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A FIXED POINT DICHOTOMY

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Abstract. We give here a dichotomic fixed point result for a certain class of mappings defined in the closed unit ball of a Hilbert space. This dichotomy states that, for any of the mappings in this class, either it has a fixed point or its Lipschitz constant with respect to any renorming of ℓ_2 has to be strictly greater than 1.
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1. Introduction

If D is a nonempty subset of a Banach space $(X, \|\cdot\|)$, a mapping $T : D \to X$ is said to be Lipschitzian whenever there is a positive constant M such that

$$\| T(x) - T(y) \| \leq M \| x - y \|, \quad x, y \in D.$$

When this happens T is said to be M-Lipschitzian in D. In an analogous manner, if T is a self-mapping in D, i.e., $T(D) \subset D$, it is said to be M-uniformly Lipschitzian provided that, for each $n \geq 1$, the iterate T^n satisfies

$$\| T^n(x) - T^n(y) \| \leq M \cdot \| x - y \|, \text{ for } x, y \in D.$$

The class of the 1-uniformly Lipschitzian self-mappings of D is just the same than the 1-Lipschitzian self-mappings of this set D, and they are usually referred to as *nonexpansive* mappings. Fixed point theory for nonexpansive mappings has been a noteworthy subject of research since 1965. (For a historical development of the early theory see [6]). A major problem in this theory is to give an answer to the following question. (See [7]) "Given a bounded closed convex subset K of the superreflexive space X, if T is a self-mapping on K such that it is nonexpansive, must T have a fixed point?". Even in the particular case of the class of the Banach spaces which are isomorphic to the classical Hilbert space ℓ_2, the above problem remains open. In this

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connection, see also the papers by S. Reich [11] and [12], and the book by K. Goebel and S. Reich [9].

A negative answer to this problem would generate a bounded closed convex subset K of ℓ_2 and a fixed point free mapping $T : K \to K$ such that T is nonexpansive with respect to some equivalent norm, say $\|\cdot\|$, on ℓ_2. With no loss of generality one may suppose that K is a subset of the closed unit ball B_{ℓ_2} of ℓ_2. In this paper we show that no such mapping can exist provided it satisfies some further and quite natural assumptions. In fact, the most well known fixed point free self-mappings of B_{ℓ_2} fulfill these assumptions. Consequently, in some sense, we give a partial positive answer to the above question.

2. Spherical mappings

In this section, X will stand for a Hilbert space, with $\langle\cdot,\cdot\rangle$ being its inner product and $\|\cdot\|$ denoting the corresponding Euclidian norm. The next definition introduces a class of self-mappings of the closed unit ball B_X for which we shall prove the following dichotomy: For each mapping in this class, either it has fixed points or its Lipschitz constant with respect to arbitrary renormings of X is strictly greater than 1. Given a self-mapping $T : B_X \to B_X$ and a subset $A \subset B_X$, we define the sets

$$A^T := \{ x \in A : (T^n(x))_n \text{ converges weakly } \},$$

$$\widetilde{A^T} := \{y \in B_X : \exists x \in A^T \text{ s.t. } (T^n(x))_n \text{ converges weakly to } y\}.$$

Notice that $\widetilde{A^T} \neq \emptyset$ implies that $A^T \neq \emptyset$. Given $\lambda \in [0,1]$, we put

$$S_\lambda := \{ x \in X : \|x\| = \lambda \}.$$

For $\lambda \in [0,1]$, the mapping $T : B_X \to B_X$ is said to be λ-*spherical* provided the following conditions are accomplished

$$(A) \qquad T(S_\lambda) \subset S_\lambda.$$

$$(B) \qquad \langle T(0), T(\cdot)\rangle \text{ is constant in } S_\lambda.$$

$$(C) \qquad \widetilde{S_\lambda^T} \cap (\{0\} \cup S_\lambda) \neq \emptyset.$$

We shall refer to the class of mappings which are λ-spherical, for some $\lambda \in [0,1]$, as *spherical* mappings. Before giving the annunciated dichotomy, we need one more definition plus an auxiliary result. If $T : B_X \to X$ is any mapping, we put

$$L(T, B_X) := \begin{cases} \inf\{M > 0 : T \text{ is } M-\text{Lips. r.t. a renorming of X}\}, & \text{for } T \text{ Lipschitz}, \\ \infty, & \text{otherwise.} \end{cases}$$

Proposition 2.1. *For a mapping $T : B_X \to X$,*

$$L(T, B_X) \geq \sup_{0<\lambda\leq 1} \frac{d(T(0), T(S_\lambda))}{\lambda},$$

where $d(\cdot,\cdot)$ stands for the distance induced by the Euclidian norm.

