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(q_1, q_2)-QUASIMETRIC SPACES. COVERING MAPPINGS AND COINCIDENCE POINTS. A REVIEW OF THE RESULTS

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Abstract. In their recent papers, A.V. Arutyunov and A.V. Greshnov introduced (q_1, q_2)-quasimetric spaces and studied their properties: investigated covering mappings between (q_1, q_2)-quasimetric spaces, established sufficient conditions for the existence of a coincidence point for two mappings acting between (q_1, q_2)-quasimetric spaces such that one is a covering mapping and the other is Lipschitz continuous, proved Banach's fixed point theorem, obtained generalizations for multivalued mappings. The class of (q_1, q_2)-quasimetric spaces is sufficiently wide; it includes quasimetric spaces, b-metric spaces, Carnot-Carathéodory spaces with Box-quasimetics, L_p-spaces with $p \in (0, 1)$, etc. The development of the theory of coincidence points of mappings on (q_1, q_2)-quasimetric spaces initiated interest in the study of more general f-quasimetric spaces and in generalizing Banach's fixed point theorem to such spaces. The present paper is a review of these results.
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1. Introduction

In the recent article [42], published in *Fixed Point Theory*, for so-called complete b-metric spaces (X, ρ) (in the terminology of our article, these are symmetric (q, q)-quasimetric spaces), a new proof was given of the fixed point theorem for contraction mappings earlier proved in the monograph [39, Chapter 12] by Kirk and Shahzad. Czerwik was the first to generalise Banach's fixed point theorem for b-metric spaces [21] (see also the recent article [37] about the proof of Czerwik's theorem mentioned above).

However, these and some other similar results embed into the general theory of (q_1, q_2)-*quasimetric* spaces, which were recently introduced and studied by Arutyunov and Greshnov in [6]–[8]. In particular, b-metric spaces, introduced by Bakhtin in 1989 (see [12]) are a particular case of (q_1, q_2)-quasimetric spaces. In addition, for $q_1 = q_2$, all known results for b-metric spaces follow from the theorems given below for (q_1, q_2)-quasimetric spaces, which hold for (q_1, q_2)-metric spaces, where $q_1 \neq q_2$.

A nontrivial example of a (q_1, q_2)-quasimetric space, where $q_1 \neq q_2$, is given by the spaces $L_p(E)$, where E is a measurable bounded set in R^n, for $0 < p < 1$. These spaces are quasinormed since we have $\|f_1 + f_2\|_p \leq 2^{-1/p'}(\|f_1\|_p + \|f_2\|_p)$ for all $f_1, f_2 \in L_p(E)$, where $p' < 0$ is the number conjugate to p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, for every $\epsilon > 0$, we have

$$\|f_1 + f_2\|_p \leq (1+\epsilon)\|f_1\|_p + C(\epsilon, p)\|f_2\|_p \quad \forall f_1, f_2 \in L_p(E),$$

where $C(\epsilon, p) = (1 - (1+\epsilon)^{p'})^{1/p'}$. Thus, $L_p(E)$ with $0 < p < 1$ is a symmetric (q_1, q_2)-quasimetric space whose quasimetric $\rho_{L_p(E)}$ is given by the formula

$$\rho_{L_p(E)}(f_1, f_2) = \|f_1 - f_2\|_p, \quad q_1 = q_2 = 2^{-1/p'}.$$

Moreover, for every $\epsilon > 0$ the (q_1, q_2)-generalized triangle inequality holds in $L_p(E)$ provided that $q_1 = 1+\epsilon$, $q_2 = C(\epsilon, p)$. Note that similar properties hold for the spaces l_p with $0 < p < 1$, which are used in the penalty method for the approximate solution of extremum problems with constraints [11].

Examples of (q_1, q_2)-quasimetric spaces for $q_1 \neq q_2$ naturally arise in analysis and geometry on Carnot–Carathéodory spaces and their generalizations (see, for example, [13], [14], [33], [43], [53]), whose Box-quasimetrics, as it turned out, are $(1, q_2)$-quasimetrics (see, for example, [27]–[30]). This plays a crucial role in the proof of an analog of Gromov's local approximation theorem (see [31], [53]). To some extent, the example of Box-quasimetrics answers the question formulated in [39, Remark 12.3].

We introduced (q_1, q_2)-quasimetric spaces and investigated their properties in [6]–[8]. Also covering mappings between (q_1, q_2)–quasimetric spaces were investigated. Sufficient conditions for the existence of a coincidence point of two mappings acting between (q_1, q_2)-quasimetric spaces such that one is a covering mapping and the other satisfies the Lipschitz condition were obtained. These results were extended to multivalued mappings. We proved that the coincidence points are stable under small perturbations of the mappings.

Earlier in [5], the local theory of coincidence points for metric spaces was constructed. A generalization of this theory to (q_1, q_2)-quasimetric spaces seems to us an interesting problem, useful in applications.

In this connection, we will briefly expose some results of [6]–[8], which are connected with theorems on the existence of fixed points and coincidence points for two mappings acting from one (q_1, q_2)-*quasimetric* space into another (q_1', q_2')-*quasimetric* space and also establish the relationship with other results.

