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KRASNOSELSKII TYPE THEOREMS IN PRODUCT BANACH SPACES AND APPLICATIONS TO SYSTEMS OF NONLINEAR TRANSPORT EQUATIONS AND MIXED FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we use a new technique for the treatment of systems based on the advantage of vector-valued norms and of the weak topology. We first present vector versions of the Leray-Schauder alternative and then some Krasnoselskii type fixed point theorems for a sum of two mappings. Applications are given to a system of nonlinear transport equations, and systems of mixed fractional differential equations.
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1. Introduction

The classical Banach contraction principle is a very useful tool in nonlinear analysis with many applications to integral and differential equations, optimization theory, and other topics. There are many generalizations of this result, one of them is due to A.I. Perov [16] and consists in replacing usual metric spaces by spaces endowed with vector-valued metrics. According to this result, if a space X is a Cartesian product $X = X_1 \times \cdots X_n$ and each component X_i is a complete metric space with the metric d_i, then instead of endowing X with some metric δ generated by $d_1, \cdots, d_n$, for instance any one of the metrics

$$\begin{aligned} \delta^p(x,y) &= \left(\sum_{i=1}^{n} d_i(x_i,y_i)^p \right)^{\frac{1}{p}}, \quad (1 \le p < \infty), \\ \delta^\infty(x,y) &= \max\{d_1(x_1,y_1), \cdots, d_n(x_n,y_n)\}, \end{aligned}$$

and applying Banach's contraction principle in the complete metric space (X, δ), better results are obtained if one considers the vector-valued metric

$$d(x,y) = (d_1(x_1,y_1), \cdots, d_n(x_n,y_n))^T$$

and one requires a *generalized contraction* (in Perov's sense) condition in the vector-matrix form

$$d(F(x), F(y)) \le Ad(x,y), \quad x, y \in X,$$

where A is a square matrix of type $n \times n$ with nonnegative elements having the spectral radius $\rho(A) < 1$. This approach is very fruitful for the treatment of systems of equations arising from various fields of applied mathematics. The advantage of using vector-valued metrics and norms instead of usual scalar ones, in connexion with several techniques of nonlinear analysis, has been pointed out in [20]. Roughly speaking, by a vector approach it is allowed that the component equations of a system behave differently, and thus more general results can be obtained.

In his Ph.D. thesis [22], A. Viorel used generalized contractions in Perov's sense and gave a vector version of Krasnoselskii's fixed point theorem [13] for a some of two operators A and B, where A is a compact map and B is a generalized contraction. Applications were given to systems of semi-linear evolution equations. Viorel's result was extended for multi-valued mappings in [17]. The proofs of these results combine a vector version of the contraction principle (Perov and Perov-Nadler theorems, respectively) with Schauder's fixed point theorem for maps that are compact with respect to the strong topology.

Alternatively, instead of the strong topology of a Banach space, one may think to use the weak topology. Fixed point results involving the weak topology have been obtained by many authors in the last decades (see, e.g., [2, 5, 4, 6, 8]). The purpose of this paper is to extend the Leray-Schauder and Krasnoselskii's fixed point theorems to sums of generalized contractions and compact maps with respect to the weak topology. Note that our technique can also be used to give vector versions of the results in [3]. Next, motivated by the papers [6], [15] and [11], we give applications of the theoretical results to a system of transport equations, and a system of mixed fractional differential equations.

The paper is organized as follows: In Section 2, we present some notations and preliminary facts that we will need in what follows. In Section 3, we first give a vector version of the Leray-Schauder fixed point theorem for weakly sequentially continuous mappings and then we extend Viorel's result by using the weak topology. In Sections 3 and 4, we apply these results to a system of transport equations and a system of mixed fractional differential equations.

2. Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper.

By a *generalized metric space* we mean a set X endowed with a *vector-valued metric* d, that is a mapping $d : X \times X \to \mathbb{R}^n_+$ which satisfies all the axioms of a usual metric, with the inequality $\leq$ understood to act componentwise. In such a space, the notions of a Cauchy sequence, convergent sequence, completeness, open and closed set, are defined in a similar way to that of the corresponding notions in a usual metric space.

A mapping $F : X \longrightarrow X$, where X is a generalized metric space with the vector-valued metric d is said to be a *generalized contraction*, or a *Perov contraction*, if there exists a matrix (called *Lipschitz matrix*) $M \in \mathcal{M}_n(\mathbb{R}_+)$ such that M^k tends to the zero matrix as $k \to \infty$ and

$$d(F(x), F(y)) \leq Md(x, y) \quad \text{for all} \quad x, y \in X.$$

Here the vector $d(x, y)$ and $d(F(x), F(y))$ are seen like all the vectors in $\mathbb{R}^n$ as column matrices. Notice that a matrix M as above is called to be *convergent to zero*, and that this property is equivalent (see [19]) to each one of the following three properties:

(a) $I - M$ is non-singular and $(I - M)^{-1} = I + M + M^2 + \cdots$.
(Here I is the unit matrix of size n).
(b) $|\lambda| < 1$ for every $\lambda \in C$ with $det(M - \lambda I) = 0$.
(c) $I - M$ is non-singular and $(I - M)^{-1}$ has nonnegative elements.