Proof. Arguing by contradiction, assume that

$$L(T, B_X) < \sup_{0<\lambda\leq 1} \frac{d(T(0), T(S_\lambda))}{\lambda}.$$

Then, there is a renorming $|||\cdot|||$ of X and a constant $M > 0$ such that T is M-Lipschitzian in B_X with respect to $|||\cdot|||$ and

$$\sup_{0<\lambda\leq 1} \frac{d(T(0), T(S_\lambda))}{\lambda} \; > \; M.$$

Thus, there is $0 < \lambda_0 \leq 1$ such that

$$\frac{d(T(0), T(S_{\lambda_0}))}{\lambda_0} \; > \; M.$$

We inductively construct the following vector sequence: Let $y_0 \in X$ be such that $\| y_0 \| = \lambda_0$, and for $n \geq 1$,

$$y_n \;\; := \;\; \lambda_0 \, \frac{T(y_{n-1}) - T(0)}{\| \, T(y_{n-1}) - T(0) \, \|}.$$

For each $n \geq 1$, the vector y_n is well defined, for, if $T(y_{n-1}) = T(0)$, then

$$\frac{d(T(0), T(S_{\lambda_0}))}{\lambda_0} \leq \frac{\| \, T(y_{n-1}) - T(0) \, \|}{\lambda_0} = 0,$$

which is a contradiction. Clearly, $(y_n)_{n=1}^{\infty}$ is contained in S_{λ_0} and, writing for simplicity

$$\mu := \frac{d(T(0), T(S_{\lambda_0}))}{M\lambda_0} > 1,$$

we have that, for each $n \geq 2$,

$$\begin{aligned}
|||\, y_n \,||| &= \frac{\lambda_0}{\| T(y_{n-1}) - T(0) \|} \, |||\, T(y_{n-1}) - T(0) \,||| \leq \; \frac{M\lambda_0}{d(T(0), T(S_{\lambda_0}))} \, |||\, y_{n-1} \,||| \\
&= \frac{1}{\mu} \, |||\, y_{n-1} \,||| = \; \frac{1}{\mu} \, |||\, \lambda_0 \frac{T(y_{n-2}) - T(0)}{\| \, T(y_{n-2}) - T(0) \, \|} \, ||| \\
&\leq \; \frac{1}{\mu} \cdot \frac{M\lambda_0}{d(T(0), T(S_{\lambda_0}))} \, |||\, T(y_{n-2}) - T(0) \,||| \\
&\leq \; \left(\frac{1}{\mu}\right)^2 |||\, y_{n-2} \,||| \leq \left(\frac{1}{\mu}\right)^3 |||\, y_{n-3} \,||| \leq \; \ldots \; \leq \; \left(\frac{1}{\mu}\right)^n |||\, y_0 \,|||.
\end{aligned}$$

Now, since $\frac{1}{\mu} < 1$, taking limits as $n \to \infty$, we obtain that $\lim_n |||\, y_n \,||| = 0$. This contradicts the fact that $|||\cdot|||$ is an equivalent renorming of X. □

Theorem 2.2. *Let T be a spherical mapping in the Hilbert space X. Then:*

Either T has a fixed point, or $L(T, B_X) > 1$.

