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AN INERTIAL SUBGRADIENT-EXTRAGRADIENT ALGORITHM FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce an iteration method for solving pseudomonotone variational inequalities and related pseudoconvex optimization problems in Hilbert spaces. The iterative scheme is based on inertial ideas and subgradient-extragradient ideas. A main feature of the method is that it formally requires only one projection step onto the feasible set. We prove a weak convergence of sequences generated by our method. In the end, some numerical examples are provided to illustrate the effectiveness and performance of the proposed algorithm. Meanwhile, we make some detailed comparisons with the known related schemes.
Key Words and Phrases: Variational inequalities, inertial extrapolation, pseudomonotonicity, pseudoconvexity, projection method, subgradient-extragradient method.
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1. Introduction

Throughout this paper, we assume that Ω is a nonempty closed convex subset of a real Hilbert space H. Its scalar product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm is denoted by $\| \cdot \|$. The variational inequality problem is formulated as finding a vector $z \in \Omega$ such that

$$\langle Fz, q - z \rangle \geq 0, \quad \forall\, q \in \Omega, \tag{1.1}$$

where $F : H \to H$ is some given operator. The solution set of variational inequality problem (1.1) is denote by $VI(\Omega, F)$. Variational inequalities can be viewed as a natural framework for unifying the treatment of equilibrium problems, and hence (1.1) has many applications in the analysis of bimatrix equilibrium points, piece-wise-linear

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resistive circuits, economic equilibrium modeling, traffic network equilibrium modeling, manufacturing system design, signal and image processing, pattern recognition and automatic control. We refer the reader to [1, 2, 3, 4, 5, 6] for recent results on the variational inequality problem and its applications.

The metric projection is the operator defined for all $z \in \Omega$, there exists a unique nearest point in Ω, denoted by $P_\Omega : H \to \Omega$, such that $P_\Omega z := \arg\min_{q\in\Omega} \|q - z\|$. Let us recall some properties of the metric projection, characterized by

(i) $\|q - P_\Omega z\|^2 + \|z - P_\Omega z\|^2 \leq \|q - z\|^2$;
(ii) $\langle P_\Omega z - z, q - P_\Omega z\rangle \geq 0, \ \ \forall q \in \Omega$.

In the context of variational inequality problems, the use of such techniques was suggested by the following equivalent knowledge of the fixed point formulation, see [7] for the details,

$$z = P_\Omega(z - \alpha F z), \ \ \alpha > 0. \tag{1.2}$$

Keep in mind that the well-known Brouwer's fixed point theorem guarantees that problem (1.2) has a solution if Ω is a bounded set. When Ω is unbounded, some sufficient conditions for the existence of the solution of (1.2) can be found in [1].

One often considers variational inequalities with F as possessing some additional properties such as the Lipschitz continuity and some certain monotonicity property. Now let us recall some related definitions and properties concerning the operator F, see [1, 8] and the references therein.

Definition 1.1. The operator $F : \Omega \to H$ is said to be

(i) weakly continuous on Ω if F is continuous on the intersection of Ω and the norm topology of H;
(ii) L-Lipschitz continuous if there exists a positive constant $L > 0$ such that

$$\|Fp - Fq\| \leq L\|p - q\|, \ \ \forall p, q \in \Omega;$$

(iii) monotone if $\langle p - q, Fp - Fq\rangle \geq 0, \ \ \forall p, q \in \Omega$;
(iv) pseudo-monotone if $\langle Fp, q - p\rangle \geq 0 \Rightarrow \langle Fq, q - p\rangle \geq 0, \ \ \forall p, q \in \Omega$.

Generally, we say the variational inequality is a monotone variational inequality, if the underlying operator F is monotone. Similarly, the variational inequality is said to be a pseudo-monotone variational inequality, if the operator F is pseudo-monotone. In addition, it is shown in [9] that a continuous operator F is convex (pseudoconvex) if and only if its generalized gradient ∇F is a monotone (pseudomonotone) operator.

2. Relation to the previous work

In recent years, the variational inequality problem has been extensively studied in both theory and practice. Much attention has been given to develop influential and efficient approaches for solving variational inequalities. Among them, the most well-known numerical solution method is the projection-type method. Due to its simplicity of implementation, this method is suitable to solve some problems arising from engineering applications such as signal processing, chain system, optimal control, multirobot systems and robot motion control. Many algorithms for solving variational inequalities are projection algorithms that employ projections onto the feasible set

Ω, in order to iteratively reach a solution. In particular, Korpelevich proposed the extragradient method for solving the monotone variational inequality problem in an Euclidean space, which generates a sequence approaching the solution by carrying out two projections per iteration. Given the current iterate p_k, calculate the next iterate p_{k+1} via

$$\begin{cases} q_k = P_\Omega(p_k - \alpha F(p_k)), \\ p_{k+1} = P_\Omega(p_k - \alpha F(q_k)). \end{cases}$$

The extragradient method has received a great deal of attention by many authors. In particular, when the potential operator F is pseudomonotone, a weaker condition than the monotonicity, it has been shown in [10] that the extragradient method for solving pseudomonotone variational inequalities is also available. However, in this way, in order to get the next iterate p_{k+1} per iteration, there is still the need to calculate two orthogonal projections onto the feasibility set Ω. This might seriously affect the efficiency of the method, if the set Ω is a general closed and convex set. Regarding the orthogonal projection, the extragradient method has been extended and improved in various ways; see [11, 12, 13, 14] and references therein. By replacing the second projection onto Ω with a projection onto a specific constructible half-space, Censor, Gibali and Reich proposed the subgradient-extragradient method in [15]. Given the current iterate p_k, calculate the next iterate p_{k+1} via

$$\begin{cases} u_k := P_\Omega(q_k - \gamma_k F q_k), \\ p_{k+1} := P_{S_k}(q_k - \gamma_k F u_k), \quad \text{where} \\ S_k := \{p \in H | \langle q_k - \gamma_k F q_k - u_k, p - u_k\rangle \leq 0\}. \end{cases}$$

Based on it, we will prove that the subgradient-extragradient method converges also when it is applied to the solving of variational inequalities governed by pseudomonotone operators, which is one of our highlights of this paper.

On the other hand, the inertial extrapolation, which was first proposed by Polyak [16] as an acceleration process, has been employed to solve various convex minimization problems recently. It is based on the heavy ball method of the two-order time dynamical system. Inertial type methods involve two iterative steps and the second iterative step is obtained with the aid of previous two iterates. They can be viewed as an efficient technique to deal with various iterative algorithms, in particular, the projection-based algorithms; see [17, 18, 19, 20].

In this paper, inspired and motivated by the mentioned works in literature and the ongoing research in these directions, we propose a subgradient-extragradient scheme combining with the inertia term for solving pseudomonotone variational inequalities, under suitable assumptions. It only needs one orthogonal projection onto the feasible set Ω at each iteration. Weak convergence theorems are established in the framework of real Hilbert spaces. Based on the methods and techniques discussed in this paper, many convex optimization problems could be extended to pseudo-convex optimization problems. Finally, we perform several numerical experiments to support the convergence of the algorithm presented in this paper. We also illustrate the computational performance of our proposed algorithm over some previously known algorithms in [10, 11, 12, 13, 14, 15, 21, 22].

3. Algorithm and convergence

Throughout this section, we make the following standing assumptions:

- The feasible set Ω is a nonempty, closed, convex subset of a real Hilbert space H;
- The operator $F : H \to H$ is pseudo-monotone, L-Lipschitz and sequentially weakly continuous with the solution set $VI(\Omega, F) \neq \emptyset$.

Our algorithm is formally designed as follows.

Algorithm 3.1 (Algorithm for variational inequality problem)

Input: Input the algorithm parameters $(\phi_i)_{i\in\mathbb{N}}$ and $(\gamma_i)_{i\in\mathbb{N}}$.
Output: Output p

```
Set k ← 1.
Initialize the data p_0, p_1 ∈ H.
while not converged do
    Update q_k := p_k + φ_k(p_k − p_{k−1}).
    Update u_k := P_Ω(Id − γ_k F)q_k.
    Update p_{k+1} := P_{S_k}(q_k − γ_k F u_k), where
    S_k := {p ∈ H | ⟨q_k − γ_k F q_k − u_k, p − u_k⟩ ≤ 0}.
    Set k ← k + 1.
end while
return p = p_k
```

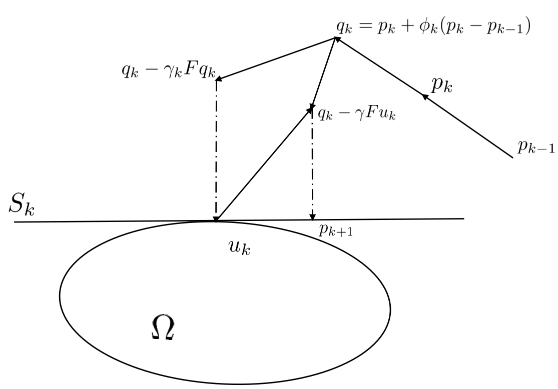


Figure 1. Iterative steps of Algorithm 3.1. The number of the projection onto the feasible set Ω is 1 per iteration.