2. (q_1, q_2)-quasimetric spaces. Covering mappings and coincidence points

As usual, a function $\rho_X : X \times X \to \mathbb{R}^+$, where X is an arbitrary set and $\mathbb{R}^+$ is the set of non-negative real numbers, is called a metric if the following properties hold:

$$\rho_X(x, y) = 0 \Leftrightarrow x = y \quad \text{(the identity axiom);}$$

$$\rho_X(x, y) = \rho_X(y, x) \quad \forall x, y \in X \quad \text{(the symmetry axiom);}$$

$$\rho_X(x, z) \leq \rho_X(x, y) + \rho_X(y, z) \quad \forall x, y, z \in X \quad \text{(the triangle axiom).}$$

If we omit the symmetry axiom (preserving the identity axiom and triangle inequality), then ρ_X is called a quasimetric and the pair (X, ρ_x) is called a quasimetric space [54].

Quasimetric spaces are studied extensively in topology, functional analysis and metric analysis (see, for example, [23], [26], [38], [45], [54]). Quasimetric and non-symmetric normed spaces were considered in [17], [35], [36], [46] (see the references therein). They have numerous applications in optimization and approximation theory, convex analysis and elsewhere.

Definition 2.1 Let q_1, q_2 be positive numbers and X any set consisting of at least two points. A function $\rho_X : X \times X \to \mathbb{R}^+$ satisfying the identity axiom is called a (q_1, q_2)-*quasimetric* if the following (q_1, q_2)-*generalized triangle inequality* holds:

$$\rho_X(x, y) \leq q_1\rho_X(x, z) + q_2\rho_X(z, y) \quad \forall x, y, z \in X.$$

(X, ρ_X) is called (q_1, q_2)-*quasimetric space*. If the (q_1, q_2)-quasimetric ρ_X satisfies the additional condition

$$\rho_X(x, y) \leq q_0\rho_X(y, x) \quad \forall x, y \in X \quad (q_0 - \text{symmetry})$$

for some $q_0 > 0$, then it is said to be q_0-symmetric and the pair (X, ρ_X) is called q_0-*symmetric* (q_1, q_2)-*quasimetric space*.

When $q_0 = 1$, the pair (X, ρ_X) is called a symmetric (q_1, q_2)-quasimetric space. When $q_1 = q_2 = 1$, ρ_X is a quasimetric and (X, ρ_X) is a quasimetric space. But if $q_0 = q_1 = q_2 = 1$, then ρ_X is a metric and (X, ρ_X) is an ordinary metric space. The (q_2, q_1)-quasimetric $\overline{\rho}_X(x, y) = \rho_X(y, x)$ is said to be conjugate to $\rho_X(x, y)$ (compare with [17]).

Definition 2.2 A (q_1, q_2)-quasimetric (and the corresponding space (X, ρ_X)) is said to be *weakly symmetric*, if $\lim\limits_{i\to\infty} \rho_X(\xi, x_i) = 0$ implies that

$$\lim_{i\to\infty} \overline{\rho}_X(\xi, x_i) = 0.$$

In another terminology, the weak symmetry condition can be found in [44]. Note that for q_0-symmetric (q_1, q_2)-quasimetric spaces we have

$$\lim_{i\to\infty} \rho_X(x_0, x_i) = 0 \Leftrightarrow \lim_{i\to\infty} \rho_X(x_i, x_0) = 0,$$

so that every q_0-symmetric space is weakly symmetric. But the converse fails (see [6, Example 2.2]).

To introduce a topology on (q_1, q_2)-quasimetric spaces and for the further use we define the sets

$$\overset{\circ}{B}_X(x, r) = \{y \in X \mid \rho_X(x, y) < r\}, \quad B_X(x, r) = \{y \in X \mid \rho_X(x, y) \leq r\},$$

which may naturally be referred to as balls centered at x of radius r. A set $U \subset X$ is said to be *open* if, for every point $u \in U$ there is a number $r_u > 0$ such that $\overset{\circ}{B}_X(u, r_u) \subset U$. Clearly, the open sets defined in this way determine a topology on which will be denoted by τ_{ρ_X} (*open balls topology*). As usual, a set is said to be *closed* if its complement is open.

Consider a sequence of points $\{x_i\} \subset X$. We say that $\{x_i\}$ *converges* to a point $x_0 \in X$ and write $x_i \to x_0$, if, for every $\varepsilon > 0$ ball $\overset{\circ}{B}_X(x_0, \varepsilon)$ contains all points x_i, starting with some of them. The point x_0 is called the *limit point* or *limit of the sequence* $\{x_i\}$. Clearly, this definition may equivalently be restated in the following form: a sequence $\{x_i\}$ converges to x_0, if $\lim\limits_{i\to\infty} \rho_X(x_0, x_i) = 0$.

We easily verify that every convergent sequence in a weakly symmetric (q_1, q_2)-quasimetric space has a unique limit. However, generally speaking, this is not the case when the space is not weakly symmetric [6], [9].

Definition 2.3 A sequence $\{x_n\}$ in a (q_1, q_2)-quasimetric space (X, ρ_X) is called a *fundamental sequence* or a *Cauchy sequence* (*left K-Cauchy* [45]), if for every $\varepsilon > 0$ there is an N such that for all $n > m > N$ we have $\rho_X(x_m, x_n) < \varepsilon$.

A (q_1, q_2)-quasimetric space (X, ρ_X) is said to be *complete* if each of its fundamental sequences has a limit (possibly non-unique).