Proof. According to the definition of a spherical mapping, there is $\lambda \in [0, 1]$ such that T is λ-spherical, i.e., T satisfies conditions $(A), (B), (C)$. We may assume that the mapping T is Lipschitzian on B_X, otherwise the dichotomy clearly follows. From condition (A), $T(S_\lambda) \subset S_\lambda$. We may also assume that $\lambda > 0$, else we would have that $\{T(0)\} = T(S_0) \subset S_0 = \{0\}$, i.e., $T(0) = 0$ and T would have a fixed point at zero. From condition (B), there is $k \in \mathbb{R}$ such that

$$\langle T(0), T(x) \rangle = k, \quad x \in S_\lambda.$$

Thus, making use of the last proposition, we have

$$\begin{aligned} L(T,B_X) \geq \frac{d(T(0),T(S_\lambda))}{\lambda} &= \frac{\inf_{\|x\|=\lambda} \| T(0)-T(x) \|}{\lambda} \\ &= \frac{\inf_{x\in S_\lambda} \sqrt{\| T(0)\|^2 + \| T(x)\|^2 - 2\langle T(0),T(x)\rangle)}}{\lambda} \\ &= \frac{\sqrt{\| T(0)\|^2 + \lambda^2 - 2k}}{\lambda}. \end{aligned} \tag{2.1}$$

Now, from condition (C), there is $u \in S_\lambda$ such that the sequence $(T^n(u))_n$ converges weakly to a point $v \in S_\lambda \cup \{0\}$. Therefore, we consider two possibilities.
One. $v \in S_\lambda$. Then, since the sequence of norms $(\| T^n(u)\|)_n$ is constantly equal to $\lambda = \| v\|$, and X has the Kadec-Klee property, it follows that $(T^n(u))_n$ converges to v in X. Clearly then, $T(v) = v$.
Two. $v = 0$. Then, since $T^n(u) \in S_\lambda$, $n \geq 0$,

$$k = \lim_n \langle T(0), T(T^{n-1}(u))\rangle = \lim_n \langle T(0), T^n(u)\rangle = \langle T(0), v\rangle = \langle T(0), 0\rangle = 0.$$

If $T(0) = 0$ we are done, so we assume that $T(0) \neq 0$. Thus, from (2.1), we have that

$$L(T,B_X) \geq \frac{\sqrt{\| T(0)\|^2 + \lambda^2}}{\lambda} > 1. \qquad \square$$

Corollary 2.3. *Let T be a spherical mapping. If $L(T,B_X) \leq 1$, then T has a fixed point. In particular, if T is nonexpansive with respect to some renorming of X, then it has fixed points.*

If C is a closed bounded subset of the Hilbert space X, it is shown in [8] that every uniformly Lipschitzian mapping $T : C \to C$ such that its uniform Lipschitz constant, with respect to the Euclidian norm, is less than $\sqrt{2}$ has fixed points. In this connection, see also Section 8 on pages 34-38 of [9]. We give next a parallel result for a particular class of self-mappings of the unit ball of X. Notice that these mappings do not need to be (uniformly) Lipschitzian. (For an example, see the mapping T_N below).

Corollary 2.4. *Let T be a λ-spherical mapping such that $T(0) \in S_\lambda$.*
If $L(T,B_X) < \sqrt{2}$, then T has a fixed point.

Proof. As seen in the proof of Theorem 2.2, we may assume that $\lambda > 0$. In view of condition (C), there is $v \in \widetilde{S_\lambda^T} \cap (S_\lambda \cup \{0\})$. Thus, for some $u \in S_\lambda$, the sequence of iterates $(T^n(u))_n$ converges weakly to v. In view of condition (B), there is $k \in \mathbb{R}$ such that $\langle T(0), T(x)\rangle = k$, $x \in S_\lambda$. Hence, since $T(S_\lambda) \subset S_\lambda$ by condition (A), it follows that

$$k = \lim_n \langle T(0), T^n(u)\rangle = \langle T(0), v\rangle. \tag{2.2}$$

Using Proposition 2.1, the proof of Theorem 2.2 and the assumption that $T(0) \in S_\lambda$, we can write

$$L(T,B_X) \geq \frac{d(T(0),T(S_\lambda))}{\lambda} = \frac{\sqrt{\| T(0)\|^2 + \lambda^2 - 2k}}{\lambda} = \sqrt{2 - \frac{2k}{\lambda^2}}$$

But recalling that $v \in \{0\} \cup S_\lambda$, it follows that $v \neq 0$, otherwise, from (2.2), we would have that $k = 0$ and, from our hypothesis, we obtain the following contradiction

$$\sqrt{2} \;>\; L(T, B_X) \;\geq\; \sqrt{2 - \frac{2k}{\lambda^2}} \;=\; \sqrt{2}.$$

Thus, $v \in S_\lambda$. Given that the sequence of iterates $(T^n(u))_n$ is contained in S_λ, it follows that it converges to v in X, yielding that $T(v) = v$.