Figure 1 illustrates iterative steps of Algorithm 3.1. The following theorem establishes the weak convergence property of sequences generated by Algorithm 3.1, under suitable assumptions.

Theorem 3.1. *Let Ω be a nonempty, closed and convex subset of a real Hilbert space H. Assume that the extrapolation factor $(\phi_k)_{k\in\mathbb{N}} \in [0,\phi]$ is nondecreasing and the relaxation parameters $(\gamma_k)_{k\in\mathbb{N}} \in \left(0, \frac{1-2a}{L}\right]$, where $a \in \left(0, \frac{1}{2}\right)$.*

Let $\lambda = \phi^2 + (1+2a)\phi - a < 0$. Then, for each initial data p_0, p_1 in H, sequences $(p_k)_{n\in\mathbb{N}}$, $(q_k)_{n\in\mathbb{N}}$ and $(u_k)_{n\in\mathbb{N}}$ generated by Algorithm 3.1 converge weakly toward the unique element of $VI(\Omega, F)$.

Before proceeding with the proof of Theorem 3.1, we establish two technical lemmas.

Lemma 3.1. [23] *Let $(\alpha_k)_{k\in\mathbb{N}}$, $(\beta_k)_{k\in\mathbb{N}}$ and $(\gamma_k)_{k\in\mathbb{N}}$ be sequences in $[0,+\infty)$ such that*

$$\alpha_{k+1} \leq \alpha_k + \beta_k(\alpha_k - \alpha_{k-1}) + \gamma_k, \ \ \forall k \geq 1, \ \sum_{k=1}^{+\infty} \gamma_k < +\infty.$$

and there exists a real constant β such that $\beta_k \in [0,\beta] \subseteq [0,1]$, $\forall k \in \mathbb{N}$. Then the following holds

(i) $\sum_{k=1}^{\infty}[\alpha_{k+1} - \alpha_k]_+ < +\infty$, *where* $[s]_+ := \max\{0, s\}$;

(ii) there exists $\alpha \in [0,+\infty)$ *such that* $\lim_{k\to+\infty} \alpha_k = \alpha$.

Lemma 3.2. *(Minty Lemma)* [24] *Consider the variational inequality problem with the operator $F : \Omega \to H$ pseudo-monotone and continuous. Thus, μ is a solution of $VI(\Omega, F)$ if and only if $\langle F(\nu), \nu - \mu\rangle \geq 0, \ \ \forall \nu \in \Omega$.*

Now we are in a position to state and prove the main result of this section.

Proof. Let $z \in VI(\Omega, F)$ be arbitrarily chosen. Therefore, we find that $\langle Fz, p-z\rangle \geq 0$ for all $p \in \Omega$, together with the pseudomonotonicity of F, we infer that $\langle Fp, p-z\rangle \geq 0$ for all $p \in \Omega$. By the definition of u_k, we immediately obtain that $z \in S_k$ and $u_k \in \Omega$. Recalling that and setting $p := u_k$, we observe that

$$\langle Fu_k, z - u_k\rangle \leq 0. \tag{3.1}$$

Invoking $z \in S_k$, then the definition of p_{k+1} entails that

$$\langle p_{k+1} - (q_k - \gamma_k F u_k), p_{k+1} - z\rangle \leq 0. \tag{3.2}$$

Setting briefly $w_k = q_k - \gamma_k F u_k$ and collecting the above results (3.1), (3.2), it asserts that

$$\begin{aligned}
\|p_{k+1} - z\|^2 =&\|p_{k+1} - w_k\|^2 + \|w_k - z\|^2 + 2\langle p_{k+1} - w_k, w_k - z\rangle \\
=&2\|p_{k+1} - w_k\|^2 + \|w_k - z\|^2 + 2\langle p_{k+1} - w_k, w_k - z\rangle - \|p_{k+1} - w_k\|^2 \\
=&\|w_k - z\|^2 + 2\langle p_{k+1} - w_k, p_{k+1} - z\rangle - \|p_{k+1} - w_k\|^2 \\
\leq&\|(q_k - \gamma_k F u_k) - z\|^2 - \|p_{k+1} - ((q_k - \gamma_k F u_k))\|^2 \\
=&\|q_k - z\|^2 - \|p_{k+1} - q_k\|^2 + 2\gamma_k\langle F u_k, z - p_{k+1}\rangle \\
=&\|q_k - z\|^2 - \|p_{k+1} - q_k\|^2 + 2\gamma_k(\langle F u_k, z - u_k\rangle + \langle F u_k, u_k - p_{k+1}\rangle) \\
\leq&\|q_k - z\|^2 - \|p_{k+1} - q_k\|^2 + 2\gamma_k\langle F u_k, u_k - p_{k+1}\rangle \\
=&\|q_k - z\|^2 - \|p_{k+1} - u_k\|^2 - \|u_k - q_k\|^2 + 2\langle q_k - u_k - \gamma_k F u_k, p_{k+1} - u_k\rangle.
\end{aligned} \tag{3.3}$$

Keeping in mind that $p_{k+1} \in S_k$, we find that

$$\begin{aligned} 2\langle q_k - u_k - \gamma_k F u_k, p_{k+1} - u_k\rangle \leq & 2\langle q_k - \gamma_k F q_k - u_k, p_{k+1} - u_k\rangle \\ & + 2\gamma_k \langle F q_k - F u_k, p_{k+1} - u_k\rangle \\ \leq & 2\gamma_k \langle F q_k - F u_k, p_{k+1} - u_k\rangle. \end{aligned} \tag{3.4}$$

The condition $(\gamma_k)_{k\in\mathbb{N}} \in \big(0, \frac{1-2a}{L}\big]$ asserts that $\frac{1-\gamma_k L}{2} \geq a$. According to the Lipschitz continuity of F, it follows from (3.3) and (3.4) that

$$\begin{aligned} \|p_{k+1} - z\|^2 \leq & \|q_k - z\|^2 - \|p_{k+1} - u_k\|^2 - \|u_k - q_k\|^2 + 2\gamma_k \langle F q_k - F u_k, p_{k+1} - u_k\rangle \\ \leq & \|q_k - z\|^2 - \|p_{k+1} - u_k\|^2 - \|u_k - q_k\|^2 + 2\gamma_k L \|q_k - u_k\| \|p_{k+1} - u_k\| \\ \leq & \|q_k - z\|^2 - (1-\gamma_k L)\|p_{k+1} - u_k\|^2 - (1-\gamma_k L)\|u_k - q_k\|^2 \\ \leq & \|q_k - z\|^2 - \frac{1-\gamma_k L}{2}\|p_{k+1} - q_k\|^2 \\ \leq & \|q_k - z\|^2 - a\|p_{k+1} - q_k\|^2. \end{aligned} \tag{3.5}$$

By the definition of q_k, we immediately obtain that

$$\begin{aligned} \|q_k - z\|^2 \leq & \|(p_k + \phi_k(p_k - p_{k-1})) - z\|^2 \\ \leq & (1+\phi_k)\|p_k - z\|^2 - \phi_k\|p_{k-1} - z\|^2 + (1+\phi_k)\phi_k\|p_k - p_{k-1}\|^2, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|p_{k+1} - q_k\|^2 = & \|p_{k+1} - (p_k + \phi_k(p_k - p_{k-1}))\|^2 \\ \geq & \|p_{k+1} - p_k\|^2 - 2\phi_k\|p_{k+1} - p_k\|\|p_k - p_{k-1}\| + \phi_k^2\|p_k - p_{k-1}\|^2 \\ \geq & (1-\phi_k)\|p_{k+1} - p_k\|^2 + (\phi_k^2 - \phi_k)\|p_k - p_{k-1}\|^2. \end{aligned} \tag{3.7}$$