Definition 2.4 A function $f : X \to \mathbb{R}$ is said to be *upper* (*lower*) *semicontinuous* at a point $x_0 \in X$, if for every $\varepsilon > 0$ there is an $r_\varepsilon > 0$ such that

$$f(x) < f(x_0) + \varepsilon \quad \big(f(x_0) < f(x) + \varepsilon\big) \quad \forall x \in \overset{\circ}{B}_X\big(x_0, r_\varepsilon\big) \Leftrightarrow \rho_X(x_0, x) < r_\varepsilon.$$

If f is simultaneously upper and lower semicontinuous at x_0, then f is *continuous* at x_0.

There are examples of quasimetrics and symmetric (q_1, q_2)-quasimetrics for which semicontinuity is violated in both arguments. At the same time, the q_0-symmetric $(1, q_2)$-quasimetric $\rho_X(x, y)$ is continuous in the second argument, while the q_0-symmetric $(q_1, 1)$-quasimetric $\rho_X(x, y)$ is continuous in the first argument.

Mention the paper [49], where sufficient conditions were obtained for the existence of a minimum for lower semicontinuous functions defined on (q_1, q_2)-quasimetric spaces, which strengthen the conditions of Caristi type.

In what follows, we consider a (q_1, q_2)-quasimetric space (X, ρ_X) and a (q_1', q_2')-quasimetric space (Y, ρ_Y).

Definition 2.5 Suppose that $\alpha > 0$. A mapping $\Psi : X \to Y$ is said to be *α-covering*, if

$$B_Y(\Psi(x), \alpha r) \subseteq \Psi(B_X(x, r)) \quad \forall r \geq 0 \quad \forall x \in X. \tag{2.1}$$

It follows from Definition 2.5 that for any points $x_0 \in X$, $y_1 \in Y$ one can find a point $x_1 \in X$ such that $y_1 = \Psi(x_1)$, $\rho_X(x_0, x_1) \leq \frac{\rho_Y(\Psi(x_0), y_1)}{\alpha}$. Hence the mapping Ψ is surjective.

Definition 2.6 A mapping $\Phi : X \to Y$ is said to be *β-Lipschitz* if

$$\rho_Y\big(\Phi(x_1), \Phi(x_2)\big) \leq \beta \rho_X(x_1, x_2) \quad \forall x_1, x_2 \in X.$$

Along with the (q_1', q_2')-quasimetric ρ_Y, we can consider the quasimetric $\tilde{\rho}_Y = \frac{\rho_Y}{\alpha}$. It is easy to see that, with respect to $\tilde{\rho}_Y$, every α-covering mapping becomes 1-covering and every β-Lipschitz mapping becomes $\frac{\beta}{\alpha}$-Lipschitz.

As usual,

$$\operatorname{gph}(F) = \{(x, y) \in X \times Y \mid y = F(x)\}$$

is the graph of a mapping $F : (X, \rho_X) \to (Y, \rho_Y)$.

Definition 2.7 We say that a mapping F *closed* if, for all sequences $\{x_i\} \subset X$ and $\{y_i\} \subset Y$ converging to x_0 and y_0 respectively and satisfying $(x_i, y_i) \in \mathrm{gph}(F)$ for all i, we have $(x_0, y_0) \in \mathrm{gph}(F)$.

Note that if the quasimetric space (X, ρ_X) contains a sequence with more than one limit (see [6]), then even the identity mapping $\mathrm{Id} : (X, \rho_X) \to (X, \rho_X)$ is non-closed although it is obviously continuous and, moreover, satisfies the Lipschitz condition.

Given a function $f : X \times X \to \mathbb{R}^+$ of two variables and a point $(x_1, x_2) \in X \times X$, we write $\varliminf_{\eta \to x_1} f(\eta, x_2)$ for its lower limit in the first variable at the point (x_1, x_2). This limit is defined as the infimum of the lower limits $\inf \varliminf_{i \to \infty} f(\eta_i, x_2)$, where the infimum is taken over all sequences $\{\eta_i\}$ that converge to x_1. The lower limit $\varliminf_{\eta \to x_2} f(x_1, \eta)$ in the second variable is defined in a similar way. Clearly, if the function is lower semicontinuous in some variable at the point under consideration, then its lower limit in this variable is equal to the value of the function at that point.

Suppose that we are given mappings $\Phi, \Psi : X \to Y$ and real numbers $\alpha > \beta \geq 0$.

Definition 2.8 A point $x \in X$ is called a *coincidence point* of Ψ and Φ if $\Psi(x) = \Phi(x)$.

For all $q_0, q_1, q_2 \geq 1$ we put

$$m_0 = \min\{j \in \mathbb{N} \mid q_2 \beta^j < \alpha^j\}, \tag{2.2}$$

and under the assumption that $q_0^2 \beta < \alpha$, we put

$$n_0 = \min\{j \in \mathbb{N} \mid q_1 (q_0^2 \beta)^j < \alpha^j\}. \tag{2.3}$$

Note that the existence of m_0 follows from the assumption $\beta < \alpha$; $S(a; n) = 1 + a + ... + a^{n-1}$.