□

3. Classical-type mappings

We consider in this section a well-known class of self-mappings of the closed unit ball of the Hilbert space ℓ_2 and we show that these mappings are spherical mappings, thus justifying their introduction. Some self-mappings of B_{ℓ_2}, which have now become classical, such that they have no fixed points, are those of Kakutani, which we denote by T_K, Nirenberg's mapping, represented as T_N, Lifschitz-Baillon's, which we denote as T_{LB} and Goebel-Kirk-Thelle's, labeled as T_{GKT}. For the sake of completeness, we list in the following lines the definitions of these mappings, with domain always all of B_{ℓ_2}.

Kakutani's mapping, see [10], is (in its generalized form), given $0 < \varepsilon \leq 1$,

$$T_K(x) \;=\; \varepsilon\,(1 - \|x\|) \cdot e_1 \;+\; Rx,$$

where R denotes the right-shift operator in ℓ_2. Nirenberg's mapping, see [13], is defined as

$$T_N(x) \;=\; \sqrt{1 - \|x\|^2} \cdot e_1 \;+\; Rx.$$

Lifschitz-Baillon's mapping, see [1], is

$$T_{LB}(x) = \begin{cases} \cos(\|x\|\,\frac{\pi}{2}) \cdot e_1 + \frac{\sin(\|x\|\,\frac{\pi}{2})}{\|x\|} \cdot Rx, & x \in B_{\ell_2} \setminus \{0\}, \\ e_1, & x = 0, \end{cases}$$

Goebel-Kirk-Thelle's mapping, see [8], is

$$T_{GKT}(x) = \frac{(1 - \|x\|) \cdot e_1 + Rx}{\sqrt{(1 - \|x\|)^2 + \|x\|^2}}.$$

In [2], [3], [4] and [5], we introduced and studied a certain type of self-mappings of B_{ℓ_2} which generalize the above particular examples. Next we define a new class of mappings, which we shall refer to as *classical-type* mappings, that contains the above classical examples. Let $\varphi, \psi : [0,1] \to \mathbb{R}$ be two continuous real functions such that they satisfy that

$$\varphi(t)^2 \;+\; \psi(t)^2\, t^2 \;\leq\; 1, \qquad 0 \leq t \leq 1.$$

In B_{ℓ_2}, the classical-type mapping associated to φ, ψ, is defined as

$$T_{\varphi,\psi}(x) \;:=\; \varphi(\|x\|) \cdot e_1 \;+\; \psi(\|x\|) \cdot Rx,$$

where e_1 stands for the first unit vector of ℓ_2, $\|\cdot\|$ is the Euclidian norm and R is the right-shift operator of ℓ_2. In view of the defining condition, it is plain that $T_{\varphi.\psi}(B_{\ell_2}) \subset B_{\ell_2}$, i.e, $T_{\varphi,\psi}$ is a self-mapping of B_{ℓ_2}. Of course, $T_{\varphi,\psi}$ is a continuous mapping on B_{ℓ_2}.

Proposition 3.1. *Classical-type mappings are spherical.*
Proof. Let $T_{\varphi,\psi}$ be a classical-type mapping. Consider the real function

$$f(\lambda) \ := \ \| T_{\varphi,\psi}(\lambda e_1) \| = \sqrt{\varphi(\lambda)^2 + \psi(\lambda)^2 \lambda^2}, \qquad \lambda \in [0,1].$$