Notice that in view of (c), a vector-matrix inequality like $x \leq Mx$ for a nonnegative vector-column

$$x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n_+$$

first yields $(I - M)x \leq 0$, and then $x \leq (I - M)^{-1}0$, whence $x = 0_{\mathbb{R}^n}$.

Recall Perov's fixed point theorem which states that any generalized contraction F on a complete generalized metric space (X, d) has a unique fixed point x^*, and for each $x \in X$ one has

$$d(F^k(x), x^*) \leq M^k(I - M)^{-1}d(x, F(x)) \quad \text{for all} \quad k \in \mathbb{N}.$$

Notice that, under the assumptions of Perov's theorem, and if J is the identity mapping of X, the mapping $J - F$ is bijective and $(J - F)^{-1}$ is continuous.

By a *vector-valued norm* on a linear space X we mean a mapping $\|\cdot\| : X \to \mathbb{R}^n_+$ which satisfies the usual axioms of a norm, with the inequality $\leq$ understood to act componentwise. Any linear space X endowed with a vector-valued norm $\|\cdot\|$ is a generalized metric space with respect to the vector-valued metric $d(x, y) = \|x - y\|$. In case that (X, d) is complete, we say that X is a *generalized Banach space*.

In particular, if $X = X_1 \times \cdots \times X_n$, where $(X_i, \|.\|_i)$ is a Banach space for $i = 1, \cdots, n$, then X is a Banach space with respect to the norm

$$|x| = \|x_1\|_1 + \cdots + \|x_n\|_n,$$

and a generalized Banach space with respect to the vector-valued norm

$$\|x\| = (\|x_1\|_1, \cdots, \|x_n\|_n)^T,$$

where $x = (x_1, \cdots, x_n)$. On such a space one can define a *vector measure of weak noncompactness* by

$$\omega(V) = (\omega_1(V_1), \cdots, \omega_n(V_n))^T$$

for $V = V_1 \times \cdots \times V_n$ and any bounded sets $V_i \subset X_i$, $i = 1, \cdots, n$, where ω_i is the De Blasi measure of weak noncompactness on X_i (see [8]). Recall that, if $(Y, \|.\|_Y)$ is any Banach space, the *De Blasi weak measure of noncompactness* $\omega_Y(C)$ of any bounded set $C \subset Y$ is given by

$$\omega_Y(C) = \inf\left\{r > 0 : \text{there is a weakly compact set } K \subset Y \text{ such that } C \subset K + \overline{B}_Y(0, r)\right\},$$

where $\overline{B}_Y(0, r) = \{y \in Y : \|y\|_Y \leq r\}$. For completeness we recall some properties of ω_Y needed below (for the proofs we refer to [1]). Let $C_1, C_2 \subset Y$ be bounded. Then

(i) Monotonicity: If $C_1 \subset C_2$, then $\omega_Y(C_1) \leq \omega_Y(C_2)$.
(ii) Regularity: $\omega_Y(C_1) = 0$ if and only if C_1 is relatively weakly compact.
(iii) Invariance under closure: $\omega_Y(\overline{C_1^{\omega}}) = \omega_Y(C_1)$, where $\overline{C_1^{\omega}}$ is the weak closure of C_1.
(iv) Semi-homogeneity: $\omega_Y(\lambda C_1) = |\lambda|\omega_Y(C_1)$ for all $\lambda \in \mathbb{R}$.
(v) Invariance under passage to the convex hull: $\omega_Y(conv(C_1)) = \omega_Y(C_1)$.
(vi) Semi-additivity: $\omega_Y(C_1 + C_2) \leq \omega_Y(C_1) + \omega_Y(C_2)$.
(vii) Cantor's intersection property: If $(C_k)_{k\geqslant 1}$ is a decreasing sequence of nonempty, bounded and weakly closed subsets of Y with $\lim\limits_{k\to+\infty} \omega_Y(C_k) = 0$, then $\bigcap\limits_{k=1}^{\infty} C_k \neq \emptyset$ and $\omega_Y\left(\bigcap\limits_{k=1}^{\infty} C_k\right) = 0$, i.e. $\bigcap\limits_{k=1}^{\infty} C_k$ is relatively weakly compact.

Throughout this paper, for a mapping $F : D \to X$, where X is the Cartesian product $X_1 \times \cdots \times X_n$ of n Banach spaces and $D = D_1 \times \cdots \times D_n$, for $D_i \subset X_i$ a weakly closed subset of X_i $(i = 1, \cdots, n)$, we shall say that F is sequentially weakly continuous if for any sequence $\left(x^k\right) \subset D$ such that $x_i^k \to x_i$ weakly in X_i, $i = 1, \cdots, n$, one has $F_i\left(x^k\right) \to F_i(x)$ weakly in X_i for $i = 1, \cdots, n$.

3. Fixed point results

We first state a useful result in terms of the vector measure of weak noncompactness.