Plugging (3.6), (3.7) into (3.5) and rearranging the terms, we infer that

$$\begin{aligned} \|p_{k+1} - z\|^2 \leq & \|q_k - z\|^2 - \frac{1-\gamma_k L}{2}\|p_{k+1} - q_k\|^2 \\ \leq & (1+\phi_k)\|p_k - z\|^2 - \phi_k\|p_{k-1} - z\|^2 - a(1-\phi_k)\|p_{k+1} - p_k\|^2 \\ & + \left((1+\phi_k)\phi_k - a(\phi_k^2 - \phi_k)\right)\|p_k - p_{k-1}\|^2 \\ = & (1+\phi_k)\|p_k - z\|^2 - \phi_k\|p_{k-1} - z\|^2 - \alpha_k\|p_{k+1} - p_k\|^2 + \beta_k\|p_k - p_{k-1}\|^2, \end{aligned} \tag{3.8}$$

where $\alpha_k = a(1-\phi_k)$ and $\beta_k = (1+\phi_k)\phi_k - a(\phi_k^2 - \phi_k)$. Let

$$\Gamma_k := \|p_k - z\|^2 - \phi_k\|p_{k-1} - z\|^2 + \beta_k\|p_k - p_{k-1}\|^2. \tag{3.9}$$

Combining (3.8) with (3.9), keeping in mind that $(\phi_k)_{k\in\mathbb{N}}$ is nondecreasing, together with the condition $\lambda = \phi^2 + (1+2a)\phi - a < 0$, we successively find that

$$\begin{aligned}
&\Gamma_{k+1} - \Gamma_k \\
\leq &\|p_{n+1} - z\|^2 - (1+\phi_k)\|p_n - z\|^2 + \phi_k\|p_{n-1} - z\|^2 \\
+&\beta_{k+1}\|p_{k+1} - p_k\|^2 - \beta_k\|p_k - p_{k-1}\|^2 \\
\leq &- \alpha_k\|p_{k+1} - p_k\|^2 + \beta_{k+1}\|p_{k+1} - p_k\|^2 \\
\leq &- (\alpha_k - \beta_{k+1})\|p_{k+1} - p_k\|^2 \\
\leq &- (a(1-\phi_{k+1}) - (1+\phi_{k+1})\phi_{k+1} + a(\phi_{k+1}^2 - \phi_{k+1}))\|p_{k+1} - p_k\|^2 \\
\leq &- (a(1-\phi) - (1+\phi)\phi - a\phi)\|p_{k+1} - p_k\|^2 \\
= &\lambda\|p_{k+1} - p_k\|^2.
\end{aligned} \tag{3.10}$$

It obviously asserts that $\Gamma_{k+1} - \Gamma_k \leq 0$, which further implies that the sequence $(\Gamma_k)_{k\in\mathbb{N}}$ is non-increasing. On the other hand, the condition $(\phi_k)_{k\in\mathbb{N}} \in [0,\phi]$ asserts that

$$\begin{aligned}
\|p_k - z\|^2 &\leq \phi_k\|p_{k-1} - z\|^2 + \Gamma_k \\
&\leq \phi\|p_{k-1} - z\|^2 + \Gamma_0 \\
&\leq \cdots \leq \phi^k\|p_0 - z\|^2 + \Gamma_0(1 + \cdots + \phi^{k-1}) \\
&\leq \phi^k\|p_0 - z\|^2 + \frac{\Gamma_0}{1-\phi}.
\end{aligned} \tag{3.11}$$

Taking account of (3.9) and (3.11), we deduce that

$$-\Gamma_{k+1} \leq \phi_{k+1}\|p_k - z\|^2 \leq \phi\left(\phi^k\|p_0 - z\|^2 + \frac{\Gamma_0}{1-\phi}\right) \leq \phi^{k+1}\|p_0 - z\|^2 + \frac{\Gamma_0\phi}{1-\phi}. \tag{3.12}$$

Recalling from (3.10) and (3.12), we additionally obtain that

$$\lambda\sum_{i=0}^{k}\|p_{i+1} - p_i\|^2 \leq \Gamma_0 - \Gamma_{k+1} \leq \Gamma_0 + \phi^{k+1}\|p_0 - z\|^2 + \frac{\Gamma_0\phi}{1-\phi} \leq \|p_0 - z\|^2 + \frac{\Gamma_0}{1-\phi}. \tag{3.13}$$

Owning to (3.8) and (3.13), together with Lemma 3.1, we check that

$$\lim_{k\to\infty}\|p_k - z\| = a. \tag{3.14}$$

Letting k tend to $+\infty$ in (3.8), we infer that

$$\|p_{k+1} - p_k\| \to 0. \tag{3.15}$$

By the definition of q_k, together with (3.15), it ensures that

$$\|p_{k+1} - q_k\| = \|p_{k+1} - p_k\| + \phi_k^2\|p_k - p_{k-1}\|^2 - 2\phi_k\langle p_{k+1} - p_k, p_k - p_{k-1}\rangle,$$

which by (3.15) amounts to

$$\lim_{k\to\infty}\|p_{k+1} - q_k\| = 0. \tag{3.16}$$

Thanks to (3.15) and (3.16), we find that

$$\lim_{k\to\infty}\|p_k - q_k\| \le \lim_{k\to\infty}\|p_k - p_{k+1}\| + \lim_{k\to\infty}\|p_{k+1} - q_k\| = 0. \tag{3.17}$$

By applying (3.14) and (3.17), we obtain that

$$\lim_{k\to\infty}\|q_k - z\| = a. \tag{3.18}$$

Coming back to (3.5), it entails that

$$(1-\gamma_k L)\|u_k - q_k\| \le \|q_k - z\|^2 - \|p_{k+1} - z\|^2. \tag{3.19}$$

Based on (3.10), (3.14) and (3.18), from the fact that $(\gamma_k)_{k\in\mathbb{N}} \in \big(0, \frac{1-2a}{L}\big]$, it obviously asserts that

$$\lim_{k\to\infty}\|u_k - q_k\| = 0. \tag{3.20}$$

In view of (3.13) and (3.16), we obtain that

$$\lim_{k\to\infty}\|u_k - p_k\| \le \lim_{k\to\infty}\|u_k - q_k\| + \lim_{k\to\infty}\|q_k - p_k\| = 0. \tag{3.21}$$

Returning to (3.14), it obviously asserts that $(p_k)_{k\in\mathbb{N}}$ is bounded. Without loss of generality, there exists a subsequence $(p_{k_i})_{i\in\mathbb{N}}$ converges weakly to p^*. Indeed, by using successively (3.17), (3.20), we conclude that $(q_{k_i})_{i\in\mathbb{N}}$ and $(u_{k_i})_{i\in\mathbb{N}}$ converge weakly toward p^* as well. Now we are in a position to prove that $p^* \in VI(\Omega, F)$. Let $z \in \Omega$ be fixed. Recalling that $z \in S_k,\ \forall k \ge 1$, we additionally obtain that

$$\langle q_{k_i} - \gamma_{k_i} F q_{k_i} - u_{k_i}, z - u_{k_i}\rangle \le 0,$$

which immediately leads to

$$\langle F q_{k_i}, z - q_{k_i}\rangle \ge \frac{1}{\gamma_{k_i}}\langle q_{k_i} - u_{k_i}, z - u_{k_i}\rangle + \langle F q_{k_i}, u_{k_i} - q_{k_i}\rangle. \tag{3.22}$$

By combining (3.20) and (3.22), we find that

$$\liminf_{i\to\infty}\langle F q_{k_i}, z - q_{k_i}\rangle \ge 0,\ \ \forall z \in \Omega.$$

We choose a sequence $(\zeta_i)_{i\in\mathbb{N}}$ of positive numbers decreasing and tending to 0. For each ζ_i, we denote by k_{j_i} the smallest positive integer such that

$$\langle F q_{k_{j_i}}, z - q_{k_{j_i}}\rangle + \zeta_i \ge 0,\ \ \forall i \ge 0. \tag{3.23}$$

From the fact that $(\zeta_i)_{i\in\mathbb{N}}$ is decreasing, we easily check that the sequence $(k_{j_i})_{i\in\mathbb{N}}$ is increasing. The rest of the proof will be divided into two cases.