Theorem 2.9 (see [6, Theorem 4.5], [7, Theorem 1]) *Let (X, ρ_X) be a complete (q_1, q_2)-quasimetric space, and Ψ an α-covering closed mapping, and Φ a β-Lipschitz mapping. Fix an arbitrary point $x_0 \in X$.*

Then Ψ and Φ have a coincidence point ξ, such that the following bound holds:

$$\varliminf_{\eta \to \xi} \rho_X(x_0, \eta) \leq \frac{q_1^2 \alpha^{m_0 - 1} S\big(q_2 \frac{\beta}{\alpha}, m_0 - 1\big) + q_1 (q_2 \beta)^{m_0 - 1}}{\alpha^{m_0} - q_2 \beta^{m_0}} \rho_Y(\Psi(x_0), \Phi(x_0)). \tag{2.4}$$

If the space (X, ρ_X) is weakly symmetric, then ξ also satisfies the bound

$$\rho_X(x_0, \xi) \leq q_1 \frac{q_1^2 \alpha^{m_0 - 1} S\big(q_2 \frac{\beta}{\alpha}, m_0 - 1\big) + q_1 (q_2 \beta)^{m_0 - 1}}{\alpha^{m_0} - q_2 \beta^{m_0}} \rho_Y(\Psi(x_0), \Phi(x_0)), \tag{2.5}$$

and if this space is q_0-symmetric with $q_0^2 \beta < \alpha$, then we also have the following bounds for ξ:

$$\overline{\rho}_X(x_0, \xi) \leq q_0 q_2^2 \frac{q_2 \alpha^{n_0 - 1} S\big(q_1 q_0^2 \frac{\beta}{\alpha}, n_0 - 1) + (q_1 q_0^2 \beta)^{n_0 - 1}}{\alpha^{n_0} - q_1 (q_0^2 \beta)^{n_0}} \rho_Y(\Psi(x_0), \Phi(x_0)), \tag{2.6}$$

$$\varliminf_{\eta \to \xi} \overline{\rho}_X(x_0, \eta) \leq q_0 q_2 \frac{q_2 \alpha^{n_0 - 1} S\big(q_1 q_0^2 \frac{\beta}{\alpha}, n_0 - 1) + (q_1 q_0^2 \beta)^{n_0 - 1}}{\alpha^{n_0} - q_1 (q_0^2 \beta)^{n_0}} \rho_Y(\Psi(x_0), \Phi(x_0)). \tag{2.7}$$

Theorem 2.9 (*on the existence of coincidence points*) generalizes the corresponding result of [3].

When the function ρ_X is lower semicontinuous in the second argument at the point ξ, the bound (2.4) is stronger than (2.5). When ρ_X is lower semicontinuous in the first argument at ξ, the bound (2.7) is stronger than (2.6). This is not the case in general (see [6, Remark 4.7]).

Corollary 2.10 (see [6, Corollary 4.9], [7, Corollary 2]) *Suppose that the space (X, ρ_X) is complete, the mapping Ψ is α-covering and closed and the mapping Φ is β-Lipschitz. Fix an arbitrary point $x_0 \in X$.*

$\mathbf{1^0}$ *Suppose that $q_2\beta < \alpha$. Then Ψ and Φ have a coincidence point ξ such that*

$$\underline{\lim}_{\eta\to\xi} \rho_X(x_0, \eta) \le \frac{q_1}{\alpha - q_2\beta}\rho_Y(\Psi(x_0), \Phi(x_0)),$$

and if the space (X, ρ_X) is weakly symmetric, then this coincidence point ξ also satisfies

$$\rho_X(x_0, \xi) \le \frac{q_1^2}{\alpha - q_2\beta}\rho_Y(\Psi(x_0), \Phi(x_0)).$$

$\mathbf{2^0}$ *Suppose that (X, ρ_X) is q_0-symmetric with $q_1 q_0^2\beta < \alpha$. Then Ψ and Φ have a coincidence point ξ such that*

$$\overline{\rho}_X(x_0, \xi) \le \frac{q_0 q_2^2}{\alpha - q_1 q_0^2\beta}\rho_Y(\Psi(x_0), \Phi(x_0)),$$

$$\underline{\lim}_{\eta\to\xi} \overline{\rho}_X(x_0, \eta) \le \frac{q_0 q_2}{\alpha - q_1 q_0^2\beta}\rho_Y(\Psi(x_0), \Phi(x_0)).$$

Suppose that $X = Y$ and $\Psi = \mathrm{Id}$ is the identity map. Then $\alpha = 1$, the condition $\beta < 1$ means that Φ is a *contraction mapping*, and the coincidence point becomes a *fixed point*. Our next assertion (*the Theorem on the fixed points of a contraction mapping*) follows from the proof of Theorem 2.9.

Theorem 2.11 (see [6, Theorem 4.8], [7, Corollary 1]) *Every closed contraction mapping from a complete (q_1, q_2)-quasimetric space to itself has a fixed point, and this point is unique.*

Observe that, in the fixed point theorem, the assumption of the closedness of the contraction mapping is substantial; namely, in [47], an example was constructed of a complete quasimetric space in which a contraction mapping has no fixed points; this mapping is not closed.