Proposition 3.1. *Let $(X_i, \|\cdot\|_i)$, $i = 1, \cdots, n$ be Banach spaces, and let*

$$X = X_1 \times \cdots \times X_n.$$

If $F : X \to X$ is weakly sequentially continuous and there is a matrix $M \in \mathcal{M}_n(\mathbb{R}_+)$ such that

$$\|F(x) - F(y)\| \leq M\|x - y\| \quad \text{for all } x, y \in X, \tag{3.1}$$

then for any bounded sets $V_i \subset X_i$, $i = 1, \cdots, n$ and $V = V_1 \times \cdots \times V_n$, one has

$$\omega(F(V)) \leq M\omega(V). \tag{3.2}$$

Proof. For each $i \in \{1, \ldots, n\}$, denote $\alpha_i = \omega_i(V_i)$. Then for any $\varepsilon_i > 0$, there exists a weakly compact subset K_i of X_i such that $V_i \subset K_i + \overline{B}_{X_i}(0, \alpha_i + \varepsilon_i)$. Hence, for every $x = (x_1, \cdots, x_n) \in V$, there is an $y = (y_1, \cdots, y_n) \in K = K_1 \times \cdots \times K_n$ such that $\|x_i - y_i\|_i \leqslant \alpha_i + \varepsilon_i$ for $i = 1, \ldots, n$. Let $F = (F_1, \cdot \cdot \cdot, F_n)$, where $F_i : X \to X_i$ and let $M = (m_{ij})_{1 \le i,j \le n}$. Then using (3.1) gives

$$\|F_i(x) - F_i(y)\|_i \le \sum_{j=1}^{n} m_{ij} \|x_j - y_j\|_j \le \sum_{j=1}^{n} m_{ij}(\alpha_j + \varepsilon_j). \tag{3.3}$$

As a result, $F_i(x) - F_i(y) \in \overline{B}_{X_i}(0, \sum_{j=1}^{n} m_{ij}(\alpha_j + \varepsilon_j))$ for $i = 1, \ldots, n$. Hence,

$$F_i(x) \in F_i(K) + \overline{B}_{X_i}\left(0, \sum_{j=1}^{n} m_{ij}(\alpha_j + \varepsilon_j)\right), \quad i = 1, \ldots, n.$$

Consequently,

$$F_i(V) \subset F_i(K) + \overline{B}_{X_i}\left(0, \sum_{j=1}^{n} m_{ij}(\alpha_j + \varepsilon_j)\right), \quad i = 1, \ldots, n. \tag{3.4}$$

Since F_i is weakly sequentially continuous and K is weakly compact, we have $F_i : K \longrightarrow X_i$ is weakly continuous. Thus, $F_i(K)$ is weakly compact. As a result

$$\omega_i\left(F_i(V)\right) \le \sum_{j=1}^{n} m_{ij}(\alpha_j + \varepsilon_j), \quad i = 1, \ldots, n. \tag{3.5}$$

Letting $\varepsilon_i \to 0$ for all i yields

$$\omega_i\left(F_i(V)\right) \le \sum_{j=1}^{n} m_{ij}\alpha_j = \sum_{j=1}^{n} m_{ij}\omega_j(V_j), \quad i = 1, \cdot \cdot \cdot, n, \tag{3.6}$$

or equivalently, in the vector form, (3.2). □

We now give some vector versions of the Leray-Schauder fixed point theorem for weakly sequentially continuous mappings.

Theorem 3.1. *Let $(X_i, \|\cdot\|_i)$, $i = 1, \cdot \cdot \cdot, n$ be Banach spaces. For each $i \in \{1, \cdot \cdot \cdot, n\}$, consider a nonempty closed and convex set $\Omega_i \subset X_i$ and a weakly open subset U_i of Ω_i with $0 \in U_i$ such that $\overline{U_i^{\omega}}$ is a weakly compact subset of Ω_i. Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$, $D = \overline{U_1^{\omega}} \times \cdots \times \overline{U_n^{\omega}}$, and $F : D \to \Omega$ a weakly sequentially continuous mapping . Then, either*

- **(i):** *F has a fixed point, or*
- **(ii):** *there exist $i \in \{1, \cdot \cdot \cdot, n\}$, a point $x = (x_1, \cdot \cdot \cdot, x_n) \in D$ with $x_i \in \partial_{\Omega_i} U_i = \overline{U_i^{\omega}} \setminus U_i$, and a number $\lambda \in (0, 1)$ with $x = \lambda F(x)$.*

Proof. Suppose (ii) does not hold. Let Σ be the set defined by

$$\Sigma = \{x \in D : \ x = \lambda F(x) \text{ for some } \lambda \in [0, 1]\}.$$