Case 1. Suppose that there exists a subsequence of positive integers $(\psi(i))_{i\in\mathbb{N}}$ such that $\psi(i) = \min\{l \in \mathbb{N} \,|\, F q_{k_{j_l}} \ne 0,\, l \ge i\}$. In this situation, since $(k_{j_i})_{i\in\mathbb{N}}$ is increasing, together with the definition of $(\psi(i))_{i\in\mathbb{N}}$, we also have that $(k_{j_{\psi(i)}})_{i\in\mathbb{N}}$ is increasing. Let

$$\tau_{k_{j_{\psi(i)}}} = \frac{F(q_{k_{j_{\psi(i)}}})}{\|F(q_{k_{j_{\psi(i)}}})\|^2}. \tag{3.24}$$

And hence, we observe that $\langle F(q_{k_{j_{\phi(i)}}}), q_{k_{j_{\psi(i)}}}\rangle = 1$, for each i. By putting together (3.23) and (3.24), we immediately obtain that, for each i

$$\langle F q_{k_{j_{\psi(i)}}}, z + \zeta_i \tau_{k_{j_{\psi(i)}}} - q_{k_{j_{\psi(i)}}}\rangle \ge 0. \tag{3.25}$$

Coming back to (3.25), the pseudomonotonicity of F asserts that

$$\langle F(z+\zeta_i\tau_{k_{j_{\psi(i)}}}), z+\zeta_i\tau_{k_{j_{\psi(i)}}} - q_{k_{j_{\psi(i)}}}\rangle \geq 0. \tag{3.26}$$

On the other hand, recalling that $(q_{k_j})_{j\in\mathbb{N}}$ converges weakly toward p^* when $j\to\infty$, it yields that the convergence $(q_{k_{j_{\psi(i)}}})_{i\in\mathbb{N}}$ weakly to p^* as $i\to\infty$. Since F is sequentially weakly continuous on Ω, $(Fq_{k_{j_{\psi(i)}}})_{i\in\mathbb{N}}$ converges weakly to $F(p^*)$, as $i\to\infty$. Without loss of generatlity, we suppose that $F(p^*)\neq 0$ (otherwise, p^* is a solution). Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$\|Fp^*\| \leq \liminf_{i\to\infty} \|F(q_{k_{j_{\psi(i)}}})\|. \tag{3.27}$$

In view of $\zeta_i\to 0$ as $i\to\infty$, together with (3.27), we obtain

$$0 \leq \lim_{i\to\infty} \|\zeta_i\tau_{k_{j_{\psi(i)}}}\| = \liminf_{i\to\infty} \frac{\zeta_i}{\|Fq_{k_{j_{\psi(i)}}}\|} \leq \frac{0}{\|Fp^*\|} = 0. \tag{3.28}$$

Hence taking the limit as $i\to\infty$ in (3.26), invoking the fact that $(q_{k_j})_{j\in\mathbb{N}}$ converges weakly to p^* when $j\to\infty$, together with (3.28), we find

$$\langle F(z), z-p^*\rangle \geq 0.$$

Applying Lemma 3.2 to this situation, we find that $p^*\in VI(\Omega,F)$.
Case 2. Otherwise, we have $\lim_{i\to\infty} Fp_{k_i}=0$. Owning to $p_{k_i}\rightharpoonup p^*$, the sequentially weakly continuity of F ensures that $\lim_{i\to\infty} Fp_{k_i}=Fp^*$.
And hence clearly, $p^*\in F^{-1}(0)$. This classically expresses that $p^*\in VI(\Omega,F)$.
Finally, we prove that the sequence $(p_k)_{k\in\mathbb{N}}$ uniquely converges weakly to p^*. To do this, it is sufficient to show that $(p_k)_{k\in\mathbb{N}}$ cannot have two distinct weak sequential cluster points in $VI(\Omega,F)$. Without loss of generality, we assume that $(p_{k_i})_{i\in\mathbb{N}}$ is another subsequence of $(p_k)_{k\in\mathbb{N}}$ converging weakly to $\hat{x}$. We have to prove that $x^*=\hat{x}$. With a similar way as above, $\hat{x}\in VI(\Omega,F)$. Indeed, for all $k\in\mathbb{N}$,

$$2\langle p_k, \hat{p}-p^*\rangle = \|p_k-p^*\|^2 - \|p_k-\hat{p}\|^2 + \|\hat{p}\|^2 - \|p^*\|^2.$$

With the help of (3.14), we deduce that the sequence $(\langle p_k,\hat{p}-p^*\rangle)_{k\in\mathbb{N}}$ also converges. Let $\lim_{k\to\infty}\langle p_k,\hat{p}-p^*\rangle=\sigma$. Passing to the limit along $(p_{k_i})_{i\in\mathbb{N}}$ and $(p_{k_j})_{j\in\mathbb{N}}$, we deduce that $\langle p^*,\hat{p}-p^*\rangle = \langle \hat{p},\hat{p}-p^*\rangle=\sigma$, which boils down to $\|\hat{p}-p^*\|^2=0$, in other words, $\hat{p}=p^*$. This completes the proof. ☐

Let us now state an extension of Theorem 3.1 as an immediate consequence of the monotonicity of F.

Remark 3.1. When working with a general monotone operator F, it is not necessary to impose the sequential weak continuity on F. Regarding this case, since F is monotone, which by (3.2) leads to

$$\langle F(z), z-q_{k_i}\rangle \geq \langle F(q_{k_i}), z-q_{k_i}\rangle \geq \frac{1}{\gamma_{k_i}}\langle q_{k_i}-u_{k_i}, z-u_{k_i}\rangle + \langle F(q_{k_i}), u_{k_i}-q_{k_i}\rangle. \tag{3.29}$$

Letting i tend to $+\infty$ in (3.29), thanks to $\lim_{k\to\infty}\|u_k-q_k\|=0$ and $\underline{\lim}_{k\to\infty}\gamma_k>0$, we have

$$\langle F(z), z-p^*\rangle \geq 0,\ \forall z\in\Omega.$$

This leads to the desired conclusion.

The shortcoming of Algorithm 3.1 is a requirement to estimate the Lipschitz constant more or less precisely. While the estimation is often quite conservative, of course, this is not practical in most cases of interest. For this reason, we give some prediction of a stepsize with its further correction along a feasible direction. Now let us state the result below.

Theorem 3.2. *Suppose that (an upper bound of) the Lipschitz constant of F is unknown. We deal with the relaxation parameters $(\gamma_k)_{k\in\mathbb{N}}$ in Algorithm 3.1 by using the following adaptive stepsize strategy. For any $k \geq 0$, the relaxation parameters $(\gamma_k)_{k\in\mathbb{N}} \in \left(0, \frac{1-2\tau}{\eta}\right]$ and*

$$\gamma_{k+1} = \begin{cases} \gamma_k, & \text{if } F(u_k) - F(q_k) = 0; \\ \min\left\{\frac{\eta\|u_k - q_k\|}{\|F(u_k)-F(q_k)\|}, \gamma_k\right\}, & \text{otherwise}, \end{cases} \tag{3.30}$$

where $\tau \in \left(0, \frac{1}{2}\right)$, $\gamma_0 > 0$. Let $\chi = \phi^2 + (1+2\tau)\phi - \tau < 0$. Then the conclusion of Theorem 3.1 still remains valid by using adaptive stepsize (3.30).

Proof. To avoid repetition, we restrict our attention to the place where arguments differ. In the upcoming statement, the assumption of the relaxation parameters in Theorem 3.1 will be removed and replaced with adaptive stepsize (3.30). By assumption, we infer that sequence $(\gamma_k)_{k\in\mathbb{N}}$ is nonincreasing and $0 < \eta\gamma_k < 1, \forall k \in \mathbb{N}$. Moreover, as $F(p_k) - F(q_k) \neq 0,\ k \geq 0$, the following holds true

$$\frac{\eta\|u_k - q_k\|}{\|F(u_k) - F(q_k)\|} \geq \frac{\eta\|u_k - q_k\|}{L\|u_k - q_k\|} = \frac{\eta}{L},$$

which ensures that $(\gamma_k)_{k\in\mathbb{N}}$ is bounded from below by $\min\{\gamma_0, \frac{\eta}{L}\} > 0$. It yields that

$$(\gamma_k)_{k\in\mathbb{N}} \in \left(0, \min\left\{\gamma_0, \frac{\eta}{L}\right\}\right).$$