In quasimetric spaces, the fixed point theorem for contraction mappings was considered by a number of authors (see, for example, [16]). For complete b-quasimetric spaces (in our terminology, these are symmetric (q, q)-quasimetric spaces), the fixed point theorem was proved in the monograph [39, Chapter 12] by Kirk and Shahzad; another proof of this theorem of Kirk and Shahzad can be found in [42]. We also mention the article [20], where, on b-metric spaces (in our terminology, these are (q, q)-quasimetric spaces), fixed point theorems were proved for some analogs of contraction mappings. Note that, in [18], Cozbaş obtained a generalization of Kirk and Shahzad's result for b-quasimetric spaces satisfying the s-relaxed triangle inequality $\rho(x,y) \leq s[\rho(x,z)+\rho(z,y)]$, for mappings f:

$$\rho(f(x), f(y)) \leq \varphi(\rho(x,y)), \tag{2.8}$$

where function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is satisfying the conditions:

(a) φ is nondecreasing,

(b) $\lim\limits_{n\to\infty} \varphi^n(t) = 0$,

(c) $\varphi(t) < \frac{t}{s}$ for all $t > 0$,

(in this connection, see Theorem 3.9 below) and also for *generalized* b-quasimetric spaces $(\rho : X \times X \to [0,\infty])$.

A separate series of results was obtained in [6, 7] for $(1, q_2)$- and $(q_1, 1)$-quasimetric spaces. Note that [39] and also Cozbaş's preprint [18] were concerned with $(1, s)$-quasimetric spaces, which were called *strong b-metric spaces* or *sb-metric spaces* therein.

Theorem 2.12 (see [6, Theorem 4.12], [7, Theorem 2]) *Suppose that the space (X, ρ_X) is complete, the mapping Ψ is α-covering and closed, and Φ is β-Lipschitz. Fix an arbitrary point $x_0 \in X$.*

1^0 *Suppose that $q_1 = 1$. Then Ψ and Φ have a coincidence point ξ such that*

$$\varliminf_{\eta\to\xi} \rho_X(x_0, \eta) \leq \frac{\alpha - \beta + q_2\beta}{\alpha(\alpha-\beta)} \rho_Y(\Psi(x_0), \Phi(x_0)). \tag{2.9}$$

2^0 *Suppose that the space (X, ρ_X) is q_0-symmetric with $q_0^2\beta < \alpha$ and $q_2 = 1$. Then there is a coincidence point ξ such that*

$$\overline{\rho}_X(x_0, \xi) \leq q_0 \frac{q_1 q_0^2 \beta + \alpha - q_0^2\beta}{\alpha(\alpha - q_0^2\beta)} \rho_Y(\Psi(x_0), \Phi(x_0)). \tag{2.10}$$

Let us compare the results of Theorems 2.9 and 2.12. The restrictions on the values of α, β, q_1 and q_2 in Theorem 2.9 are weaker than those in Theorem 2.12. At the same time, the bound (2.9) obtained in part 1) of Theorem 2.12 under the additional assumption $q_1 = 1$ is better than the bound (2.4) in Theorem 2.9 provided that this assumption holds (see details in [6]); we can similarly prove that the bound (2.10) in part 2) of Theorem 2.12, which was proved under the additional assumption $q_2 = 1$, is better than the bound (2.6) in Theorem 2.9 (provided that this assumption holds).

In [6], we constructed Example 4.13, which shows that the bounds (2.6), (2.7) in Theorem 2.9 and (2.10) in part 2) of Theorem 2.12 are unimprovable. In [6], we constructed Example 4.14, which shows that the bounds (2.9) in part 1) of Theorem 2.12 is also unimprovable.

3. Multivalued mappings of (q_1, q_2)-quasimetric spaces. Coincidence points and stability

Suppose that we are given a(q_1, q_2)-quasimetric space (X, ρ_X) and (q_1', q_2')-quasimetric space (Y, ρ_Y). As usual, the distance between two sets $U, V \subset X$ is defined by the formula

$$\mathrm{dist}_X(U,V) = \inf\{\rho_X(u,v) \mid u \in U,\ v \in V\}.$$

In (q_1, q_2)-quasimetric spaces as well as in metric spaces, the Hausdorff deviation $h_X^+(U,V)$ of a set U from a set V is defined by the formula

$$h_X^+(U,V) = \inf\{\varepsilon \geq 0 \mid U \subset N_\varepsilon(V)\},$$

$$N_\varepsilon(V) = \bigcup_{v \in V} \{x \in X \mid \rho_X(x,v) < \varepsilon\}.$$

If U is unbounded, then $h_X^+(U,V)$ can take the value $+\infty$.

For arbitrary sets $U, V \subset X$ we easily see that

$$h_X^+(U,V) \geq \operatorname{dist}_X(U,V),$$

and $h_X^+(U,V) = \sup\limits_{u \in U} \operatorname{dist}_X(u,V)$. If the sets $U, V \subset X$ are closed, then

$$h_X^+(U,V) = 0 \Leftrightarrow U \subseteq V.$$

The (q_1, q_2)-generalized triangle inequality need not hold for the distance dist_X between sets. However, it always holds for the Hausdorff deviation h_X^+:

$$h_X^+(U,W) \leq q_1 h_X^+(U,V) + q_2 h_X^+(V,W), \quad \forall U, V, W \subset X.$$

Property 3.1 ([6], [8, Property 5.1]) *For arbitrary sets $U, V, W \subset X$ we have*

$$\operatorname{dist}_X(U,W) \leq q_1 \operatorname{dist}_X(U,V) + q_2 h_X^+(V,W). \tag{3.1}$$

For metric spaces, (3.1) was proved in [2, §2.2].

For arbitrary closed sets $U, V \subset X$ we put

$$h_X(U,V) = \max\left\{h_X^+(U,V),\ h_X^+(V,U)\right\},$$

where $h_X(U,V) = +\infty$, if at least one of the deviations $h_X^+(U,V)$ or $h_X^+(V,U)$ is $+\infty$.