The set Σ is non-empty because $0 \in D$. We will show that Σ is weakly compact. The weak sequentially continuity of F implies that Σ is weakly sequentially closed. For that, let $(x_n)_n$ be a sequence of Σ such that $x_n \to x$ weakly, $x \in D$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \lambda_n F(x_n)$. Since $\lambda_n \in [0,1]$, we can extract a subsequence $(\lambda_{n_j})_j$ such that $\lambda_{n_j} \to \lambda \in [0,1]$. So, $\lambda_{n_j} F(x_{n_j}) \to \lambda F(x)$ weakly. Hence $x = \lambda F(x)$ and $x \in \Sigma$. Let $x \in D$. Since $\overline{\Sigma^\omega}$ is weakly compact by the Eberlein-Smulian theorem ([10], Theorem 8.12.4, p. 549), there exists a sequence $(x_n)_n \subset \Sigma$ such that $x_n \to x$ weakly, so $x \in \Sigma$. Hence $\overline{\Sigma^\omega} = \Sigma$ and Σ is a weakly closed subset of the weakly compact set D. Therefore, Σ is weakly compact. Because X endowed with its weak topology is a Hausdorff locally convex space, we have that X is completely regular ([21], p. 16). Since $\Sigma \cap (\Omega \setminus U_1 \times \cdots \times U_n) = \emptyset$, then by ([12], p. 146), there is a weakly continuous function $\varphi : \Omega \to [0,1]$, such that $\varphi(x) = 1$ for $x \in \Sigma$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U_1 \times \cdots \times U_n$. Let $F^* : \Omega \to \Omega$ be the mapping defined by

$$F^*(x) = \varphi(x)F(x),$$

Because $\partial_{\Omega_i} U_i = \partial_{\Omega_i} \overline{U_i^\omega}$, φ is weakly continuous and F is weakly sequentially continuous, we have that F^* is weakly sequentially continuous. In addition

$$F_i^*(\Omega) \subset \overline{conv}(F_i(D) \cup \{0\}).$$

Let $D_i^* = \overline{conv}(F_i(D) \cup \{0\})$ and $D^* = D_1^* \times \cdots \times D_n^*$. It follows, using the Krein-Smulian theorem (see [9], p. 434) and the weakly sequential continuity of F that D^* is a weakly compact convex set. Moreover $F^*(D^*) \subset D^*$. Since F^* is weakly sequentially continuous, it follows using the Arino et al's. theorem [2] that F^* has a fixed point $x_0 \in \Omega$. If $x_0 \notin U_1 \times \cdots \times U_n, \varphi(x_0) = 0$ and $x_0 = 0$, which contradicts the hypothesis $0 \in U_1 \times \cdots \times U_n$. Then $x_0 \in U_1 \times \cdots \times U_n$ and $x_0 = \varphi(x_0)F(x_0)$, which implies that $x_0 \in \Sigma$, and so $\varphi(x_0) = 1$ and the proof is complete. □

In the next result, the weak compactness of the sets $\overline{U_i^\omega}$ is removed and replaced by a stronger condition on F. The proof is standard and we omit it.

Theorem 3.2. *Let $(X_i, \|\cdot\|_i)$, $i = 1, \cdots, n$ be Banach spaces. For each $i \in \{1, \cdots, n\}$, consider a nonempty closed and convex set $\Omega_i \subset X_i$ and a weakly open subset U_i of Ω_i with $0 \in U_i$. Let $\Omega = \Omega_1 \times \cdots \times \Omega_n$, $D = \overline{U_1^\omega} \times \cdots \times \overline{U_n^\omega}$, and $F : D \to \Omega$ a weakly sequentially continuous mapping such that $F(D)$ is relatively weakly compact. Then the alternative result given by Theorem 3.1 holds.*

Theorem 3.2 will now be exploited to derive a Krasnoselskii type fixed point theorem which is the analogue for the weak topology of Viorel's theorem [22], and a vector version of Theorem 3.4 in [4].

Theorem 3.3. *Let X_i, Ω_i, U_i $(i = 1, \cdots, n)$, Ω and D be as in Theorem 3.1, and $X = X_1 \times \cdots \times X_n$. Let $A : D \longrightarrow X$ and $B : X \longrightarrow X$ be two weakly sequentially continuous mappings such that:*

(a) *$A(D)$ is relatively weakly compact;*
(b) *B is a Perov contraction;*
(c) *$(J - B)^{-1} A(D) \subset \Omega$.*

Then, either

(i): *$A+B$ has a fixed point, or*

(ii): *there exist $i \in \{1, \cdots, n\}$, a point $x = (x_1, \cdots, x_n) \in D$ with $x_i \in \partial_{\Omega_i} U_i = \overline{U_i^{\omega}} \backslash U_i$, and a number $\lambda \in (0,1)$ such as $x = \lambda A(x) + \lambda B(\frac{x}{\lambda})$.*

Proof. For any given $x \in D$, let $F_x : X \longrightarrow X$ be defined by

$$F_x(y) = A(x) + B(y), \quad y \in X.$$

Using (b) we have

$$\|F_x(y) - F_x(z)\| = \|B(y) - B(z)\| \leqslant M\|y - z\|, \quad \text{for all } y, z \in X,$$

where M is the Lipschitz matrix of B. This shows that F_x is a Perov contraction with the same Lipschitz matrix M. Perov's theorem guarantees the existence of a unique point $y_x \in X$ such that $y_x = A(x) + B(y_x)$. Let $F : D \to X$ be defined as

$$F(x) = y_x, \quad x \in D.$$

From (c), we have $F(D) \subset \Omega$. Notice that

$$F(x) = (J - B)^{-1} A(x), \quad x \in D.$$

Our next task is to show that the mapping $F := (J-B)^{-1}A$ fulfills the conditions of Theorem 3.2. Indeed, since from (a), the set $A(D)$ is relatively weakly compact, it is also a bounded set. Next using