Remembering that, when $\gamma_0 \leq \frac{\eta}{L}$, we have that $(\gamma_k)_{k\in\mathbb{N}}$ is a constant sequence, which leads to a fixed stepsize strategy. As a straightforward consequence, the limit $\lim_{k\to\infty}\gamma_k$ exists and it is a positive real number. Setting $\xi_k = \frac{\gamma_k}{\gamma_{k+1}}$, this entails that the limit $\lim_{k\to\infty}\xi_k = 1$. The condition $(\gamma_k)_{k\in\mathbb{N}} \in \left(0, \frac{1-2\tau}{\eta}\right]$ boils down to $\frac{1-\eta\gamma_k}{2} \geq \tau$. Based on (3.6) and (3.7), we rewrite inequalities (3.5) and (3.8) as

$$\begin{aligned}\|p_{k+1} - z\|^2 \leq& \|q_k - z\|^2 - \|p_{k+1} - u_k\|^2 - \|u_k - q_k\|^2 + 2\gamma_k\langle Fq_k - Fu_k, p_{k+1} - u_k\rangle \\ \leq& \|q_k - z\|^2 - \|p_{k+1} - u_k\|^2 - \|u_k - q_k\|^2 + 2\eta\gamma_k\|q_k - u_k\|\|p_{k+1} - u_k\| \\ \leq& \|q_k - z\|^2 - (1-\eta\gamma_k)\|p_{k+1} - u_k\|^2 - (1-\eta\gamma_k)\|u_k - q_k\|^2 \\ \leq& \|q_k - z\|^2 - \tau\|p_{k+1} - q_k\|^2 \\ \leq& (1+\phi_k)\|p_k - z\|^2 - \phi_k\|p_{k-1} - z\|^2 - \tau(1-\phi_k)\|p_{k+1} - p_k\|^2 \\ &+ \left((1+\phi_k)\phi_k - \tau(\phi_k^2 - \phi_k)\right)\|p_k - p_{k-1}\|^2 \\ =& (1+\phi_k)\|p_k - z\|^2 - \phi_k\|p_{k-1} - z\|^2 - \varrho_k\|p_{k+1} - p_k\|^2 + \sigma_k\|p_k - p_{k-1}\|^2,\end{aligned} \tag{3.31}$$

where $\varrho_k = \tau(1-\phi_k)$ and $\sigma_k = (1+\phi_k)\phi_k - \tau(\phi_k^2 - \phi_k)$. Put

$$\Lambda_k := \|p_k - z\|^2 - \phi_k\|p_{k-1} - z\|^2 + \sigma_k\|p_k - p_{k-1}\|^2. \tag{3.32}$$

According to (3.31), (3.32), the condition $\chi = \phi^2 + (1+2\tau)\phi - \tau < 0$, together with the nonincreasing property of the sequence $(\gamma_k)_{k\in\mathbb{N}}$, it entails that

$$\begin{aligned}
&\Lambda_{k+1} - \Lambda_k \\
\leq&\|p_{n+1} - z\|^2 - (1+\phi_k)\|p_n - z\|^2 + \phi_k\|p_{n-1} - z\|^2 \\
&+\sigma_{k+1}\|p_{k+1} - p_k\|^2 - \sigma_k\|p_k - p_{k-1}\|^2 \\
\leq& - \varrho_k\|p_{k+1} - p_k\|^2 + \sigma_{k+1}\|p_{k+1} - p_k\|^2 \\
\leq& - (\tau(1-\phi_{k+1}) - (1+\phi_{k+1})\phi_{k+1} + \tau(\phi_{k+1}^2 - \phi_{k+1}))\|p_{k+1} - p_k\|^2 \\
\leq& - (\tau(1-\phi) - (1+\phi)\phi - \tau\phi)\|p_{k+1} - p_k\|^2 \\
=&\chi\|p_{k+1} - p_k\|^2.
\end{aligned}$$

It obviously asserts that $\Lambda_{k+1} - \Lambda_k \leq 0$, which further implies that the sequence $(\Lambda_k)_{k\in\mathbb{N}}$ is non-increasing. In such a case, it suffices to proceed in much the same way as in Theorem 3.1. And hence, the last proof of the weak convergence of the sequence $(p_k)_{k\in\mathbb{N}}$ follows immediately from the corresponding lines of Theorem 3.1. Hence we obtain the desired result. □

4. Applications

In this section, we present several illustrative numerical examples in order to demonstrate the performance, efficiency and applicability of the proposed algorithm. Note that all examples are considered in finite dimensional spaces, thus there is non sense to use any of strong algorithms to obtain the solution of variational inequalities. In all tests, we use our proposed algorithm to solve this problem by letting the extrapolation factors $\theta_k = 0.099$ and the relaxation parameters $\gamma_k = 0.02$. All the experiments are performed on a PC with Intel (R) Core (TM) i5-8250U CPU @1.60GHz, under the Matlab computing environment.

Example 4.1. [25] First, we consider the variational inequality via the following property

$$\Psi(p) = (\varphi_1(p), \varphi_2(p), \cdots, \varphi_m(p))^T,$$

where $p = (p_1, p_2, \cdots, p_m)^T$ and $\varphi_i(p) = e^{p_i} - 1,\ i = 1, 2, \cdots, m$. Note that $\Psi(\cdot)$ is a pseudomonotone and Lipschitz continuous operator on the feasible set

$$\Omega = \{p \in \mathbb{R}^m | -1 \leq p_1, p_2, \cdots, p_m \leq 4\}.$$

Our problem here is to find a point $p^* \in \Omega$ such that $\langle \Psi(p^*), p - p^* \rangle \geq 0, \forall p \in \Omega$. Let us take $m = 240$ as a special case and denote $p = (p_1, p_2, \cdots, p_{240})^T$. In this experiment, we choose the initial data generated from the uniform distribution over $(0,1)^{240}$ and we take the number of iterations $k = 1000$ as the stopping criterion.

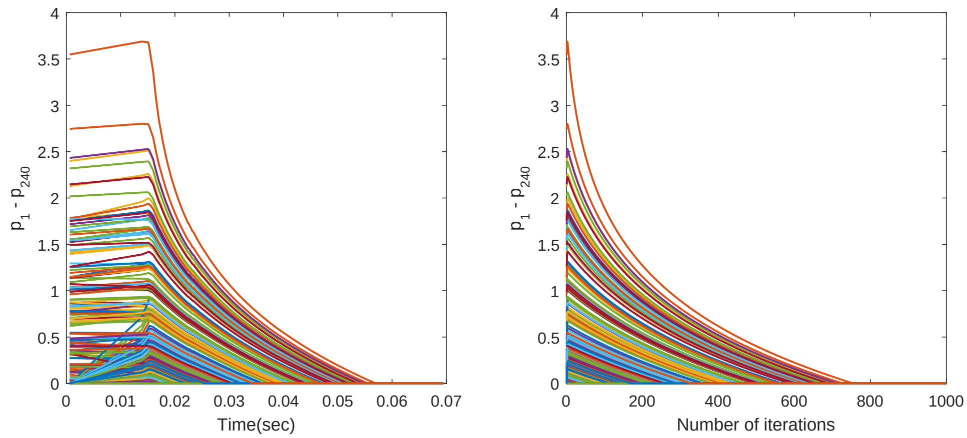


FIGURE 2. Behaviors of $p_1, p_2, \cdots, p_{240}$ with the number of iterations $k = 1000$. Numerical results for Algorithm 3.1.

From the results reported in Figure 2, one has shown that the values of $p_1 - p_{240}$ (y-axis) with the number of iterations $k = 1000$ (x-axis). Obviously, this problem has a unique solution $p^* = (0, 0, \cdots, 0)^T$.

Note that the constrained optimization arises in a broad variety of engineering and scientific applications. And hence, we consider the following general nonsmooth optimization problem expressed as follows

$$\begin{aligned} \text{minimize} \ & \Upsilon(p) \\ \text{subject to} \ & \Theta p = b, \\ & p \in \Omega, \end{aligned} \tag{4.1}$$

where $\Upsilon : \mathbb{R}^n \to \mathbb{R}$ is an objective function,

$$p = (p_1, p_2, \cdots, p_n)^T \in \mathbb{R}^n, \ b = (b_1, b_2, \cdots, b_m)^T \in \mathbb{R}^m,$$

$\Theta \in \mathbb{R}^{m\times n}$ is a full row-rank matrix (i.e., $rank(\Theta) = m \leq n$) and Ω is a nonempty, closed and convex set in $\mathbb{R}^n$. In our experiments, the objective function of problem (4.1) is not necessarily convex everywhere, and only needs to be pseudoconvex on a set defined by the constraints. Moreover, problem (4.1) has two special cases described as follows

$$\left\{ \begin{array}{ll} \text{minimize} & \Upsilon(p) \\ \text{subject to} & p \in \Omega; \end{array} \right. \qquad \left\{ \begin{array}{ll} \text{minimize} & \Upsilon(p) \\ \text{subject to} & \Theta p = b. \end{array} \right. \tag{4.2}$$

The left one is with only bound constraints, the right one is with only equality constraints. As for problem (4.2), we study Examples 4.2-4.6 where our proposed algorithm is effective.