Unlike the Hausdorff deviation h_X^+, the function h_X, being the symmetrization of h_X^+, is itself symmetric. Therefore h_X is a symmetric $(\hat{q}_1, \hat{q}_2)$-quasimetric on the space of all closed bounded subsets of the space (X, ρ_X), where the constants $\hat{q}_1$, $\hat{q}_2$, may, generally speaking, be different from q_1, q_2, because of the "symmetrization effect" ([6, Example 3.3]). It is natural to call h_X *the Hausdorff $(\hat{q}_1, \hat{q}_2)$-quasimetric*.

When speaking of a multi-valued mapping, we mean a mapping $F : X \rightrightarrows Y$ sending each point $x \in X$ to a *non-empty closed* subset $F(x) \subset Y$. Suppose that we are given multi-valued mappings Ψ, $\Phi : X \rightrightarrows Y$ and numbers $\alpha > \beta \geq 0$.

Definition 3.1 A point ξ is called a *coincidence point* of multi-valued mappings Φ, Ψ, if $\Phi(\xi) \cap \Psi(\xi) \neq \emptyset$.

Definition 3.2 Ψ is said to be *α-covering* if

$$\bigcup_{y \in \Psi(x)} B_Y(y, \alpha r) \subseteq \Psi(B_X(x,r)) \quad \forall r \geq 0 \quad \forall x \in X. \tag{3.2}$$

Definition 3.3 A multi-valued mapping Φ is said to be *β-Lipschitz* if

$$h_Y\big(\Phi(x_1), \Phi(x_2)\big) \leq \beta \rho_X(x_1, x_2) \quad \forall x_1, x_2 \in X.$$

Let F be a multi-valued mapping. As usual, its graph is

$$\operatorname{gph}(F) = \{(x,y) \in X \times Y \mid y \in F(x)\}$$

and then closedness of F is defined as in Definition 2.7.

In what follows we use the same quantities m_0, n_0 as in (2.2), (2.3).

Theorem 3.2 ([6, Theorem 5.7], [8, Theorem 1]) *Let Ψ be an α-covering closed multi-valued mapping and Φ a β-Lipschitz multi-valued mapping. Suppose also that at least one of the graphs* gph (Φ), gph (Ψ) *is a complete space. Fix an arbitrary point $x_0 \in X$ and a number $\varepsilon > 0$.*

Then Ψ and Φ have a coincidence point ξ such that

$$\varliminf_{\eta\to\xi}\rho_X(x_0,\eta) \le \frac{q_1^2\alpha^{m_0-1}S\big(q_2\frac{\beta}{\alpha}, m_0-1\big)+q_1(q_2\beta)^{m_0-1}}{\alpha^{m_0}-q_2\beta^{m_0}}\operatorname{dist}_Y(\Psi(x_0),\Phi(x_0))+\varepsilon. \quad (3.3)$$

If the space (X, ρ_X) is weakly symmetric, then ξ also satisfies

$$\rho_X(x_0,\xi) \le q_1\frac{q_1^2\alpha^{m_0-1}S\big(q_2\frac{\beta}{\alpha}, m_0-1\big)+q_1(q_2\beta)^{m_0-1}}{\alpha^{m_0}-q_2\beta^{m_0}}\operatorname{dist}_Y(\Psi(x_0),\Phi(x_0))+\varepsilon, \quad (3.4)$$

and if this space is q_0-symmetric with $q_0^2\beta < \alpha$, then ξ also satisfies

$$\overline{\rho}_X(x_0,\xi) \le q_0q_2^2\frac{q_2\alpha^{n_0-1}S\big(q_1q_0^2\frac{\beta}{\alpha}, n_0-1\big)+(q_1q_0^2\beta)^{n_0-1}}{\alpha^{n_0}-q_1(q_0^2\beta)^{n_0}}\operatorname{dist}_Y(\Psi(x_0),\Phi(x_0))+\varepsilon, \quad (3.5)$$

$$\varliminf_{\eta\to\xi}\overline{\rho}_X(x_0,\eta) \le q_0q_2\frac{q_2\alpha^{n_0-1}S\big(q_1q_0^2\frac{\beta}{\alpha}, n_0-1\big)+(q_1q_0^2\beta)^{n_0-1}}{\alpha^{n_0}-q_1(q_0^2\beta)^{n_0}}\operatorname{dist}_Y(\Psi(x_0),\Phi(x_0))+\varepsilon. \quad (3.6)$$

In the case when $X = Y$ is complete, $\Psi(x) \equiv \{x\}$, $\beta < 1$ and Φ is closed, Theorem 3.2 is a fixed point theorem for multi-valued mappings. Theorem 3.2 is a multi-valued analogue of Theorem 2.9. One can similarly obtain a multi-valued analogue of Theorem 2.12.

We claim that under rather general conditions the coincidence points of two mappings are stable with respect to small (in the sense defined below) perturbations of these mappings. The following assertion is informative for single-valued as well as multi-valued mappings.