$$\|(J-B)^{-1}(x) - (J-B)^{-1}(y)\| \leq (I - M)^{-1}\|x - y\| \quad \text{for all } x, y \in X,$$

we see that $F(D) = (J-B)^{-1}A(D)$ is also bounded. We now claim that $F(D)$ is relatively weakly compact. Indeed, from

$$F(D) \subset A(D) + B(F(D)), \tag{3.7}$$

we obtain

$$\omega(F(D)) \leq \omega(A(D) + B(F(D))). \tag{3.8}$$

Further, taking into account that $A(D)$ is relatively weakly compact and using the property (vi) of ω_i we deduce that

$$\omega(F(D)) \leq \omega(A(D)) + \omega(B(F(D))) = \omega(B(F(D))). \tag{3.9}$$

Now, by Proposition 3.1 and inequality (3.9), we get

$$\omega(F(D)) \leq M\omega(F(D)).$$

So $(I - M)\omega(F(D)) \leq 0_{\mathbb{R}^n}$. Since matrix M is convergent to zero, we then have $\omega(F(D)) = 0_{\mathbb{R}^n}$ and so $\omega_i(F_i(D)) = 0$ for all $i \in \{1, \cdots, n\}$. Consequently, $F(D)$ is relatively weakly compact as claimed.

Next, we show that $F : D \to \Omega$ is weakly sequentially continuous. To do so, let $(x^k)_k \subset D$ be such that $x_i^k \to x_i$ weakly as $k \to \infty$, for $i = 1, \cdots, n$. Because $F(D)$ is relatively weakly compact, it follows by the Eberlein-Smulian theorem ([9], p. 430) that there exists a subsequence of (x^k) (still denoted by (x^k)) and $y \in \Omega$ such that $F_i(x^k) \to y_i$ weakly, for $i = 1, \cdots, n$. Now the weak sequentially continuity of B

guarantees that $B\left(F(x^k)\right) \to B\,(y)$ weakly. Also, from the equality $BF = -A + F$, it follows that

$$-A(x^k) + F(x^k) \to -A(x) + y \quad \text{weakly.}$$

So $y = F(x)$. It is now easy to see that the whole sequence $\left(F\left(x^k\right)\right)$ weakly converges to $F\,(x)$, which proves that F is weakly sequentially continuous. Finally, we note that the fixed points of F are the same as the fixed points of $A + B$, and that the equation $x = \lambda F\,(x)$, where $x \in D$, is equivalent to the equation

$$x = \lambda A(x) + \lambda B\left(\frac{x}{\lambda}\right). \qquad \square$$

Now we state a variant of the previous result where the assumptions on mapping B are relaxed.

Theorem 3.4. *Let X_i, Ω_i, U_i $(i = 1, \cdots, n)$, Ω, D and X be as in Theorem 3.3. Let $A : D \longrightarrow X$ and $B : \Omega \longrightarrow X$ be two weakly sequentially continuous mappings such that:*

(a) *$A(D)$ is relatively weakly compact;*
(b) *$A(D) \subset (J - B)\,(\Omega)$;*
(c) *If $(J - B)(x_k) \to y$ weakly, then $(x_k)_k$ has a weakly convergent subsequence;*
(d) *$J - B$ is invertible.*

Then the alternative of Theorem 3.3 holds.

Proof. For any given $y \in D$, define $F : D \to \Omega$ by $F(y) := (J - B)^{-1}A(y)$. F is well defined by assumption (b). We show that $F(D)$ is relatively weakly compact. For any $(y_n)_n \subset F(D)$, we choose $(x_n)_n \subset D$ such that $y_n = F(x_n)$. Taking into account assumption (a), together with the Eberlein-Smulian's theorem (see [9], p. 430), we get a subsequence $(y_{\varphi_1(n)})_n$ of $(y_n)_n$ such that $(J-B)y_{\varphi_1(n)} \to z$ weakly, for some $z \in \Omega$. Thus, by assumption (c), there exists a subsequence $y_{\varphi_1(\varphi_2(n))})$ converging weakly to $y_0 \in \Omega$. Hence, $F(D)$ is relatively weakly compact. Next, we show that $F : D \to \Omega$ is weakly sequentially continuous. To do so, let $(x^k)_k \subset D$ be such that $x_i^k \to x_i$ weakly as $k \to \infty$, for $i = 1, \cdots, n$. Because $F(D)$ is relatively weakly compact, it follows by the Eberlein-Smulian theorem [[9], p. 430] that there exists a subsequence of (x^k) (still denoted by $\left(x^k\right)$) and $y \in \Omega$ such that $F_i(x^k) \to y_i$ weakly, for $i = 1, \cdots, n$. Now the weak sequentially continuity of B guarantees that $B\left(F(x^k)\right) \to B\,(y)$ weakly. Also, from the equality $BF = -A + F$, it follows that

$$-A(x^k) + F(x^k) \to -A(x) + y \quad \text{weakly.}$$

So $y = F(x)$. It is now easy to see that the whole sequence $\left(F\left(x^k\right)\right)$ weakly converges to $F\,(x)$, which proves that F is weakly sequentially continuous.
Consequently, using Theorem 3.2 we get either $A + B$ has a fixed point or there exist $i \in \{1, \cdots, n\}$, a point $x = (x_1, \cdots, x_n) \in D$ with $x_i \in \partial_{\Omega_i} U_i = \overline{U_i^{\omega}} \setminus U_i$, and a number $\lambda \in (0, 1)$ such as $x = \lambda A(x) + \lambda B(\frac{x}{\lambda})$. $\square$

