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LARGE TIME BEHAVIOR OF SOLUTIONS TO A SYSTEM OF COUPLED NONLINEAR OSCILLATORS VIA A GENERALIZED FORM OF SCHAUDER-TYCHONOFF FIXED POINT THEOREM

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Dedicated to Academician Dorin Ieşan on the occasion of his eighties anniversary

Abstract. In this paper we investigate the stability of the null solution of a system of ODEs describing the motion of two coupled damped nonlinear oscillators. We also show that for any solution (x, y) of the system we have $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0$, for small initial data in the case when the uniqueness of solutions is not guaranteed. Our proofs are mainly based on a generalized form of Schauder-Tychonoff fixed point theorem. The theoretical results are illustrated with numerical simulations.

Key Words and Phrases: Coupled oscillators, uniform stability, asymptotic stability, fixed point theorem.
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1. Introduction

Consider a mechanical system consisting of 2 blocks having the same mass, m, as shown in Figure 1. We suppose that the stiffnesses of the springs are represented by the functions $k_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \overline{1,2}$ (where $\mathbb{R}_+ := [0, +\infty)$) and the functions $\widehat{f}_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \overline{1,2}$ denote the friction coefficients of the horizontal surface. We assume that, when the two blocks are in their equilibrium positions, the springs are also in their equilibrium positions.

Let the displacements of the blocks from their equilibrium positions be x, y. We suppose that the system moves under the action of some external forces $\widehat{g}_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i \in \overline{1,2}$, depending on the time and the displacements.

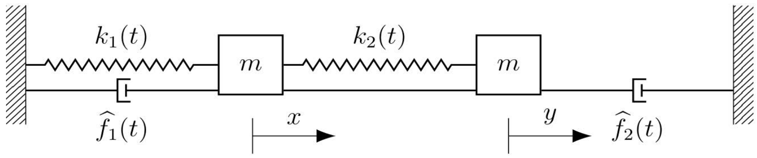

FIGURE 1. Two 1-D coupled damped nonlinear oscillators

Therefore we can associate with the above physical application the following system of ODEs describing the motion of the oscillators (see, e.g., [16])

$$\begin{cases} \ddot{x} + 2f_1(t)\dot{x} + \beta(t)x - \gamma(t)y + g_1(t,x,y) = 0, \\ \ddot{y} + 2f_2(t)\dot{y} + \delta(t)y - \gamma(t)x + g_2(t,x,y) = 0, \end{cases} \tag{1.1}$$

where

$$\beta(t) := \frac{1}{m}(k_1(t) + k_2(t)), \ \delta(t) := \frac{1}{m}k_2(t),$$

$$f_1(t) := \frac{1}{m}\widehat{f}_1(t), \ f_2(t) := \frac{1}{m}\widehat{f}_2(t), \ \gamma(t) := \frac{1}{m}k_2(t),$$

$$g_1(t,x,y) := -\frac{1}{m}\widehat{g}_1(t,x,y), \ g_2(t,x,y) := -\frac{1}{m}\widehat{g}_2(t,x,y).$$

In [15], [16] we provided stability results for different systems of nonlinear coupled oscillators. Our approaches were based on elementary differential inequalities and on the classical Lyapunov's method. For other results regarding the asymptotic stability of the equilibria of coupled damped nonlinear oscillators, we refer the reader to [10], [17]-[20], and the references therein. For fundamental concepts and results in stability theory we refer the reader to [2], [6], [7].

The general case of a single damped nonlinear oscillator is described by the following equation which is well-known in the literature

$$\ddot{x} + 2f^*(t)\dot{x} + \beta^*(t)x + g^*(t,x) = 0, \quad t \in \mathbb{R}_+. \tag{1.2}$$

T.A. Burton and T. Furumochi [4] introduced a new method, based on the Schauder fixed point theorem (see, e.g., [21, p. 218], [25, Theorem 2.A, Corollary 2.13]), to study the stability of the null solution of Eq. (1.2) in the case $\beta^*(t) = 1$. In [1], [13] we reported new stability results for the same equation using some Bernoulli type differential inequalities, and in [14], [24] we considered Eq. (1.2) under more general assumptions, which required more sophisticated arguments. For other investigations regarding the asymptotic stability of the equilibrium of a single damped nonlinear oscillator, we refer the reader to [8]-[9], [12]-[11], [22], and the references therein.

A powerful tool to deduce the existence of solutions to initial value problems is Schauder-Tychonoff fixed point theorem (see, e.g., [21, p. 218], [25, Corollary 9.6]), which is an extension to locally convex spaces of the Schauder fixed point theorem.

In the present paper we will present some results on the stability of the equilibrium of system (1.1) by using a generalized form of Schauder-Tychonoff theorem (see Theorem 3.1 below), on the metrizable locally convex space of the continuous functions defined on a half-line $[t_0, +\infty)$, endowed with a countable family of seminorms as chosen as to determine the uniform convergence on the compact subsets of this interval (see Theorem 2.1 below). We also prove that for any solution (x, y) to system (1.1) we have $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0$, for small initial data, in the case when the nonlinearities are not necessarily locally Lipschitz functions (hence uniqueness is not guaranteed).

2. General framework and main result

The following hypotheses will be admitted:

(H1) $f_1,\ f_2 \in C^1(\mathbb{R}_+)$, $f_1(t) \geq 0$, $f_2(t) \geq 0$, $\forall t \in \mathbb{R}_+$;

(H2) there exist three constants $h,\ K_1,\ K_2 \geq 0$ such that

$$\left|\dot{f}_i(t) + f_i^2(t)\right| \leq K_i \widetilde{f}(t),\ \forall t \geq h,\ \forall i \in \{1, 2\}, \tag{2.3}$$

where $\widetilde{f}(t) := \min\{f_1(t), f_2(t)\}$, $\forall t \in \mathbb{R}_+$;

(H3) $\int_0^{+\infty} \widetilde{f}(t)\,\mathrm{d}t = +\infty$;

(H4) $\gamma \in C(\mathbb{R}_+)$, $\gamma(t) \geq 0$, $\forall t \in \mathbb{R}_+$, and $\int_0^{+\infty} \gamma(t)\,\mathrm{d}t < +\infty$;