Theorem 3.3 ([6, Theorem 5.8], [8, Theorem 3]) *Suppose that (Y, ρ_Y) is a q_0'-symmetric (q_1', q_2')-quasimetric space and x_0 is a coincidence point of multi-valued mappings $\Phi, \Psi : X \rightrightarrows Y$. Consider sequences $\{\Psi_i\}$, $\{\Phi_i\}$ of multi-valued mappings satisfying the following conditions. For every i the mapping Ψ_i is α-covering and closed, the mapping Φ_i is β-Lipschitz, and at least one of the graphs* gph (Ψ_i), gph (Φ_i) *is a complete set. Suppose also that*

$$h_Y^+(\Psi(x_0),\Psi_i(x_0)) \to 0, \quad h_Y^+(\Phi(x_0),\Phi_i(x_0)) \to 0, \quad (3.7)$$

and there is a sequence $\{\varepsilon_i\}$ of positive numbers converging to zero. Put

$$h_i(x_0) = q_1'q_2'q_0'h_Y^+\big(\Psi(x_0),\Psi_i(x_0)\big) + q_2'h_Y^+\big(\Phi(x_0),\Phi_i(x_0)\big).$$

Then, for every i the mappings Ψ_i and Φ_i have a coincidence point ξ_i such that

$$\varliminf_{\eta\to\xi_i}\rho_X(x_0,\eta) \le \frac{q_1^2\alpha^{m_0-1}S\big(q_2\frac{\beta}{\alpha},m_0-1\big)+q_1q_2^{m_0-1}\beta^{m_0-1}}{\alpha^{m_0}-q_2\beta^{m_0}}h_i(x_0)+\varepsilon_i,$$
$$\lim_{i\to\infty}\big(\varliminf_{\eta\to\xi_i}\rho_X(x_0,\eta)\big)=0. \quad (3.8)$$

If the space (X,ρ_X) is weakly symmetric, then we also have

$$\rho_X(x_0,\xi_i) \le q_1\frac{q_1^2\alpha^{m_0-1}S\big(q_2\frac{\beta}{\alpha},m_0-1\big)+q_1q_2^{m_0-1}\beta^{m_0-1}}{\alpha^{m_0}-q_2\beta^{m_0}}h_i(x_0)+\varepsilon_i,$$
$$\lim_{i\to\infty}\rho_X(x_0,\xi_i)=0, \quad (3.9)$$

and if this space is q_0-symmetric with $q_0^2\beta<\alpha$, then for every ξ_i we have

$$\overline{\rho}_X(x_0,\xi_i) \le q_0q_2^2\frac{q_2\alpha^{n_0-1}S\big(q_1q_0^2\frac{\beta}{\alpha},n_0-1\big)+q_1^{n_0-1}(q_0^2\beta)^{n_0-1}}{\alpha^{n_0}-q_1(q_0^2\beta)^{n_0}}h_i(x_0)+\varepsilon_i,$$
$$\lim_{i\to\infty}\overline{\rho}_X(x_0,\xi_i)=0, \quad (3.10)$$

$$\varliminf_{\eta\to\xi_i}\overline{\rho}_X(x_0,\eta) \le q_0q_2\frac{q_2\alpha^{n_0-1}S\big(q_1q_0^2\frac{\beta}{\alpha},n_0-1\big)+q_1^{n_0-1}(q_0^2\beta)^{n_0-1}}{\alpha^{n_0}-q_1(q_0^2\beta)^{n_0}}h_i(x_0)+\varepsilon_i,$$
$$\lim_{i\to\infty}\big(\varliminf_{\eta\to\xi_i}\overline{\rho}_X(x_0,\eta)\big)=0. \quad (3.11)$$

For metric spaces, Theorem 3.3 was proved in [4]. Problems close to Theorem 3.3 were discussed in [10]; the article [48] generalized the results of [10] to the case of (q_1,q_2)-quasimetric spaces.

4. f-quasimetric spaces

The following particular cases cases of (q_1,q_2)-quasimetric spaces are sufficiently well studied:

(1) when the (q_1,q_2)-quasimetric is symmetric (or generalized symmetric) and the (q_1,q_2)-triangle inequality coincides with the generalized triangle inequality ($q_1=q_2$);

(2) when the (q_1,q_2)-triangle inequality coincides with the usual triangle inequality.

In the first case, these spaces are actively studied in analysis and geometry (see, for example, [1], [19], [34], [40], [41], [51], [52]); in some works connected with topology and functional analysis, they are called b-spaces [21], [22], [39], [50]. In the second case, we have quasimetric spaces (see, for example, [54]).

Note that (q_1,q_2)-metric spaces are an important particular case of so-called f-quasimetric spaces, actively studied from the beginning of the 20th century to the present time (see, for instance, [9], [15], [24], [25]).

We assume that M is an arbitrary set consisting of at least two points, and $\rho : M\times M\to R^+\cup 0$ is a function satisfying the identity axiom:

$$\rho(x,y)=0 \Leftrightarrow x=0.$$

We call ρ the distance from x to y.

Definition 4.1 A distance function ρ is called *f-quasimetric* if the following f-triangle inequality holds

$$\exists\sigma > 0\ \forall x, y, z \in X : \rho(x, y) < \sigma,\ \rho(y, z) < \sigma \Rightarrow \rho(x, z) \leq f(\rho(x, y), \rho(y, z))$$

for some non-negative function f defined on $\mathbb{R}^+ \times \mathbb{R}^+$ such that $f(x, y) \to_{x^2+y^2\to 0} 0$. The pair (M, ρ) is called an *f-quasimetric space*.

