], [110] and the references therein.

6. Coincidence point property and fixed point property

For X and Y two nonempty sets and $f, g \in \mathbb{M}(X,Y)$ we denote by

$$C(f,g) := \{x \in X \mid f(x) = g(x)\},$$

the coincidence point set of f and g.
Definition 6.1. Let $(\mathcal{U}, M)$ be a set-mapping pair and $X, Y \in \mathcal{U}$. A mapping $p \in \mathbb{M}(X,Y)$ is a coincidence producing mapping on $M(X,Y)$, if for all $g \in M(X,Y)$, $C(f,g) \neq \emptyset$.
Remark 6.1. In some particular structures this notion appears under various names. For example:

(1) W. Holsztynski [61]. Let X and Y be topological spaces. A mapping $f \in C(X,Y)$ is universal mapping on $C(X,Y)$ if for all $g \in C(X,Y)$, $C(f,g) \neq \emptyset$. In [116] H. Schirmer uses the name, coincidence producing instead of universal.
(2) W.A. Kirk [69]. Let X and Y be two metric spaces. A nonexpansive mapping, $f : X \to Y$, is universal nonexpansive mapping if for all nonexpansive mapping, $g : X \to Y$, $C(f,g) \neq \emptyset$.
(3) M. Furi, M. Martelli and A. Vignoli [50]. Let X and Y be two Banach spaces. A mapping $f \in C(X,Y)$ is a strong surjection if for all $g \in C(X,Y)$ with $\overline{g(X)} \in P_{cp}(Y)$, $C(f,g) \neq \emptyset$.

For more considerations on this notion and some basic properties see A. Buică [29] and the references therein. See also [106], [7], [8].

Remark 6.2. Let $X, Y \in \mathcal{U}$. If all constant mappings from X to Y are in $M(X,Y)$ and $p \in \mathbb{M}(X,Y)$ is a coincidence producing mapping on $M(X,Y)$, then p is surjective.

Remark 6.3. If $p \in \mathbb{M}(X,Y)$ is a coincidence producing on $M(X,Y)$ such that, $g \circ p \in M(X,Y)$ for all $g \in M(X,Y)$, then $Y \in \mathcal{S}_{max}$ of $(\mathcal{U}, M)$.

Example 6.1. $\mathcal{U} :=$ the class of all connected Hausdorff topological spaces and $M(X,Y) := C(X,Y)$ for all $X, Y \in \mathcal{U}$. In this case all surjective $p \in M(X,Y)$, where $X \in \mathcal{U}$, and $Y \in \mathcal{U}$ is endowed with a total order, $\leq$, such that $(Y, \tau_Y, \leq)$ is an ordered topological space, are coincidence producing mappings on $M(X,Y)$. For example we can take, $Y := [a,b] \subset \mathbb{R}$ with the usual topology and usual order structure.

Indeed, let $f \in M(X,Y)$ and

$$A := \{x \in X \mid p(x) \leq f(x)\}$$

and

$$B := \{x \in X \mid p(x) \geq f(x)\}.$$

Since p and f are continuous, and p is surjective, $A, B \in P_{cl}(X)$. Since, $\leq$ is a total order we have that, $X = A \cup B$. Since X is connected, $A \cap B \neq \emptyset$, i.e., $C(p,f) \neq \emptyset$.

Example 6.2. Let $(\mathcal{U}, M)$ be a set-mapping pair, $X \in \mathcal{S}_{max}$ of $(\mathcal{U}, M)$ and $p \in M(X,X)$ be a bijective mapping such that, $p^{-1} \circ h \in M(X,X)$, for all $h \in M(X,X)$. Then p is a coincidence producing mapping on $M(X,X)$.

Definition 6.2. A f.p.s., $(\mathcal{U}, \mathcal{S}, M)$ is with coincidence point property if $X \in \mathcal{S}$, $f, g \in M(X,X)$, $f \circ g = g \circ f$ imply that, $C(f,g) \neq \emptyset$.

Example 6.3. Each f.p.s., $(\mathcal{U}, \mathcal{S}, M)$ with common fixed point property is with coincidence point property.

Example 6.4. $\mathcal{U} := P(\mathbb{R})$, $M(X,Y) := C(X,Y)$ and $\mathcal{S} := P_{cp,cv}(\mathbb{R})$. The fixed point structure, $(\mathcal{U}, \mathcal{S}, M)$ is with coincidence point property. This follows from a coincidence point theorem of Horn [62].

Problem 6.1. To give examples of fixed point structures with coincidence point property.

This problem is a difficult one. A relevant example is the following conjecture:

Horn's Conjecture. ([62], [122]). Let $\mathbb{B}$ be a Banach space, and $X \in P_{cp,cv}(\mathbb{B})$. If $f, g \in C(X,X)$ and $f \circ g = g \circ f$, then, $C(f,g) \neq \emptyset$.

The following conjecture is a particular case of Horn's conjecture.

Schauder-Browder-Nussbaum conjecture. ([122], [107]). Let $\mathbb{B}$ be a Banach space, $Y \in P_{b,cl,cv}(\mathbb{B})$ and $f : Y \to Y$ be a mapping. We suppose that:

(1) $f \in C(Y,Y)$;
(2) there exists $n \in \mathbb{N}^*$ such that, $\overline{f^n(Y)}$ is compact.

Then, $F_f \neq \emptyset$.

In general we have that:

Remark 6.5. Let $(\mathcal{U}, \mathcal{S}, M)$ be a f.p.s. with the coincidence property. Let $X \in \mathcal{S}$ and $f \in \mathbb{M}(X,X)$. If there exists $k \in \mathbb{N}^*$ such that, f^k and f^{k+1} are in $M(X,X)$, then $F_f \neq \emptyset$.

7. Fixed point structure on a set-multivalued mapping pair

The problem is to give similar notions, problems and results in the case of multivalued mappings. For the fixed point structures for multivalued mappings on a set with structure see [104] and [107] an dthe references therein.

As starting references look to: [57], [82], [12], [54], [48], [104], [107], [110], [90], [112], [5], [46].

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