(H5) $\beta,\ \delta \in C^1(\mathbb{R}_+)$, $\beta,\ \delta$ are decreasing, and

$$\beta_0 := \lim_{t\to+\infty} \beta(t) > 0,\ \delta_0 := \lim_{t\to+\infty} \delta(t) > 0$$

are such that

$$\frac{K_1}{\sqrt{\beta_0}} + \frac{K_2}{\sqrt{\delta_0}} < 1; \tag{2.4}$$

(H6) $g_i \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$, $i \in \{1, 2\}$ and fulfill the relations

$$|g_1(t, x, y)| \leq r_1(t)\,\mathrm{o}(|x|), \quad \forall t \in \mathbb{R}_+,\ \forall y \in \mathbb{R}, \tag{2.5}$$

$$|g_2(t, x, y)| \leq r_2(t)\,\mathrm{o}(|y|), \quad \forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{R}, \tag{2.6}$$

where $r_i \in C(\mathbb{R})$, $r_i(t) \geq 0$, $\forall t \in \mathbb{R}_+$, $\int_0^{+\infty} r_i(t)\,\mathrm{d}t < +\infty$, $i \in \{1, 2\}$, and "$\mathrm{o}(|x|)$" denotes the usual Landau symbol for $x \to 0$, i.e., $\lim_{x\to 0} \frac{\mathrm{o}(|x|)}{|x|} = 0$ (the same definition for "$\mathrm{o}(|y|)$").

(H7) g_i, $i \in \{1, 2\}$ are locally Lipschitzian with respect to x, y.

Remark 2.1. *If (H1) and (H2) hold, then $f_i,\ \dot{f}_i$ are bounded, $i \in \{1, 2\}$. Indeed, by (H2) we see that*

$$(t \geq h,\ f_i(t) > K_i) \Longrightarrow \dot{f}_i(t) < 0,$$

which, combined with (H1), implies $f_i(t) \leq M_i := \max\{f_i(h), K_i\}$, $\forall t \geq h$. So, using again (H2), we obtain $\left|\dot{f}_i(t)\right| \leq 2M_i^2$, $\forall t \geq h$. This concludes the proof, since, by (H1), $f_i,\ \dot{f}_i \in C[0, h]$, $i \in \{1, 2\}$.

Our main result is the following.

Theorem 2.1. *i) Suppose that hypotheses (H1)-(H6) are fulfilled. Then for every solution (x, y) to system (1.1), we have*

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0,$$

for small initial data.

ii) If hypotheses (H1)-(H7) are fulfilled, then the null solution of (1.1) *is asymptotically stable.*

iii) If hypotheses (H1)-(H2), (H4)-(H7) are fulfilled, then the null solution of (1.1) *is uniformly stable.*

Remark 2.2. *Let us note that in order to prove the first assertion of Theorem 2.1 the hypothesis (H7), which ensures the uniqueness of the solution of any initial value problem associated to system* (1.1)*, is not needed. So while ii) and iii) are comparable to the stability results reported in* [16]*, the statement i) is new and is obtained by using a generalized form of Schauder-Tychonoff theorem (see Section* 3 *below). This shows that the fixed point method is efficient in studying the behavior at infinity of the solutions of system* (1.1) .

3. Proof of Theorem 2.1

By using the transformation (similar to that introduced by T.A. Burton and T. Furumochi in [4])

$$\begin{cases} \dot{x} = u - f_1(t)\,x \\ \dot{u} = \left[\dot{f}_1(t) + f_1^2(t) - \beta(t)\right] x - f_1(t)\,u + \gamma(t)\,y - g_1(t,x,y) \\ \dot{y} = v - f_2(t)\,y \\ \dot{v} = \gamma(t)\,x + \left[\dot{f}_2(t) + f_2^2(t) - \delta(t)\right] y - f_2(t)\,v - g_2(t,x,y) \end{cases} \tag{3.7}$$

system (1.1) becomes

$$\dot{z} = A(t)\,z + B(t)\,z + F(t,z)\,, \tag{3.8}$$

where

$$z = \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix},\ A(t) = \begin{pmatrix} -f_1(t) & 1 & 0 & 0 \\ -\beta(t) & -f_1(t) & \gamma(t) & 0 \\ 0 & 0 & -f_2(t) & 1 \\ \gamma(t) & 0 & -\delta(t) & -f_2(t) \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dot{f}_1(t) + f_1^2(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dot{f}_2(t) + f_2^2(t) & 0 \end{pmatrix},\ F(t,z) = \begin{pmatrix} 0 \\ -g_1(t,x,y) \\ 0 \\ -g_2(t,x,y) \end{pmatrix},$$

and our large time behavior question reduces to the large time behavior of the solutions to system (3.8).

Let $t_0 \geq 0$ be arbitrarily fixed and let

$$Z(t,t_0) = \begin{pmatrix} a_1(t,t_0) & b_1(t,t_0) & c_1(t,t_0) & d_1(t,t_0) \\ a_2(t,t_0) & b_2(t,t_0) & c_2(t,t_0) & d_2(t,t_0) \\ a_3(t,t_0) & b_3(t,t_0) & c_3(t,t_0) & d_3(t,t_0) \\ a_4(t,t_0) & b_4(t,t_0) & c_4(t,t_0) & d_4(t,t_0) \end{pmatrix},\ t \geq t_0,$$

be the fundamental matrix to the linear system

$$\dot{z} = A(t)\,z, \tag{3.9}$$

which equals the identity matrix for $t = t_0$.