Example 4.2. [26] Consider the following convex optimization problem with a feasible set

$$\begin{aligned} &\text{minimize } \Upsilon(p) = 1 + p_1^2 - e^{-p_2^2} \\ &\text{subject to } p \in \Omega, \end{aligned} \tag{4.3}$$

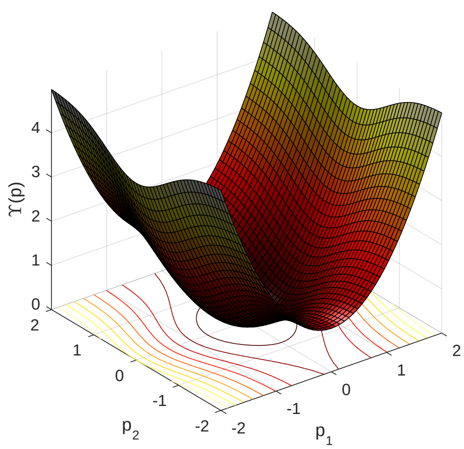
where

$$p = (p_1, p_2)^T \text{ and } \Omega = \{p \in \mathbb{R}^2 | -2 \leq p_1, p_2 \leq 2\}.$$

It is easy to check that the objective function $\Upsilon(p)$ is convex in the nonempty, closed and convex set Ω. And hence, the generalized gradient

$$F = \nabla \Upsilon(p) = (2p_1, 2p_2 e^{-p_2^2})^T, \ \forall p = (p_1, p_2)^T$$

is pseudomonotone in Ω. In the following experiment, we randomly choose the starting data in the range of $(0,1)^2$ and we take the iteration number $k = 400$ as the termination criterion.



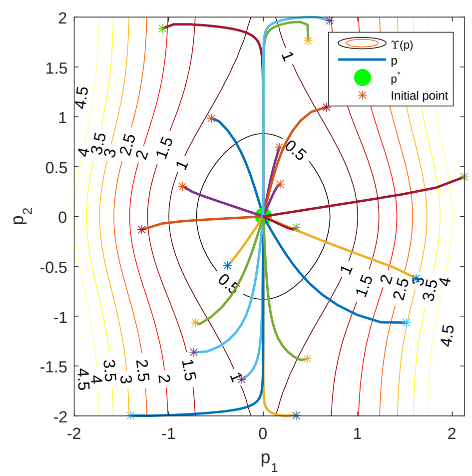
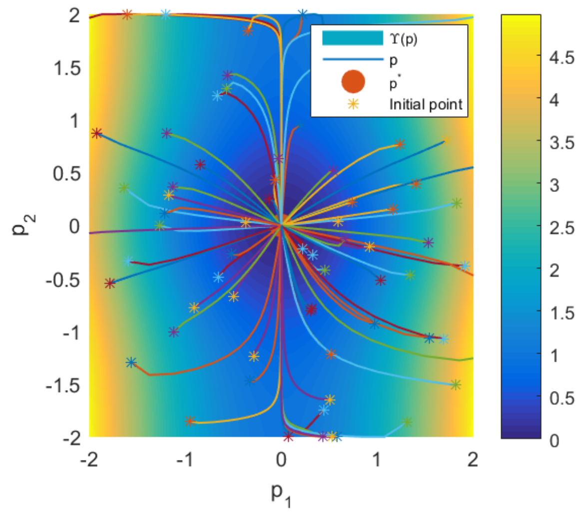


FIGURE 3. The isometric view of the objective function of $\Upsilon(p)$ in Example 4.2.

FIGURE 4. Behaviors of $(p_1, p_2, \Upsilon(p))^T$ with 20 initial points for $k = 400$. Numerical results for Algorithm 3.1.

Figure 3 depicts the isometric view of $\Upsilon(p)$ in 3-D space. To illustrate the computational performance, Figure 4 displays the changing processes of $(p_1, p_2, \Upsilon(p))^T$ with the number of iterations that are convergent to the exact solution $(0, 0, 0)^T$, which also is the global minimum solution of the objective function in the whole space.



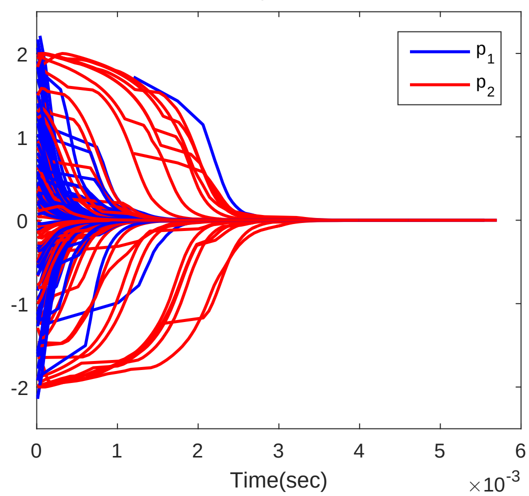


FIGURE 5. Behaviors of $(p_1, p_2, \Upsilon(p))^T$ in terms of many initial points, and require $k = 400$ iterations in Example 4.2.

FIGURE 6. Behaviors of p_1, p_2 with the execution time. Numerical results for Algorithm 3.1.

In the second experiment, we randomly choose many starting points in the range of $(0,1)^2$ and we take the iteration number $k = 350$ as the stopping criterion. The numerical result is shown in two different versions. Figure 5 plots the changing processes of $(p_1, p_2, \Upsilon(p))^T$ with many initial points at star symbol and the unique convergent point at dot symbol. Meanwhile, as depicted in Figure 6, it is obvious that the convergence of the value $(p_1, p_2)^T$ to $(0,0)^T$, which is the optimal solution of problem (4.3) when $\Upsilon(p) = 0$.

Example 4.3. Consider the following nonlinear optimization problem via

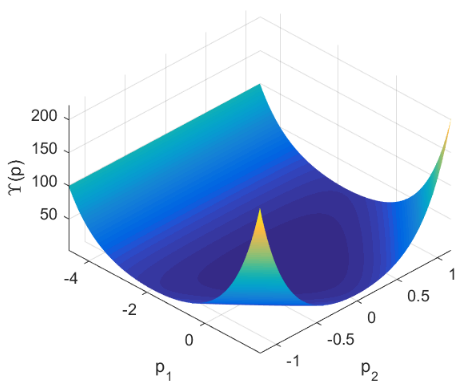
$$\begin{aligned} &\text{minimize } \ \Upsilon(p) = exp(p_1 + 3p_2 - 0.1) + exp(p_1 - 3p_2 - 0.1) + exp(-p_1 - 0.1) \\ &\text{subject to } \ -3 \le p_1, p_2 \le 3, \end{aligned} \tag{4.4}$$

where $p = (p_1, p_2)^T \in \mathbb{R}^2$. One sees that the objective function is convex on Ω with its generalized gradient

$$\nabla\Upsilon(p) = \begin{pmatrix} exp(p_1 + 3p_2 - 0.1) + exp(p_1 - 3p_2 - 0.1) - exp(-p_1 - 0.1) \\ 3exp(p_1 + 3p_2 - 0.1) - 3exp(p_1 - 3p_2 - 0.1) \end{pmatrix}$$

$$\forall p = (p_1, p_2)^T,$$

pseudomonotone on Ω. In addition, $\nabla\Upsilon(p)$ is also Lipschitz continuous with respect to $p \in \Omega$. In this experiment, the starting points are generated from the uniform distribution over $(0,1)^2$. And we take the iteration number $k = 700$ as the stopping criterion.



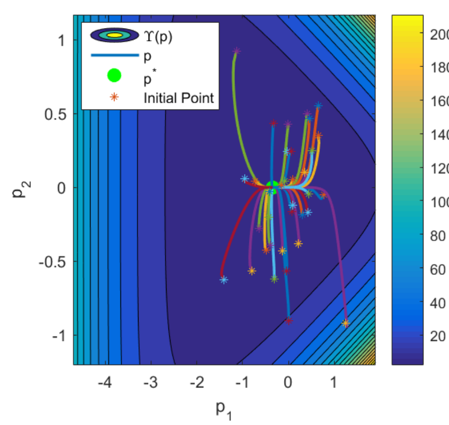


FIGURE 7. The isometric view of the objective function of $\Upsilon(p)$ in Example 4.3.