Remark 3.1. Any Perov contraction $B : \Omega \longrightarrow X$, with $B(\Omega)$ bounded, satisfies condition *(c)* in Theorem *3.4*. To prove this, assume that $(J - F)\,(x_k) \to y$ weakly, for some $(x_k)_k \subset \Omega$ and $y \in X$. Writing x_k as $x_k = (J - B)(x_k) + B(x_k)$ and using the subadditivity of the De Blasi measure of weak noncompactness, we get

$$\omega(\{x_k\}) \leq \omega(\{(J - B)(x_k)\}) + \omega(\{B(x_k)\}).$$

Since $\omega(\{(J-B)(x_k)\}) = 0_{\mathbb{R}^n}$, we obtain $\omega(\{x_k\}) \leq \omega(\{B(x_k)\})$. On the other hand, if M is the Lipschitz matrix of B, then

$$\omega(\{B(x_k)\}) \leq M\omega(\{x_k\}).$$

It follows that $(I - M)\omega(\{x_k\}) \leq 0_{\mathbb{R}^n}$, and then $\omega(\{x_k\}) = 0_{\mathbb{R}^n}$. Consequently, $\{x_k\}$ is relatively weakly compact and then by the Eberlein-Smulian's theorem, it has a weakly convergent subsequence. Hence, condition (c) is satisfied.

As a consequence of Theorem 3.4 and Remark 3.1, we have the following result.

Corollary 3.1. *Let X_i, Ω_i, U_i $(i = 1, \cdots, n)$, Ω, D and X be as in Theorem 3.3. Assume that $A : D \longrightarrow X$ and $B : \Omega \longrightarrow X$ are two weakly sequentially continuous mappings such that:*

(1) *$A(D)$ is relatively weakly compact;*
(2) *B is a Perov contraction and $B(\Omega)$ is bounded;*
(3) *$A(D) + B(\Omega) \subset \Omega$.*

Then the alternative of Theorem 3.3 holds.

Notice that the vector versions of the original theorems applied to the product space $X = X_1 \times \cdots \times X_n$ allow to use different measures of noncompactness on the factor spaces X_i, such is the case in paper [7].

4. Application I: Solutions of a system of nonlinear transport equations

We consider the following system:

$$\begin{cases} v_3 \dfrac{\partial \Psi_1}{\partial x}(x,v) + \sigma_1(x, v, \Psi_1(x,v), \Psi_2(x,v)) - \lambda_1 \Psi_1(x,v) \\ = \displaystyle\int_K r_1(x, v, v', \Psi_1(x,v'), \Psi_2(x,v'))dv' \\ v_3 \dfrac{\partial \Psi_2}{\partial x}(x,v) + \sigma_2(x, v, \Psi_1(x,v), \Psi_2(x,v)) - \lambda_2 \Psi_2(x,v) \\ = \displaystyle\int_K r_2(x, v, v', \Psi_1(x,v'), \Psi_2(x,v'))dv' \end{cases} \tag{4.1}$$

where $(x, v) \in D = (0,1) \times K$ with K the unit sphere of $\mathbb{R}^3$, $x \in (0,1)$, $v = (v_1, v_2, v_3) \in K$, $r_j(.,.,.,.), j = 1,2$ is a nonlinear function of Ψ_j, $\sigma_j(.,.,.,.)$, $j = 1,2$ is a function on $[0,1] \times K \times \mathbb{C}^2$ and $\lambda_j, j = 1,2$ is a complex number. The boundary conditions are modeled by

$$\Psi_{j_{|D^i}} = H^j(\Psi_{j_{|D^0}}), \quad \text{for } j = 1,2 \tag{4.2}$$

where D^i (resp. D^0) is the incoming (resp. outgoing) part of the space boundary and are given by

$$D^i = D_1^i \cup D_2^i = \{0\} \times K^1 \cup \{1\} \times K^0,$$

$$D^0 = D_1^0 \cup D_2^0 = \{0\} \times K^0 \cup \{1\} \times K^1,$$

for

$$K^0 = K \cap \{v_3 < 0\} \quad \text{and} \quad K^1 = K \cap \{v_3 > 0\}.$$

We shall treat the problem (4.1)-(4.2) in the following functional setting: let

$$X := L^1(D; dxdv),$$

and

$$X^i := L^1\Big(D^i, |v_3|dv\Big) := L^1\Big(D_1^i, |v_3|dv\Big) \oplus L^1\Big(D_2^i, |v_3|dv\Big) := X_1^i \oplus X_2^i,$$