As in [16], we deduce

$$\begin{aligned}
\beta(t)\, a_{11}^2(t,t_0) + a_{21}^2(t,t_0) + \delta(t)\, a_{31}^2(t,t_0) + a_{41}^2(t,t_0) &\le \beta(t_0)\, \mathrm{e}^{I(t,t_0)}, && (3.10)\\
\beta(t)\, a_{12}^2(t,t_0) + a_{22}^2(t,t_0) + \delta(t)\, a_{32}^2(t,t_0) + a_{42}^2(t,t_0) &\le \mathrm{e}^{I(t,t_0)}, && (3.11)\\
\beta(t)\, a_{13}^2(t,t_0) + a_{23}^2(t,t_0) + \delta(t)\, a_{33}^2(t,t_0) + a_{43}^2(t,t_0) &\le \delta(t_0)\, \mathrm{e}^{I(t,t_0)}, && (3.12)\\
\beta(t)\, a_{14}^2(t,t_0) + a_{24}^2(t,t_0) + \delta(t)\, a_{34}^2(t,t_0) + a_{44}^2(t,t_0) &\le \mathrm{e}^{I(t,t_0)}, && (3.13)
\end{aligned}$$

for all $t \ge t_0$, where

$$I(t,t_0) := \int_{t_0}^{t} \left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}} \right] \mathrm{d}u,\ \forall t \ge t_0$$

and

$$\zeta(t) := \min\{\beta(t), \delta(t)\},\ \forall t \in \mathbb{R}_+.$$

Let $\|\cdot\|_0$ be the norm in $\mathbb{R}^4$ defined by

$$\|z\|_0 = \left(\beta_0 x^2 + u^2 + \delta_0 y^2 + v^2\right)^{1/2},\ \text{for } z = (x,u,y,v)^\top, \tag{3.14}$$

which is equivalent to the Euclidean norm.

For $z_0 \in \mathbb{R}^4$, from (3.10) – (3.13) and (H4), we deduce for all $t \ge t_0$

$$\|Z(t,t_0)\, z_0\|_0 \le \lambda \|z_0\|_0 \sqrt{\beta(t_0) + \delta(t_0) + 2}\, \exp\left(\int_{t_0}^{t} \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u \right), \tag{3.15}$$

where $\lambda := \max\{1, 1/\sqrt{\beta_0}, 1/\sqrt{\delta_0}\}$,

$$\begin{aligned}
\left\| Z(t,t_0)\, Z(s,t_0)^{-1} e_2 \right\|_0 &\le \exp\left(\int_s^t \left[-\widetilde{f}(u) + \tfrac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u \right),\\
\left\| Z(t,t_0)\, Z(s,t_0)^{-1} e_4 \right\|_0 &\le \exp\left(\int_s^t \left[-\widetilde{f}(u) + \tfrac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u \right),
\end{aligned} \tag{3.16}$$

for all $t \ge s \ge t_0 \ge 0$, where $e_2 = (0,1,0,0)^\top$, $e_4 = (0,0,0,1)^\top$,

$$\left\| Z(t,t_0)\, Z(s,t_0)^{-1} z_0 \right\|_0 \le \Lambda \|z_0\|_0 \exp\left(\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u \right), \tag{3.17}$$

for all $t \ge s \ge t_0 \ge 0$, where $\Lambda := \max\left\{ \sqrt{\frac{\beta(0)}{\beta_0}}, \sqrt{\frac{\delta(0)}{\delta_0}} \right\}$.

For $t_0 \ge 0$ we consider the functional space

$$C_c(t_0) := \left\{ z : [t_0, +\infty) \to \mathbb{R}^4,\ z \text{ continuous} \right\},$$

which becomes a metrizable locally convex space with respect to the countable family of seminorms

$$\|z\|_n := \sup_{t \in [t_0, n]} \{\|z(t)\|_0\},\ n \in \mathbb{N},\ n > t_0.$$

The metric is given by

$$d(z_1, z_2) := \sum_{\substack{n\in\mathbb{N} \\ n>t_0}} \frac{1}{2^n} \frac{\|z_1 - z_2\|_n}{1 + \|z_1 - z_2\|_n}, \ \forall z_1,\ z_2 \in C_c(t_0)$$

and since it is complete, the space $C_c(t_0)$ is actually a Fréchet space.

Notice that the topology defined by this family of seminorms is the topology of the uniform convergence on compact subsets of $[t_0, +\infty)$; in addition, a family $\mathcal{A} \subset C_c(t_0)$ is relatively compact if and only if it is equicontinuous and uniformly bounded on the compacts subsets of $[t_0, +\infty)$ (Arzelà-Ascoli Theorem).

Let $t_0 \geq 0$ and $z_0 \in \mathbb{R}^4$. We define on $C_c(t_0)$ the operator

$$(Hw)(t) := Z(t, t_0) z_0 + \int_{t_0}^{t} Z(t, t_0) Z^{-1}(s, t_0) [B(s) w(s) + F(s, w(s))] \,\mathrm{d}s, \quad (3.18)$$

for all $w \in C_c(t_0)$ and for all $t \geq t_0$.

If is obvious that the set of solutions to system (3.8) fulfilling the initial condition $z(t_0) = z_0$ is equal to the set of the fixed points of H.

From (2.5), (2.6) we infer that there exist $l_i > 0$, $i \in \{1, 2\}$, such that

$$\begin{aligned} |g_1(t, x, y)| &\leq r_1(t) |x|, \text{ if } |x| < l_1, \\ |g_2(t, x, y)| &\leq r_2(t) |y|, \text{ if } |y| < l_2. \end{aligned} \quad (3.19)$$

Let $q(t)$ be the unique solution to the initial value problem

$$\begin{aligned} \dot{q}(t) &= \left[-\widetilde{f}(t) + \frac{\left|\dot{f}_1(t) + f_1^2(t)\right|}{\sqrt{\beta_0}} + \frac{\left|\dot{f}_2(t) + f_2^2(t)\right|}{\sqrt{\delta_0}} \right. \\ &\qquad \left. + \frac{r_1(t)}{\sqrt{\beta_0}} + \frac{r_2(t)}{\sqrt{\delta_0}} + \frac{\gamma(t)}{2\sqrt{\zeta(t)}} \right] q(t), \ t \geq t_0, \\ q(t_0) &= \lambda \|z_0\|_0 \sqrt{\beta(t_0) + \delta(t_0) + 2} \end{aligned} \quad (3.20)$$

and let us consider the set

$$B(t_0, \rho) := \{ w \in C_c(t_0) \mid \|w(t)\|_0 \leq \rho \text{ and } \|w(t)\|_0 \leq q(t), \ \forall t \geq t_0 \},$$

for $t_0 \geq 0$ and $\rho > 0$. Since $w(t) := \min_{t\geq t_0} \{\rho, q(t)\} (0, 1, 0, 0)^\top$ is contained in $B(t_0, \rho)$, it follows that the set $B(t_0, \rho)$ is nonempty. Obviously $B(t_0, \rho)$ is a nonempty closed convex subset of $C_c(t_0)$, $\forall t_0 \geq 0$, $\forall \rho > 0$.