FIGURE 8. Behaviors of $(p_1, p_2, \Upsilon(p))^T$ with 35 initial points for $k = 700$. Numerical results for Algorithm 3.1.

Figure 7 plots the isometric view of $\Upsilon(p)$ in a 3-D space. Figure 8 depicts the changing processes of $(p_1, p_2, \phi(p))^T$ with 35 random initial data at star symbol and the unique convergent point at dot symbol. The behaviors of $(p_1, p_2) \in \mathbb{R}^2$ that are convergent to the exact solution

$$p^* = (-0.3466, 0)^T,$$

which is also the optimal solution of the convex optimization problem (4.4) when

$$\phi(p) = -6.7590exp(-5).$$

Example 4.4. Consider a quadratic optimization program as follows

$$\begin{aligned} &\text{minimize } \ \Upsilon(p) = \frac{1}{2}p^T\Theta p + \beta^T p \\ &\text{subject to } \ p \in \Omega, \end{aligned} \tag{4.5}$$

via the following properties

$$\Theta = \begin{pmatrix} 10 & -18 & 2 \\ -18 & 40 & -1 \\ 2 & -1 & 3 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 12 \\ -47 \\ -8 \end{pmatrix},$$

where

$$p = (p_1, p_2, p_3)^T,$$

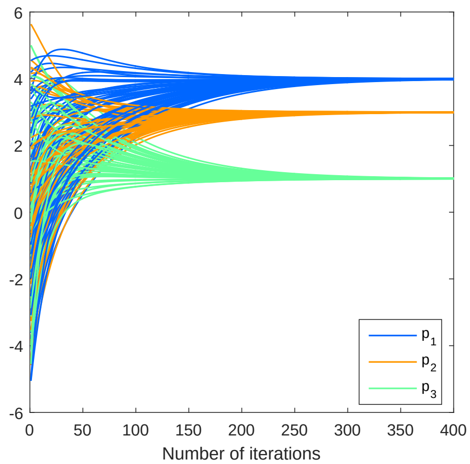
Θ is a 3×3 positive semidefinite matrix, β is a 3-vector and Ω is a nonempty closed convex subset of $\mathbb{R}^3$ defined by

$$\Omega := \{p \in \mathbb{R}^3 | -5 \le p_1, p_2, p_3 \le 5\}.$$

It is known that the objective function $\Upsilon(p)$ is pseudoconvex with the gradient

$$F := \nabla\Upsilon = \Theta p + \beta$$

Lipschitz continuous on the feasible region Ω. We study this problem for many different choices of starting points generated from the uniform distribution over $(0, 1)^3$. Meanwhile, we take the iteration number $k = 400$ as the stopping criterion.



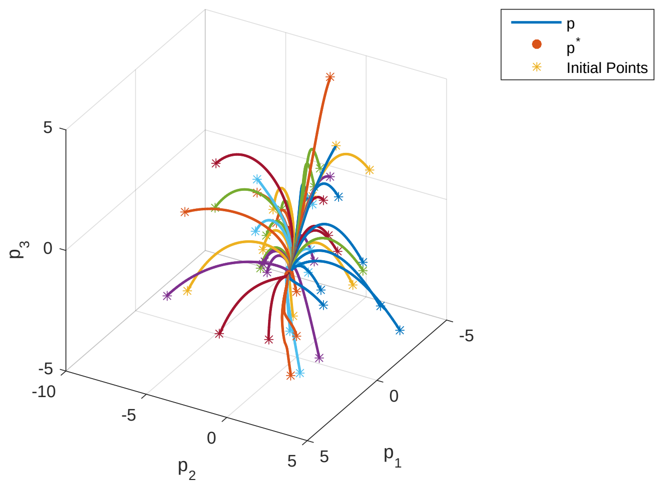


FIGURE 9. Behaviors of three elements of p in terms of many random initial points, and require $k = 400$ iterations in Example 4.3.

FIGURE 10. Behaviors of $(p_1, p_2, p_3)^T$ with 50 random initial points in 3-D space. Numerical results for Algorithm 3.1.

From the results reported in Figure 9 and Figure 10, one has shown that the changing processes of$(p_1, p_2, p_3)^T \in \mathbb{R}^3$ in two different versions with the number of iterations. Obviously, the convergence of the value $(p_1, p_2, p_3) \in \mathbb{R}^3$ to the exact point $(4, 3, 1)^T$, which is the optimal solution of problem (4.5) with $\Upsilon(p) = -50.5$.

Let us recall some previously known algorithms firstly. The extragradient method (EGM) was proposed by Korpelevich [10]. The subgradient-extragradient method (PSEM) was proposed by Censor, Gibali, Reich in [15], as one extension of the extragradient method. An alternative to the extragradient method or its modification (TSENG) was proposed by Tseng in [11] and was extended in [12]. The algorithms (Y-EAPM,L-EAPM) based on the extragradient method, combining with the approximate method were proposed in [13, 14]. The algorithm (SEGM) based on the Popov extragradient method, by means of the subgradient-extragradient method was presented by Malitsky and Semenov in [21]. A modification of the subgradient-extragradient method, combining with positive features of the Halpern method (HSEM) was proposed in [22]. Next, we will present the numerical results to illustrate the practicability and the competitive performance of our proposed algorithm, in comparison with algorithms mentioned above.

Example 4.5. [27] Consider the following problem governed by the Gaussian function with linear equality constraint

$$\begin{aligned} &\text{minimize } \Upsilon(p) = -exp\left(-\sum_{i=1}^{2}\frac{p_i^2}{\tau_i^2}\right) \\ &\text{subject to } \Psi p = \beta, \end{aligned} \tag{4.6}$$

where

$$p = (p_1, p_2)^T \in \mathbb{R}^2,\ \tau = (1,1)^T,$$

$$\Psi = (1,2) \text{ and } \beta = 1.2.$$

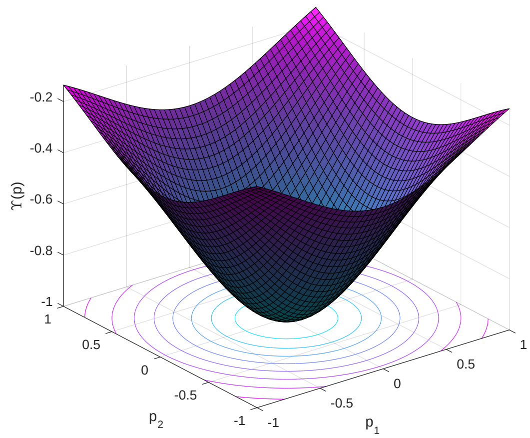
One sees that the objective function is locally Lipschitz continuous and strictly pseudoconvex on $\mathbb{R}^2$. Thus its gradient

$$F = \nabla\Upsilon(p) = \left(\frac{2p_1}{\tau_1^2}exp\left(-\sum_{i=1}^{2}\frac{p_i^2}{\tau_i^2}\right), \frac{2p_2}{\tau_2^2}exp\left(-\sum_{i=1}^{2}\frac{p_i^2}{\tau_i^2}\right)\right)^T,\ \forall p = (p_1,p_2)^T$$

is Lipschitz and pseudomonotone on the equality constraint. We denote k by the number of iterations. Since we do not know the exact solution of the problem, we use sequence

$$E_k = \|p_k - P_\Omega(p_k - \gamma_k F p_k)\|,\ \forall k = 0,1,2,\cdots$$

to measure the error of the k-th iteration for Algorithms 3.1, EGM, TSENG, PSEM. According to the nearest point (metric) projection, if the error distance $E_k < \varepsilon$, then p_k can be considered as a ε-solution of the problem, which also serves as the role of checking whether or not the proposed algorithm converges to the solution. The initial elements are generated from the uniform distribution over $(0,1)^2$. We takes the iteration number $k = 200$ as the stopping criterion in the following experiment.