Clearly, f is a continuous function such that $f([0,1]) \subset [0,1]$ and so there is $\lambda_0 \in [0,1]$ such that $f(\lambda_0) = \lambda_0$. We show next that $T_{\varphi,\psi}$ is λ_0-spherical. To do so, if $x \in S_{\lambda_0}$,

$$\begin{aligned} \| \ T_{\varphi,\psi}(x) \ \| \ &= \| \ \varphi(\| x \|) e_1 \ + \ \psi(\| x \|) R x \ \| \\ &= \| \ \varphi(\lambda_0) e_1 + \psi(\lambda_0) R x \ \| \\ &= \ \sqrt{\varphi(\lambda_0)^2 + \psi(\lambda_0)^2 \lambda_0^2} \\ &= \ f(\lambda_0) = \ \lambda_0, \end{aligned}$$

i.e., condition (A) is satisfied.
To check that condition (B) is also satisfied, note that, if $x \in S_{\lambda_0}$,

$$\langle T_{\varphi,\psi}(0), T_{\varphi,\psi}(x) \rangle \ = \ \langle \varphi(0) e_1, \ \varphi(\| x \|) e_1 + \psi(\| x \|) R x \rangle \ = \ \varphi(0)\varphi(\lambda_0).$$

Finally, we see that condition (C) is accomplished, that is, we have to prove that $\widetilde{S_{\lambda_0}^{T_{\varphi,\psi}}} \cap (\{0\} \cup S_{\lambda_0}) \neq \emptyset$. In other words, we have to show that there is a point $u \in S_{\lambda_0}$ whose sequence of iterates $(T^n_{\varphi,\psi}(u))_n$ weakly converges either to zero or to an element of S_{λ_0}. If $x \in S_{\lambda_0}$, it can be seen with not much difficulty that, for $n \geq 2$,

$$T^n_{\varphi,\psi}(x) \ \ = \ \varphi(\lambda_0) \sum_{j=0}^{n-1} \psi(\lambda_0)^j \cdot e_{j+1} \ + \ \psi(\lambda_0)^n \cdot R^n x. \tag{3.1}$$

We may assume that $\lambda_0 \in]0,1]$, otherwise, if $\lambda_0 = 0$, we would have that $S_{\lambda_0} = \{0\}$, $\varphi(\lambda_0) = \varphi(0) = f(0) = 0$ and $T^n_{\varphi,\psi}(0) = 0$, $n \geq 1$, and so $(T^n_{\varphi,\psi}(0))_n$ clearly converges to zero. Now, from the equality $\varphi(\lambda_0)^2 + \psi(\lambda_0)^2 \lambda_0^2 = \lambda_0^2$, since $\lambda_0 > 0$, it follows that $|\psi(\lambda_0)| \leq 1$. Thus, we may consider two possibilities.
One. $|\psi(\lambda_0)| = 1$. Then, $\varphi(\lambda_0) = 0$ and so, after (3.1) we have that, for any $x \in S_{\lambda_0}$, $T^n_{\varphi,\psi}(x) = \psi(\lambda_0)^n \cdot R^n x$, $n \geq 2$, and the sequence $(T^n_{\varphi,\psi}(x))_n$ converges weakly to zero.
Two. $|\psi(\lambda_0)| < 1$. We first see that the vector $w \ := \ \varphi(\lambda_0)\,(1, \ \psi(\lambda_0), \ \psi(\lambda_0)^2, \ ... \,)$ lies in S_{λ_0}. For this, given that w corresponds to a geometric progression whose ratio has absolute value less than one,

$$\| w \|^2 = \ \frac{\varphi(\lambda_0)^2}{1 - \psi(\lambda_0)^2} \ = \ \lambda_0^2, \ \text{ i.e., } \ w \in S_{\lambda_0}.$$

Now, again from (3), if $m \geq 1$, we have that, for any $x \in S_{\lambda_0}$,

$$\lim_n \langle e_m, T^n_{\varphi,\psi}(x) \rangle \ \ = \ \ \varphi(\lambda_0)\psi(\lambda_0)^{m-1} \ = \ \langle e_m, w \rangle.$$

Hence, the bounded sequence $(T^n_{\varphi,\psi}(x))_n$ converges weakly to $w \in S_{\lambda_0}$. We have thus shown that, for any $x \in S_{\lambda_0}$, the sequence $(T^n_{\varphi,\psi}(x))_n$ weakly converges, either to zero, or to a point of S_{λ_0}. □

Corollary 3.2. *If a classical-type mapping $T_{\varphi,\psi}$ is such that $L(T_{\varphi,\psi}, B_{\ell_2}) \leq 1$, then it has fixed points.*

As a consequence of Corollary 2.4, we have the following result.