Given $x \in M$ and $r > 0$, define the ball centered at x with radius r:

$$O(x, r) = \{y \in X \mid \rho(x, y) < r\}.$$

Define open sets in M as follows. A set $A \subset M$ is open if for each $x \in A$ there exists $r > 0$ such that $O(x, r) \subset A$. Open sets define a topology τ. Topology τ is T_1 because for arbitrary $x \in M$ set $M \setminus \{x\}$ is obviously open. Also the topology τ is first-countable that all balls $O(x, r)$ are not open

The existence of not open balls $O(x, r)$ was mentioned in [34].

Example. (Proposed by S. Zhukovskiy, [9, Example 1.6]) Let $M = \mathbb{R}$. Consider the symmetric $(2, 2)$-quasimetric

$$\rho(a, b) = \begin{cases} |a - b|, & (a - b) \quad \text{is rational,} \\ 2|a - b|, & (a - b) \quad \text{is irrational.} \end{cases}$$

One can check that every rational number x satisfying the inequality $r/2 < |x| < r$ belongs to the ball $O(0, r)$, but $x \notin intO(0, r)$. So, $O(0, r)$ is not open. Analogous arguments are valid for the balls centered at every point $x \in M$. Note that the topology τ in this example coincides with the standard topology of $\mathbb{R}$.

Call f-quasimetrics ρ_1, ρ_2 defined on one set X *bi-Lipschitz equivalent* if there exists $L > 0$ such that

$$\frac{\rho_1(x, y)}{L} \leq \rho_2(x, y) \leq L\rho_1(x, y).$$

Property 4.2 ([32]) *For every (q_1, q_2)-quasimetric space (X, ρ_X) there exists a (q_1', q_2')-quasimetric ρ_X' on X bi-Lipschitz equivalent to ρ such that any set*

$$\overset{o}{B'}_X(x, r) = \{y \in X \mid \rho_X'(x, y) < r\}$$

is open in the topology $\tau_{\rho'}$.

Definition 4.3 A distance function ρ is called weakly symmetric if

$$\lim_{n\to\infty} \rho(x_0, x_n) = 0 \Rightarrow \lim_{n\to\infty} \rho(x_n, x_0) = 0.$$

A space (X, ρ) is called quasimetrizable (metrizable) if its topology is generated by a quasimetric (metric).

Theorem 4.4 ([9, Theorem 3.1]) *If ρ is a f-quasimetric, then (M, ρ) is quasimetrizable. If ρ is a weakly symmetric f-quasimetric, then (M, ρ) is metrizable.*

Property 4.5 ([29]) *There exist symmetric (q_1, q_2)-quasimetric spaces (X, ρ_X) such that X has no metrics bi-Lipschitz equivalent to ρ_X.*

In the recent paper [55] by Zhukovskiy, Banach's Contraction Mapping Theorem was generalized to complete f-quasimetric spaces. Let us briefly discuss the results of [55].

Definition 4.6 Let (M,ρ) be an f-quasimetric space. Suppose that to any $R \geq r > 0$ there is associated $\beta(r,R) \in [0,1)$. A mapping $G : M \to M$ is called a β-generalized contraction if

$$\forall R \geq r > 0 \ \forall x,y \in M \ r \leq \rho(x,y) \leq R \Rightarrow \rho(G(x),G(y)) \leq \beta(r,R)\rho(x,y).$$

We have the following

Theorem 4.7 ([55]) *Suppose that an f-quasimetric space is complete. If a mapping $G : M \to M$ is a β-generalized contraction ans satisfies the condition*

$$\forall x,y \in M \quad \lim_{i\to\infty} \rho(y, G^i(x)) = 0, \quad G(x) \neq y \Rightarrow y \neq x \tag{4.1}$$

then G has a unique fixed point $\tilde{x} \in M$ and also

$$\lim_{i\to\infty} \rho(\tilde{x}, x_i) = \lim_{i\to\infty} \rho(x_i, \tilde{x}) = 0, \quad x_i = G(x_{i-1})$$

for any initial value $x_0 \in M$.

It is assumed in Theorem 4.7 that condition (4.1) is fulfilled; this condition is not required in the familiar theorems about contraction mappings of metric spaces. Note that the contraction mapping in a complete metric space having no fixed point considered in [47] is not closed and does not satisfy condition (4.1). Of interest are sufficient conditions under which a β-generalized contraction mapping satisfies condition (4.1).

Property 4.8 ([55]) 1^0 *If a β-generalized contraction operator G in an f-quasimetric space (M,ρ) is closed then it satisfies condition (4.1).*

2^0 *Suppose that any Cauchy sequence in a complete f-quasimetric space has finitely many limits. Then a β-generalized contraction operator G satisfies condition (4.1).*

Theorem 4.9 ([55]) *Suppose we are given an increasing right continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(d) < d$ for all $d > 0$. If an f-quasimetric space (M,ρ) is complete and for a mapping $G : M \to M$ satisfying condition (4.1) we have*

$$\forall x,y \in M \quad \rho(G(x),G(y)) \leq \varphi(\rho(x,y))$$

then this mapping has a unique fixed point $\tilde{x} \in M$ and

$$\lim_{i\to\infty} \rho(\tilde{x}, x_i) = \lim_{i\to\infty} \rho(x_i, \tilde{x}) = 0, \quad x_i = G(x_{i-1})$$

for any fixed value $x_0 \in M$.

Theorem 4.9 generalizes the results of [18, Theorem 3.2].

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