endowed with the norm

$$\|\Psi\|_{X^i} = \|\Psi_1^i\|_{X_1^i} + \|\Psi_2^i\|_{X_2^i} = \int_{K^1} |\Psi(0,v)||v_3|dv + \int_{K^0} |\Psi(1,v)||v_3|dv,$$

and

$$X^0 := L^1\Big(D^0, |v_3|dv\Big) := L^1\Big(D_1^0, |v_3|dv\Big) \oplus L^1\Big(D_2^0, |v_3|dv\Big) := X_1^0 \oplus X_2^0,$$

endowed with the norm

$$\|\Psi\|_{X^0} = \|\Psi_1^0\|_{X_1^0} + \|\Psi_2^0\|_{X_2^0} = \int_{K^0} |\Psi(0,v)||v_3|dv + \int_{K^1} |\Psi(1,v)||v_3|dv.$$

For each $j \in \{1, 2\}$, let H^j be the following linear bounded boundary operator defined by:

$$\begin{cases} H^j : X_1^0 \oplus X_2^0 \longrightarrow X_1^i \oplus X_2^i \\ H^j \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} H_{11}^j & H_{12}^j \\ H_{21}^j & H_{22}^j \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{cases}$$

where $H_{l,k}^j \in \mathcal{L}(X_l^0, X_k^i)$, for $l, k, j = 1, 2$. The boundary condition can be written as $\Psi^i = H^j(\Psi^0)$ for $j = 1, 2$. Now for each $j \in \{1, 2\}$ we define the streaming operator T_{H^j} with domain including the boundary conditions

$$\begin{cases} T_{H^j} : D(T_{H^j}) \subseteq X \longrightarrow X, \\ \qquad\qquad \Psi \longmapsto T_{H^j}\Psi(x,v) = v_3 \dfrac{\partial \Psi}{\partial x}(x,v) \\ D(T_{H^j}) = \big\{\Psi \in X \text{ such that } \Psi^i = H^j(\Psi^0)\big\}, \end{cases}$$

where $\Psi^0 = (\Psi_1^0, \Psi_2^0)^T$ and $\Psi^i = (\Psi_1^i, \Psi_2^i)^T$ where $\Psi_1^0, \Psi_2^0, \Psi_1^i$ and Ψ_2^i are given by

$$\begin{cases} \Psi_1^i(v) = \Psi(0,v), & \text{for } v \in K^1, \\ \Psi_2^i(v) = \Psi(1,v), & \text{for } v \in K^0, \\ \Psi_1^0(v) = \Psi(0,v), & \text{for } v \in K^0, \\ \Psi_2^0(v) = \Psi(1,v), & \text{for } v \in K^1. \end{cases}$$

Remark 4.1. For each $j \in \{1, 2\}$, the derivative of Ψ in the definition of T_{H^j} is meant in distributional sense.

For each $j \in \{1,2\}$, let λ_0^j be the real defined by

$$\lambda_0^j := \begin{cases} 0 & \text{if } \|H^j\| \leq 1, \\ -\log(\|H^j\|) & \text{if } \|H^j\| > 1. \end{cases}$$

Proposition 4.1. *For each $j \in \{1,2\}$, we have*

$$\Big\{\lambda \in \mathbb{C} \text{ such that } \; Re(\lambda) < \lambda_0 = \inf(\lambda_0^1, \lambda_0^2)\Big\} \subset \rho(T_{H^j}).$$

Proof. See reference ([5] Proposition 3.1). □

For our subsequent analysis, we need this hypothesis: For each $j \in \{1,2\}$,

$$(\mathcal{A}_1) \quad r_j\big(x, v, v', \Psi_1(x,v'), \Psi_2(x,v')\big) = \kappa_j(x,v,v') f_j\big(x, v', L_j(\Psi_1,\Psi_2)(x,v')\big),$$

with $L_j := (L^1([0,1]\times K))^2 \longrightarrow L^\infty([0,1]\times K)$ is a continuous linear map and

$$\begin{cases} f_j : [0,1]\times K \times \mathbb{C}^2 \longrightarrow \mathbb{C} \\ \qquad (x,v,u_1,u_2) \longmapsto f_j(x,v,u_1,u_2). \end{cases}$$

is a mesurable function. The function $\kappa_j(.,.,.), j = 1,2$ is a measurable function from $[0,1]\times K \times K$ into $\mathbb{R}$. It defines a continuous linear operator $F_j, j = 1,2$ by

$$\begin{aligned} F_j : \quad X \quad &\longrightarrow X \\ \Psi \quad &\longmapsto F_j(\Psi)(x,v) = \int_K \kappa_j(x,v,v')\Psi(x,v')dv' \end{aligned} \tag{4.3}$$

Note that $dx \otimes dv - ess - \sup_{(x,v)\in[0,1]\times K} \int_K |\kappa_j(x,v,v')|dv' = \|F_j\| < \infty.$

Definition 4.1. A collision operator $F_j, j = 1,2$ in form (4.3) is said to be regular if the set

$$\Big\{\kappa_j(x,.,v') \text{ such that } (x,v') \in [0,1]\times K\Big\}$$

is a relatively weakly compact subset of $L^1(K, dx)$.