Corollary 3.3. *Let $T_{\varphi,\psi}$ be a classical-type mapping such that*

$$\varphi(t)^2 + t^2\psi(t)^2 \ = \ 1, \quad t \in [0,1].$$

Then, if $L(T_{\varphi,\psi}, B_{\ell_2}) < \sqrt{2}$, the mapping $T_{\varphi,\psi}$ has a fixed point.

Remark. From our previous study, we can now completely describe the classical-type mappings. Let $T_{\varphi,\psi}$ be a classical-type mapping. Let $Fix(T_{\varphi,\psi})$ stand for the set of fixed points of $T_{\varphi,\psi}$. Then, $T_{\varphi,\psi}$ is under one of the following situations:

1) $0 \in Fix(T_{\varphi,\psi})$.

2) Considering the function $f(\lambda) = \sqrt{\varphi(\lambda)^2 + \lambda^2\psi(\lambda)^2}$, $\lambda \in [0,1]$, let

$$F(f) \ := \ \{\lambda \in [0,1] : \ f(\lambda) = \lambda\}.$$

For each $\lambda \in F(f)$, putting $w(\lambda) := \varphi(\lambda)(1, \psi(\lambda), \psi(\lambda)^2, ...)$, then

$$Fix(T_{\varphi,\psi}) \ = \ \{ \ w(\lambda) \ : \ \lambda \in F(f), \ \ |\psi(\lambda)| < \ 1 \ \} \ \neq \ \emptyset.$$

3) $Fix(T_{\varphi,\psi}) = \emptyset$ and $L(T_{\varphi,\psi}, B_{\ell_2}) > 1$.

Corollary 3.4. *Let $T_{\varphi,\psi}$ be a classical-type mapping. If $L(T_{\varphi,\psi}, B_{\ell_2}) \leq 1$ (in particular, if $T_{\varphi,\psi}$ is nonexpansive with respect to a renorming of ℓ_2), then*

$$\emptyset \ \neq \ Fix(T_{\varphi,\psi}) \ \subset \ \{0\} \cup \{ \ w(\lambda) \ : \ \lambda \in F(f), \ \ |\psi(\lambda)| < \ 1 \ \}.$$

References

[1] J.B. Baillon, *Quelques aspects de la théorie des points fixed dans les espaces de Banach, I*, Séminaire d'Analyse Fonctionelle de l'École Polytechnique, **7**(1978-79), 1-13.

[2] J. Ferrer, E. Llorens-Fuster, *On a class of maps which cannot be made nonexpansive after renormings*, J. Nonlinear Convex Anal., **18**(2)(2017), 197-214.

[3] J. Ferrer, E. Llorens-Fuster, *Lower bounds for the Lipschitz constants of some classical fixed point free maps*, J. Math. Anal. Appl., **465**(2018), no. 1, 297-308.

[4] J. Ferrer, E. Llorens-Fuster, *On never nonexpansive mappings in reflexive Banch spaces*, Banach J. Math. Anal., **14**(2020), no. 1, 76-97.

[5] J. Ferrer, E. Llorens-Fuster, *A short proof that some mappings of the unit ball of ℓ_2 are never nonexpansive*, Amer. Math. Monthly, **127**(2020), no. 4, 354-358.

[6] K. Goebel, W.A. Kirk, *Classical theory of nonexpansive mappings*, Handbook of Metric Fixed Point Theory, 49-91, Kluwer Acad. Publ., Dordrecht, 2001.

[7] K. Goebel, W.A. Kirk, *Some problems in metric fixed point theory*, J. Fixed Point Theory Appl., **4**(2008), no. 1, 13-25.

[8] K. Goebel, W.A. Kirk, R.I. Thele, *Uniformly Lipschitzian semigroups in Hilbert spaces*, Canad. J. Math., **26**(1974), 1245-1256.

[9] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[10] S. Kakutani, *Topological properties of the unit sphere of a Hilbert space*, Proc. Imp. Acad. Tokyo, **19**(1943), 269-271.

[11] S. Reich, *The fixed point property for nonexpansive mappings, I*, Amer. Math. Monthly, **83**(1976), 266-268.

[12] S. Reich, *The fixed point property for nonexpansive mappings, II*, Amer. Math. Monthly, **87**(1980), 292-294.

[13] D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, 1974.

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