Lemma 3.1. *There exists $l > 0$, such that for all $t_0 \geq 0$ and for all $\rho \in (0, l)$, there exists $a > 0$, such that for all z_0 with $\|z_0\|_0 \in (0, a)$ the operator H maps $B(t_0, \rho)$ into $B(t_0, \rho)$.*

Proof. Let $l := \min\left\{\sqrt{\beta_0} l_1, \sqrt{\delta_0} l_2\right\}$, $\rho \in (0, l)$, $t_0 \geq 0$, and $z_0 \in \mathbb{R}^4 \setminus \{0\}$ with $\|z_0\|_0$ small enough. Consider an arbitrary $w \in B(t_0, \rho)$ with

$$w(t) := (x(t), u(t), y(t), v(t))^\top, \ \forall t \geq t_0.$$

From relations (3.16) and hypothesis (H2) we get for all $t \geq t_0$

$$\begin{aligned}\left\|\int_{t_0}^{t} Z(t,t_0) Z^{-1}(s,t_0) B(s) w(s)\,\mathrm{d}s\right\|_0 &\leq \int_{t_0}^{t} \exp\left(\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right) \\ &\quad \times \left[\left|\dot{f}_1(s) + f_1^2(s)\right| |x(s)| \right. \\ &\quad \left. + \left|\dot{f}_2(s) + f_2^2(s)\right| |y(s)|\right]\mathrm{d}s \\ &\leq \int_{t_0}^{t} \exp\left(\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right) \\ &\quad \times \left[\frac{\left|\dot{f}_1(s) + f_1^2(s)\right|}{\sqrt{\beta_0}} + \frac{\left|\dot{f}_2(s) + f_2^2(s)\right|}{\sqrt{\delta_0}}\right] \\ &\quad \times \|w(s)\|_0\,\mathrm{d}s, \end{aligned} \tag{3.21}$$

$$\begin{aligned}\left\|\int_{t_0}^{t} Z(t,t_0) Z^{-1}(s,t_0) F(s,w(s))\,\mathrm{d}s\right\|_0 &\leq \int_{t_0}^{t} \exp\left(\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right) \\ &\quad \times \left[|g_1(s,x(s),y(s))| \right. \\ &\quad \left. + |g_2(s,x(s),y(s))|\right]\mathrm{d}s \end{aligned} \tag{3.22}$$

Since $\|w(s)\|_0 \leq \rho$, $\forall s \geq t_0$, from (3.22) and (3.19) it follows that

$$\begin{aligned}\left\|\int_{t_0}^{t} Z(t,t_0) Z^{-1}(s,t_0) F(s,w(s))\,\mathrm{d}s\right\|_0 &\leq \int_{t_0}^{t} \exp\left(\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right) \\ &\quad \times \left[\frac{r_1(s)}{\sqrt{\beta_0}} + \frac{r_2(s)}{\sqrt{\delta_0}}\right] \|w(s)\|_0\,\mathrm{d}s \end{aligned} \tag{3.23}$$

for all $t \geq t_0$.

From (3.18), (3.15), (3.21) and (3.23) we deduce

$$\begin{aligned}\|(Hw)(t)\|_0 &\leq \lambda \|z_0\|_0 \sqrt{\beta(t_0) + \delta(t_0) + 2}\exp\left(\int_{t_0}^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right) \\ &\quad + \int_{t_0}^{t} \exp\left(\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right)\left[\frac{\left|\dot{f}_1(s) + f_1^2(s)\right|}{\sqrt{\beta_0}}\right. \\ &\quad \left. + \frac{\left|\dot{f}_2(s) + f_2^2(s)\right|}{\sqrt{\delta_0}} + \frac{r_1(s)}{\sqrt{\beta_0}} + \frac{r_2(s)}{\sqrt{\delta_0}}\right] \|w(s)\|_0\,\mathrm{d}s, \end{aligned} \tag{3.24}$$

for all $t \geq t_0$.

Since $\|w(s)\|_0 \le q(s)$, $\forall s \ge t_0$, we obtain

$$\begin{aligned}\|(Hw)(t)\|_0 &\le \lambda \|z_0\|_0 \sqrt{\beta(t_0)+\delta(t_0)+2}\exp\left(\int_{t_0}^t \left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right)\\ &\quad +\int_{t_0}^t \exp\left(\int_s^t \left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right)\left[\frac{\left|\dot{f}_1(s)+f_1^2(s)\right|}{\sqrt{\beta_0}}\right.\\ &\quad \left.+\frac{\left|\dot{f}_2(s)+f_2^2(s)\right|}{\sqrt{\delta_0}}+\frac{r_1(s)}{\sqrt{\beta_0}}+\frac{r_2(s)}{\sqrt{\delta_0}}\right]q(s)\,\mathrm{d}s =: \sigma(t),\ \forall t\ge t_0.\end{aligned}$$

But from (3.20) we have

$$\begin{aligned}\left[\frac{\left|\dot{f}_1(t)+f_1^2(t)\right|}{\sqrt{\beta_0}}+\frac{\left|\dot{f}_2(t)+f_2^2(t)\right|}{\sqrt{\beta_0}}+\frac{r_1(t)}{\sqrt{\beta_0}}+\frac{r_2(t)}{\sqrt{\delta_0}}\right]q(t) &= q(t)\left[\widetilde{f}(t)-\frac{\gamma(t)}{2\sqrt{\zeta(t)}}\right]\\ &\quad +\dot{q}(t),\end{aligned}$$

for all $t \ge t_0$. Hence,

$$\begin{aligned}\sigma(t) &= \lambda\|z_0\|_0\sqrt{\beta(t_0)+\delta(t_0)+2}\exp\left(\int_{t_0}^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right)\\ &\quad +\int_{t_0}^t \exp\left(\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right)\dot{q}(s)\,\mathrm{d}s\\ &\quad +\int_{t_0}^t \exp\left(\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right)q(s)\left[\widetilde{f}(s)-\frac{\gamma(s)}{2\sqrt{\zeta(s)}}\right]\mathrm{d}s,\end{aligned}$$

for all $t \ge t_0$. Integrating by parts $\int_{t_0}^t \exp\left(\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u\right)\dot{q}(s)\,\mathrm{d}s$, we easily infer that $\sigma(t)=q(t)$, $\forall t\ge t_0$ and hence $\|(Hw)(t)\|_0 \le q(t)$, $\forall t \ge t_0$.