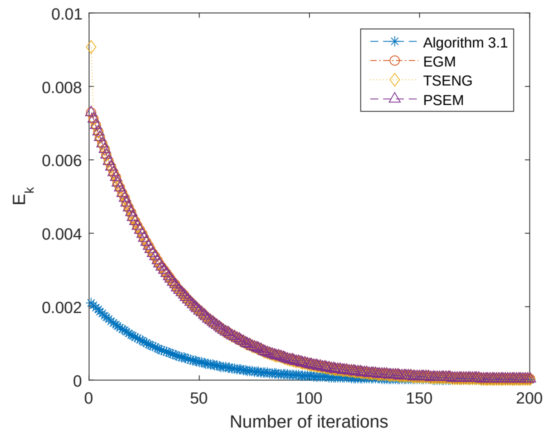


FIGURE 11. The isometric view of inverted 2-D normalized Gaussian function with $\tau = (1,1)^T$.

FIGURE 12. Comparison of the convergence behaviors of the errors $(E_k)_{k\in\mathbb{N}}$ with the number of iterations (resp.) $k = 200$. Numerical results for different algorithms.

TABLE 1. Comparison results between proposed Algorithm 3.1, EGM, TSENG, PSEM.

Iter.	Algorithm 3.1	EGM	TSENG	PSEM
50	**4.9686e-04**	0.0019	0.0019	0.0019
100	**1.1359e-04**	4.3313e-04	4.3313e-04	4.3313e-04
150	**2.5957e-05**	9.9007e-05	9.9008e-05	9.9007e-05
200	**5.9317e-06**	2.2626e-05	2.2626e-05	2.2626e-05

Figure 11 shows the isometric view of $\Upsilon(p)$ in 3-D space. To illustrate the computational performance, we make a comparison on the convergence speed between those algorithms, with the same random initial condition and the same number of iterations. We code the numerical results in Figure 12 and Table 1. One can check that, our proposed algorithm is more efficient at each iteration, in contrast with that of the extragradient method and modified extragradient methods (EGM, TSENG, PSEM), with respect to the convergence behavior of the error E_k. That is, the four numerical methods have less convergence speed than Algorithm 3.1, which can be explained by the presence of the inertial extrapolation term at each iteration. Thus the inertial step plays the key role in the acceleration process. Besides, the convergence of $(E_k)_{k\in\mathbb{N}}$ to 0 means that the iterative sequence converges to the solution of the variational inequality problem. Further, the convergence of the value $(p_1, p_2)^T$ to the exact solution $(0.24, 0.48)^T$, which is the optimal solution of problem (4.6) when $\Upsilon(p) = -0.7498$.

Example 4.6. Consider the quadratic fractional programming problem via

$$\begin{array}{ll} \min & \Upsilon(p) = \frac{p^T\Theta p+\beta^T p+\beta_0}{\omega^T p+\omega_0} \\ s.t. & \Psi p = \varrho, \end{array} \tag{4.7}$$

where Θ is an $m \times m$ matrix, Ψ is a $2 \times m$ matrix, both β and ω are 2-vectors, ϱ is a 2-vector and $\beta_0, \omega_0 \in \mathbb{R}$. When $m = 4$, we set $\omega_0 = 2$ and

$$\Theta = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \ \omega = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\Psi = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & -2 \end{pmatrix}, \ \varrho = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \beta_0 = -4.$$

One sees that the matrix Θ is symmetric positive definite, and hence the objective function $\Upsilon(p)$ is pseudo-convex on Ω [28]. Let F be the gradient function of Υ which can be written in the following explicit form

$$F := \nabla\Upsilon = \frac{(\omega^T p + \omega_0)(2\Theta p + \beta) - \omega(p^T\Theta p + \beta^T p + \beta_0)}{(\omega^T p + \omega_0)^2}.$$

It is known that Ω is a closed and convex subset of $\mathbb{R}^4$ and F is pseudomonotone on Ω. In this test, one randomly chooses many initial elements in the range of $(0, 1)^m$. Further, we take the iteration number $k = 40$ as the termination criterion in the following experiment.

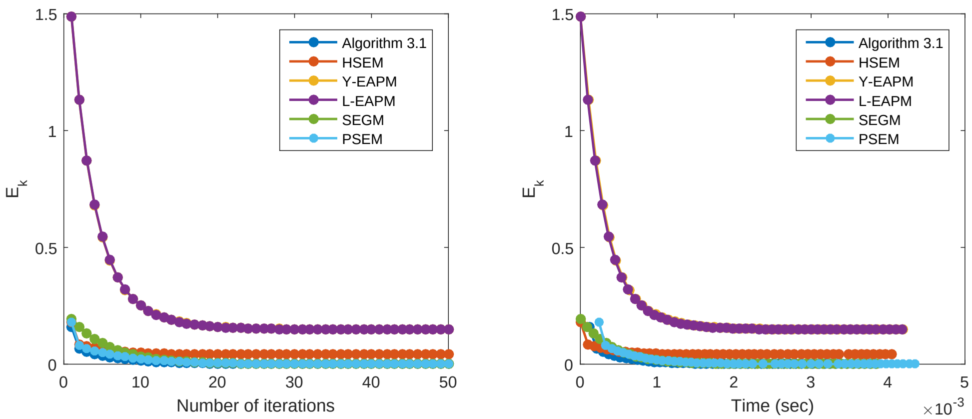


FIGURE 13. Comparison of the convergence behaviors of the errors $(E_k)_{k\in\mathbb{N}}$ with the number of iterations (resp.) $k = 50$. Numerical results for Algorithm 3.1, HSEM, Y-EAPM, L-EAPM, SEGM and PSEM.

TABLE 2. Comparison results between proposed Algorithm 3.1, HSEM, Y-EAPM, L-EAPM, SEGM and PSEM.

Method	Algorithm3.1		HSEM		Y-EAPM		L-EAPM		SEGM		PSEM	
Iter.	Time	E_k	Time	E_k	Time	E_k	Time	E_k	Time	E_k	Time	E_k
10	**8.1351e-04**	**0.0147**	7.4467e-04	0.0480	8.0725e-04	0.2509	7.9303e-04	0.2506	7.2192e-04	0.0341	9.1932e-03	0.0227
20	**1.5644e-03**	**0.0025**	1.5354e-03	0.0425	1.6572e-03	0.1600	1.6412e-03	0.1594	1.5297e-03	0.0051	1.6686e-03	0.0048
30	**2.3495e-03**	**0.0005**	2.3444e-03	0.0419	2.5071e-03	0.1504	2.4889e-03	0.1501	2.3046e-03	0.0008	2.6146e-03	0.0010
40	**3.0993e-03**	**0.0001**	3.2603e-03	0.0418	3.3536e-03	0.1494	3.3365e-03	0.1491	3.0794e-03	0.0001	3.6056e-03	0.0002
50	**3.8411e-03**	**0.0000**	4.0459e-03	0.0418	4.1978e-03	0.1493	4.1842e-03	0.1490	3.8542e-03	0.0000	4.3486e-03	0.0000

As the same case of Example 4.5, we use the sequence

$$E_k = \|p_k - P_\Omega(p_k - \beta_k \Phi p_k)\|, \ \forall k = 0, 1, 2, 3 \cdots$$

to measure the error of the k-th iteration of Algorithms 3.1, EGM, HSEM, MEGM, Y-EAPM, L-EAPM and SEGM. To illustrate the convergence and computational performance of all the algorithms, one has shown that the values $(E_k)_{k\in\mathbb{N}}$ (y-axis) with the number of iterations $k = 50$ (x-axis). For comparison on the convergence speed between those algorithms, with the same random initial data and the same number of iterations, we code the test results in Figure 13 and Table 2. One can check that, our proposed algorithm is more efficient in CPU-Time, with respect to the convergence behavior of the error E_k. That is, the mentioned iterative schemes have less convergence speed than Algorithm 3.1, which can be explained by the presence of the initial extrapolation term and the absence of one projection step of a point onto the feasible set per iteration. Meanwhile, from the changing processes of the values $(E_k)_{k\in\mathbb{N}}$, one finds that Algorithm 3.1 has a better behavior, in contrast to that of other algorithms. It achieves a more stable and higher precision with the number of iterations. Besides, the convergence of the values $(E_k)_{k\in\mathbb{N}}$ to 0 means that the iterative sequence converges to the solution of the variational inequality problem. Above all, Algorithm 3.1 has substantially a better performance, compared with that of other algorithms. In addition, we have that the unique convergent point $p^* = (1.1834, 1.8774, 0.2442, -1.6642)^T$ is the optimal solution of problem (4.7).

We want to note that, in this experiment, since both the operator F and projections are cheap, the running time is quite small, measured only with some error of order 10^{-3}. For this reason, our results cannot achieve a drastic contrast of algorithms' performance.

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