Therefore

$$\begin{aligned}\|(Hw)(t)\|_0 &\le \lambda\|z_0\|_0\sqrt{\beta(t_0)+\delta(t_0)+2}\exp\left(-\int_{t_0}^t \widetilde{f}(s)\,\mathrm{d}s\right)\\ &\quad \times\exp\left(\frac{1}{\sqrt{\beta_0}}\int_{t_0}^t\left|\dot{f}_1(s)+f_1^2(s)\right|\mathrm{d}s\right)\\ &\quad \times\exp\left(\frac{1}{\sqrt{\delta_0}}\int_{t_0}^t\left|\dot{f}_2(s)+f_2^2(s)\right|\mathrm{d}s\right)\\ &\quad \times\exp\left(\frac{1}{\sqrt{\beta_0}}\int_{t_0}^t r_1(s)\,\mathrm{d}s\right)\exp\left(\frac{1}{\sqrt{\delta_0}}\int_{t_0}^t r_2(s)\,\mathrm{d}s\right)\\ &\quad \times\exp\left(\int_{t_0}^t\frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right),\ \forall t\ge t_0.\end{aligned}\tag{3.25}$$

Corresponding to the position of t_0 with respect to h from hypothesis (H2), we distinguish two cases.

Case 1: $t_0 \in [0,h)$. Since $f_i \in C^1[0,h]$ and $r_i,\ \beta,\ \delta,\ \gamma \in C[0,h],\ \forall i \in \{1,2\}$, from (3.25) we deduce that there exists a constant $D>0$, such that

$$\|(Hw)(t)\|_0 \le \lambda \|z_0\|_0 \sqrt{\beta(t_0)+\delta(t_0)+2}\exp(Dh),\ \forall t \in [t_0,h].$$

By (3.25) and hypothesis (H2) we derive

$$\begin{aligned}\|(Hw)(t)\|_0 \quad \le \quad & \lambda \|z_0\|_0 \sqrt{\beta(t_0)+\delta(t_0)+2}\exp(Dh) \\ & \times \exp\left(\left(\frac{K_1}{\sqrt{\beta_0}}+\frac{K_2}{\sqrt{\delta_0}}-1\right)\int_h^t \widetilde{f}(s)\,\mathrm{d}s\right) \\ & \times \exp\left(\frac{1}{\sqrt{\beta_0}}\int_h^t r_1(s)\,\mathrm{d}s+\frac{1}{\sqrt{\delta_0}}\int_h^t r_2(s)\,\mathrm{d}s\right) \\ & \times \exp\left(\int_h^t \frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right),\ \forall t \ge h.\end{aligned}$$

Let

$$\begin{aligned}a := \quad & \rho\exp(-Dh)\exp\left(-\frac{1}{\sqrt{\beta_0}}\int_h^{+\infty} r_1(s)\,\mathrm{d}s-\frac{1}{\sqrt{\delta_0}}\int_h^{+\infty} r_2(s)\,\mathrm{d}s\right) \\ & \times\exp\left(-\int_h^{+\infty}\frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right)/\left(\lambda\sqrt{\beta(t_0)+\delta(t_0)+2}\right).\end{aligned}$$

From (2.4) we deduce that if $\|z_0\|_0 < a$, then $\|(Hw)(t)\|_0 < \rho, \quad \forall t \ge t_0$.

Case 2: $t_0 \ge h$. We obtain similarly

$$\begin{aligned}\|(Hw)(t)\|_0 \quad \le \quad & \lambda\|z_0\|_0\sqrt{\beta(t_0)+\delta(t_0)+2}\exp\left(\left(\frac{K_1}{\sqrt{\beta_0}}+\frac{K_2}{\sqrt{\delta_0}}-1\right)\int_{t_0}^t \widetilde{f}(s)\,\mathrm{d}s\right) \\ & \times\exp\left(\frac{1}{\sqrt{\beta_0}}\int_{t_0}^t r_1(s)\,\mathrm{d}s+\frac{1}{\sqrt{\delta_0}}\int_{t_0}^t r_2(s)\,\mathrm{d}s\right) \\ & \times\exp\left(\int_{t_0}^t\frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right),\ \forall t \ge t_0\end{aligned}$$

and with the same a as in Case 1, $\|z_0\|_0 < a$ implies $\|(Hw)(t)\|_0 < \rho, \quad \forall t \ge t_0$.

Therefore the operator H maps $B(t_0,\rho)$ into $B(t_0,\rho)$. □

Taking into account Lemma 3.1, in order to prove the part i) of Theorem 2.1, it suffices to prove that system (3.8) admits solutions defined on $\mathbb{R}_+$ for initial data small enough. To this aim, we will apply the following generalized form of Schauder-Tychonoff fixed point theorem, which can be found, e.g., in [3, Appendix, Singbal Theorem], [5, Theorem 2.1 (b)].

Theorem 3.1. *Let E be a Hausdorff locally convex topological vector space. Let S be a nonempty closed convex set in E and let H be a continuous mapping of S into S, such that HS is contained in a compact set. Then H has a fixed point in S.*

Let $t_0 \geq 0$ and $\rho \in (0, l)$, where l is given by Lemma 3.1. Setting $E = C_c(t_0)$, H given by (3.18), and $S = B(t_0, \rho)$, we are going to prove the continuity of H and the relative compactness of $HB(t_0, \rho)$.

Consider $n \in \mathbb{N}$, $n > t_0$ fixed.

Let $(w_m)_{m \in \mathbb{N}} \subset B(t_0, \rho)$ be a sequence such that $w_m \to w$ in $C_c(t_0)$, as $m \to \infty$; that is, $\forall n \in \mathbb{N}$, $n > t_0$, $\forall \varepsilon > 0$, $\exists m_0$, $\forall m > m_0$, $\forall t \in [t_0, n]$, $\|w_m(t) - w(t)\|_0 < \varepsilon$.

From $(3.16) - (3.18)$, we deduce

$$\begin{aligned}
\|(Hw_m)(t) - (Hw)(t)\|_0 &\leq \int_{t_0}^{t} \left\| Z(t, t_0) Z^{-1}(s, t_0) B(s) [w_m(s) - w(s)] \right\|_0 \mathrm{d}s \\
&\quad + \int_{t_0}^{t} \left\| Z(t, t_0) Z^{-1}(s, t_0) [F(s, w_m(s)) \right. \\
&\quad \left. - F(s, w(s))] \right\|_0 \mathrm{d}s \\
&\leq \left(\frac{K_1}{\sqrt{\beta_0}} + \frac{K_2}{\sqrt{\delta_0}} \right) \int_{t_0}^{t} \exp\left(\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u \right) \\
&\quad \times \widetilde{f}(s) \|w_m(s) - w(s)\|_0 \mathrm{d}s \\
&\quad + \Lambda \int_{t_0}^{t} \exp\left(\int_s^t \left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}} \right] \mathrm{d}u \right) \\
&\quad \times \|F(s, w_m(s)) - F(s, w(s))\|_0 \mathrm{d}s,
\end{aligned}$$

for all $t \in [t_0, n]$. Hence there are some positive constants α_n and β_n, such that

$$\begin{aligned}
\|(Hw_m)(t) - (Hw)(t)\|_0 &\leq \alpha_n \int_{t_0}^{n} \|w_m(s) - w(s)\|_0 \mathrm{d}s \\
&\quad + \beta_n \int_{t_0}^{n} \|F(s, w_m(s)) - F(s, w(s))\|_0 \mathrm{d}s.
\end{aligned}$$

Since $F(t, z)$ is uniformly continuous for $t \in [t_0, n]$ and $\|z\|_0 \leq \rho$, it follows that the sequence $F(t, w_m(t))$ converges uniformly on $[t_0, n]$ to $F(t, w(t))$, which finally proves the continuity of H.

It remains to prove that $HB(t_0, \rho)$ is relatively compact. To this aim, we need to prove that for each $n \in \mathbb{N}$, the set $\left\{ (Hw)(t)|_{t \in [t_0, n]},\ w \in B(t_0, \rho) \right\}$ is uniformly bounded and equicontinuous.

Let $n \in \mathbb{N}$, $n > t_0$ be fixed.

Let $w \in B(t_0, \rho)$ be arbitrary; since $Hw \in B(t_0, \rho)$, it follows that $\|(Hw)(t)\|_0 \leq \rho$, $\forall t \in [t_0, n]$. Therefore, the set $HB(t_0, \rho)$ is uniformly bounded in $C_c(t_0)$.

Let $w \in B(t_0, \rho)$ be arbitrary and $z = Hw \in B(t_0, \rho)$. By differentiating (3.18) with respect to $t \in [t_0, n]$, we obtain

$$\dot{z}(t) = A(t) z(t) + B(t) w(t) + F(t, w(t)),\ \forall t \in [t_0, n]. \tag{3.26}$$

Since the functions f_i, β, δ, γ, $\left| \dot{f}_i + f_i^2 \right|$, $i \in \{1, 2\}$ are bounded on $[t_0, n]$ and $F(t, w)$ is bounded for $t \in [t_0, n]$, $\|w\|_0 \leq \rho$, it follows that there are some positive constants

φ_n, ψ_n, such that

$$\|\dot{z}(t)\| \le \varphi_n \rho + \psi_n, \qquad \forall t \in [t_0, n].$$

Therefore the family of the derivatives of the functions from $HB(t_0,\rho)$ is uniformly bounded and we deduce that $HB(t_0,\rho)$ is equicontinuous on the compact subsets of $\mathbb{R}_+$.

By applying Theorem 3.1, it follows that H admits fixed points in $B(t_0,\rho)$. Thus a solution $z(t)$ with initial data small enough exists on the whole $\mathbb{R}_+$. Since for all $t \ge t_0$

$$\begin{aligned} q(t) \quad &= \quad \lambda \|z_0\|_0 \sqrt{\beta(t_0)+\delta(t_0)+2} \exp\left(\left(\frac{K_1}{\sqrt{\beta_0}}+\frac{K_2}{\sqrt{\delta_0}}-1\right)\int_{t_0}^t \widetilde{f}(s)\,\mathrm{d}s\right) \\ &\times \exp\left(\frac{1}{\sqrt{\beta_0}}\int_{t_0}^t r_1(s)\,\mathrm{d}s + \frac{1}{\sqrt{\delta_0}}\int_{t_0}^t r_2(s)\,\mathrm{d}s + \int_{t_0}^t \frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right), \end{aligned}$$

from hypotheses (H3)-(H5) and Lemma 3.1, it follows that $\lim_{t\to+\infty}\|z(t)\|_0 = 0$ and so $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty}\dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty}\dot{y}(t) = 0$.

ii) If g_1, g_2 are locally Lipschitzian with respect to x, y, then the solution exists on the whole $\mathbb{R}_+$ for small initial data and is unique. So we can proceed with the stability question for the null solution to system (1.1), which, due to the boundedness of the functions f_i, β, δ, γ, $\left|\dot{f}_i + f_i^2\right|$, g_i, $i \in \{1,2\}$, reduces to the stability of the null solution $z(t) = 0$ to system (3.8).

By virtue of i), to prove the asymptotic stability, we should prove that the null solution to system (3.8) is stable.

Let $\varepsilon > 0$ and $\rho \in (0,l)$ be fixed. Consider $t_0 \ge 0$ and $z_0 \in \mathbb{R}^4 \setminus \{0\}$, with $\|z_0\|_0 < a$, where l and a are given by Lemma 3.1. If $z(t,t_0,z_0)$ is the solution of (3.8) which equals z_0 for $t = t_0$, then we have for all $t \ge t_0$

$$z(t,t_0,z_0) = Z(t,t_0)z_0 + \int_{t_0}^t Z(t,t_0)Z^{-1}(s,t_0)\left[B(s)z(s,t_0,z_0) + F(s,z(s,t_0.z_0))\right]\mathrm{d}s$$

and we already know from i) that $\|z(t,t_0,z_0)\|_0 \le \rho$, $\forall t \ge t_0$.

We distinguish again two cases.

Case 1: $t_0 \in [0,h)$. We deduce, as in the proof of Lemma 3.1,

$$\|z(t,t_0,z_0)\|_0 \le \lambda\|z_0\|_0\sqrt{\beta(t_0)+\delta(t_0)+2}\exp(Dh), \ \forall t \in [t_0,h]$$

and for all $t \ge h$

$$\begin{aligned} \|z(t,t_0,z_0)\|_0 \quad &\le \quad \lambda\|z_0\|_0\sqrt{\beta(t_0)+\delta(t_0)+2}\exp(Dh) \\ &\times \exp\left(\left(\frac{K_1}{\sqrt{\beta_0}}+\frac{K_2}{\sqrt{\delta_0}}-1\right)\int_h^t \widetilde{f}(s)\,\mathrm{d}s\right) \\ &\times \exp\left(\int_h^t \frac{r_1(s)}{\sqrt{\beta_0}}\mathrm{d}s + \int_h^t \frac{r_2(s)}{\sqrt{\delta_0}}\mathrm{d}s\right)\exp\left(\int_h^t \frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right). \end{aligned}$$

Let

$$\eta = \eta(t_0, \varepsilon) := \varepsilon \exp(-Dh) \exp\left(-\frac{1}{\sqrt{\beta_0}} \int_{t_0}^{+\infty} r_1(s)\,\mathrm{d}s - \frac{1}{\sqrt{\delta_0}} \int_{t_0}^{+\infty} r_2(s)\,\mathrm{d}s\right) \times \exp\left(-\int_{t_0}^{+\infty} \frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right) / \left(\lambda\sqrt{\beta(t_0)+\delta(t_0)+2}\right).$$

Then we can derive that if $\|z_0\|_0 < \min\{\eta, a\}$, then $\|z(t, t_0, z_0)\|_0 < \varepsilon, \quad \forall t \geq t_0$. Since the boundedness of the functions f_i, β, δ, γ, $\left|\dot{f}_i + f_i^2\right|$, g_i, $i \in \{1,2\}$ we get that $\|\dot{z}(t, t_0, z_0)\|_0$ is also small.

Case 2: $t_0 \geq h$. We have for all $t \geq t_0$

$$\|z(t, t_0, z_0)\|_0 \leq \lambda \|z_0\|_0 \sqrt{\beta(t_0)+\delta(t_0)+2} \exp\left(\left(\frac{K_1}{\sqrt{\beta_0}} + \frac{K_2}{\sqrt{\delta_0}} - 1\right) \int_{t_0}^{t} \widetilde{f}(s)\,\mathrm{d}s\right) \times \exp\left(\int_{t_0}^{t} \frac{r_1(s)}{\sqrt{\beta_0}}\mathrm{d}s + \int_{t_0}^{t} \frac{r_2(s)}{\sqrt{\delta_0}}\mathrm{d}s\right) \exp\left(\int_{t_0}^{t} \frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right).$$

Hence, with the same η as before, $\|z_0\|_0 < \min\{\eta, a\}$ implies $\|z(t, t_0, z_0)\|_0 < \varepsilon, \quad \forall t \geq t_0$. Since $\|\dot{z}(t, t_0, z_0)\|_0$ is also small, it follows that the null solution of (3.8) is stable.

iii) Let us note that the uniform stability of the null solution to system (3.8) can be deduced in the same manner as for the stability, if we consider

$$a := \rho \exp(-Dh) \exp\left(-\frac{1}{\sqrt{\beta_0}} \int_{h}^{+\infty} r_1(s)\,\mathrm{d}s - \frac{1}{\sqrt{\delta_0}} \int_{h}^{+\infty} r_2(s)\,\mathrm{d}s\right) \times \exp\left(-\int_{h}^{+\infty} \frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right) / \left(\lambda\sqrt{\beta(0)+\delta(0)+2}\right),$$

$$\eta_1 = \eta_1(\varepsilon) := \varepsilon \exp(-Dh) \exp\left(-\frac{1}{\sqrt{\beta_0}} \int_{0}^{+\infty} r_1(s)\,\mathrm{d}s - \frac{1}{\sqrt{\delta_0}} \int_{0}^{+\infty} r_2(s)\,\mathrm{d}s\right) \times \exp\left(-\int_{0}^{+\infty} \frac{\gamma(s)}{2\sqrt{\zeta(s)}}\mathrm{d}s\right) / \left(\lambda\sqrt{\beta(0)+\delta(0)+2}\right).$$

The proof of Theorem 2.1 is now complete. □

Remark 3.1. Let us note that we can also deduce the part i) of Theorem 2.1 by using [23, Lemma 1], where Theorem 3.1 is stated specifically for spaces of continuous functions of type $C_c(t_0)$.

Example 3.1. An example of functions f_1, f_2, β, δ, γ, g_1, g_2 is:

$$f_1(t) = \frac{1}{2(t-1)},\ \forall t \geq 2,\ f_2(t) = \frac{1}{t-2},\ \forall t \geq 3,$$

$$\beta(t) = 1 + \mathrm{e}^{-t},\ \delta(t) = 2 + \mathrm{e}^{-t},\ \gamma(t) = \mathrm{e}^{-t},\ \forall t \geq 0,$$

$$g_1(t,x,y) = \frac{x^2}{(t+2)^2},\ g_2(t,x,y) = \frac{2y^2}{(t+1)^2},\ \forall t \geq 0,\ \forall x,\ y \in \mathbb{R},$$

where f_1, f_2 are extended to smooth nonnegative functions defined on $\mathbb{R}_+$, e.g.,

$$f_1(t) = -\frac{3t^2}{8} + t,\ \forall t \in [0,2), \quad f_2(t) = -\frac{4t^2}{9} + \frac{5t}{3},\ \forall t \in [0,3).$$

It is easily seen that these functions satisfy the assumptions (H1)-(H7), with

$$K_1 = K_2 = 1/2,\ h = 3,\ \beta_0 = 1,\ \delta_0 = 2,\ r_1(t) = \frac{1}{(t+2)^2},\ r_2(t) = \frac{2}{(t+1)^2}.$$

For small initial data, the solution to system (1.1) and its derivative are plotted, using `Matlab`, in Figure 2 on some time intervals.

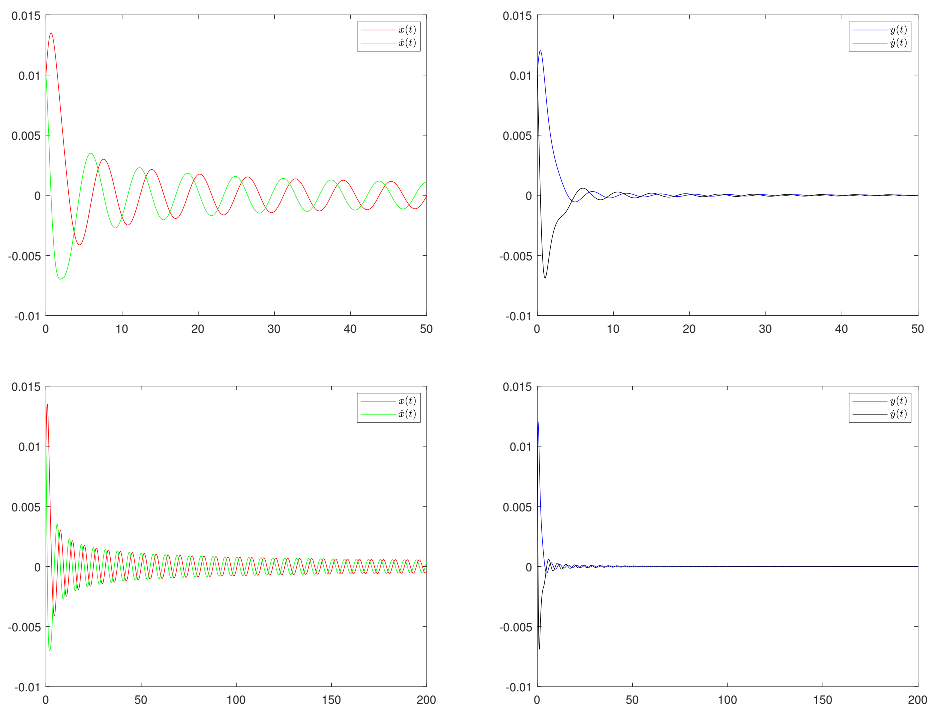

FIGURE 2. The solution to system (1.1), with the initial data $z_0 = [0.01, 0.01, 0.01, 0.01]$.

If one changes only the functions f_i, $i \in \{1,2\}$, with

$$f_1(t) = \begin{cases} \frac{1}{(t-2)^2},\ t \geq 3, \\ -\frac{7t^2}{9} + \frac{8t}{3},\ t \in [0,3), \end{cases} \quad f_2(t) = \begin{cases} \frac{2}{t^2},\ \forall t \geq 1, \\ -6t^2 + 8t,\ t \in [0,1), \end{cases}$$

then hypotheses (H1)-(H2), (H4)-(H7) are satisfied with $K_1 = 2/5$, $K_2 = 1/4$, $h = 7$, and the same β_0, δ_0, $r_1(t)$, $r_2(t)$ as before. For small initial data, the solution to system (1.1) and its derivative are plotted, using `Matlab`, in Figure 3 on the same time intervals. In this case, hypothesis (H3) is not fulfilled, the null solution to (1.1) is uniformly stable and it is not asymptotically stable, as can be easily observed on the plottings.

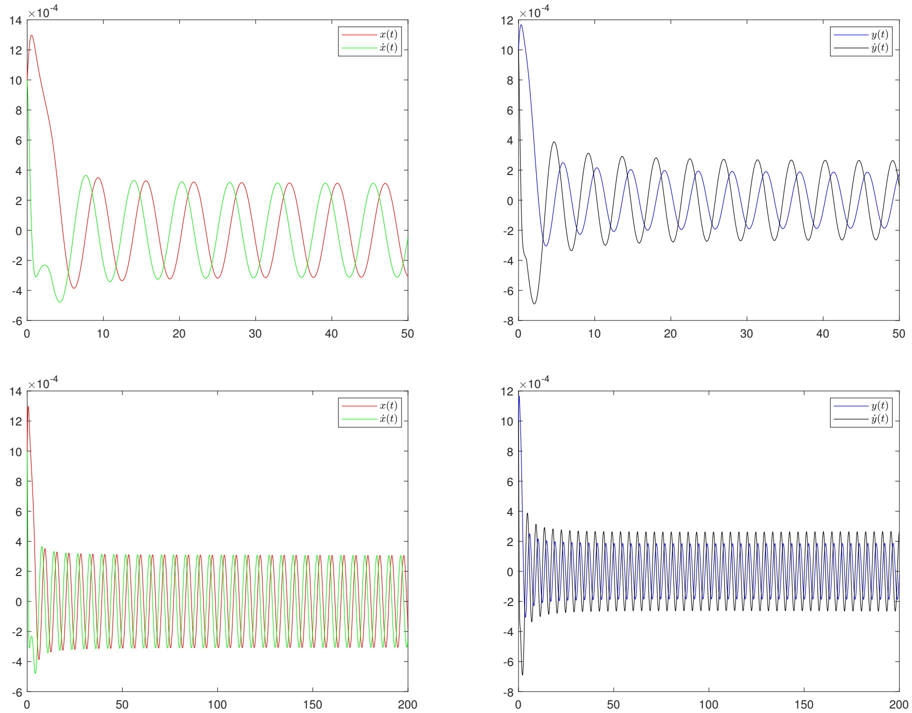

FIGURE 3. The solution to system (1.1), with the initial data $z_0 = [0.001, 0.001, 0.001, 0.001]$.

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