We need also the following result which is an immediate consequence of Lemme 4.1 in [6] for $\sigma \equiv 0$.

Lemma 4.1. *If the collision operator $F_j, j = 1,2$ is regular on X, then $(T_{H^j} - \lambda I)^{-1}F_j$ is weakly compact on X, for $Re(\lambda) < \lambda_0$.*

Definition 4.2. A function $f : [0,1]\times K \times \mathbb{C}^2 \to \mathbb{C}$ is a Carathéodory map if the following conditions are satisfied

$$\begin{cases} (x,v) \longmapsto f(x,v,u_1,u_2) \text{ is measurable on } [0,1]\times K, \text{ for all } (u_1,u_2) \in \mathbb{C}^2. \\ u \longmapsto f(x,v,u_1,u_2) \text{ is continuous on } \mathbb{C}^2, \text{ for almost all } (x,v) \in [0,1]\times K. \end{cases}$$

If f satisfies the Carathéodory conditions, we can define the operator $\mathcal{N}_f$ on the set of functions $(\Psi_1,\Psi_2) : [0,1]\times K \longrightarrow \mathbb{C}^2$ by

$$\mathcal{N}_f(\Psi_1,\Psi_2)(x,v) = f(x,v,\Psi_1(x,v),\Psi_2(x,v)), \text{ for every } \; (x,v) \in [0,1]\times K.$$

The operator $\mathcal{N}_f$ is called the Nemytskii operator generated by f. We assume that

$$(\mathcal{A}_2) \qquad \begin{cases} \text{For each } j \in \{1,2\}, f_j \text{ is a Carathéodory map satisfying} \\ |f_j(x,v,u_1,u_2)| \leqslant a_j(x,v)h_j(\|(u_1,u_2)\|_{L^1\times L^1}), \\ \text{where } a_j \in L^1([0,1]\times K, dxdv) \text{ and} \\ h_j \in L^\infty_{loc}(\mathbb{R}^+) \text{ a non-decreasing function.} \end{cases}$$

The interest that an operator satisfies the property $(\mathcal{A}_2)$ lies in the following lemma:

Lemma 4.2. *For each $j \in \{1,2\}$, let $L_j : (L^1([0,1]\times K, dxdv))^2 \longrightarrow L^\infty([0,1] \times K, dxdv)$ be a continuous linear map and let $f_j : [0,1]\times K \times \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a map satisfying the hypothesis $(\mathcal{A}_2)$. Then the map*

$$\Phi_j := \mathcal{N}_{f_j} \circ L_j : (L^1([0,1]\times K, dxdv))^2 \longrightarrow L^1([0,1]\times K, dxdv)$$

is weakly sequentially continuous.

Proof. Let $(u_n, v_n) \rightharpoonup (u,v)$ in $(L^1([0,1]\times K, dxdv))^2$. By the Eberlein-Smulian Theorem, the set $G = \{(u_n,v_n),(u,v)\}_{n=1}^\infty$ is weakly compact. Let us show that $\Phi_j(G), j = 1,2$ is relatively weakly compact in $L^1([0,1]\times K, dxdv)$. Clearly $\Phi_j(G)$ is bounded, once

$$\|\Phi_j(u_1,u_2)\|_{L^1} \leqslant \|a_j\|_{L^1} h_j(\|L_j\|\|(u_1,u_2)\|_{L^1\times L^1}).$$

Which also shows that $\Phi_j(G)$ is uniformly integrable. Since $\mathbb{C}^2$ is reflexive, we get, according to Dunford's Theorem ([3] Theorem 7.10), that $\Phi_j(G)$ is relatively weakly compact in $L^1([0,1]\times K, dxdv)$. Up to a subsequence, $\Phi_j(u_n,v_n) \rightharpoonup w_j \in L^1([0,1] \times K, dxdv)$. The idea is to show that actually $w_j = \Phi_j(u,v)$. We know $L_j(u_n,v_n)(x,\xi) \rightharpoonup L_j(u,v)(x,\xi)$ in $\mathbb{C}$ for a.e. $(x,\xi) \in [0,1]\times K$. Since f is a Carathéodory map, then $\Phi_j(u_n,v_n)(x,\xi) \rightharpoonup \Phi_j(u,v)(x,\xi)$ in $\mathbb{C}$ for almost every $(x,\xi) \in [0,1]\times K$. Now we shall conclude that $w_j = \Phi_j(u,v)$ a.e. To this end, we start by throwing away a set A_0 of measure zero such that, for each $j \in \{1,2\}$ the space

$$F_j := \overline{span}\Big(w_j(([0,1]\times K)\setminus A_0) \cup \Phi_j(u,v)(([0,1]\times K)\setminus A_0)\Big)$$

is a separable and reflexive Banach space. The existence of such a A_0 is due to Pettis' Theorem. Let now $\{\varphi_k\}$ be a dense sequence of continuous linear functionals in F_j. By Ergorov's Theorem, for each φ_k fixed, there exists a negligible set A_k, such that $\varphi_k(w_j) = \varphi_k(\Phi_j(u,v))$ in $([0,1]\times K)\setminus A_k$. Finally we define $A = \cup_{k=0}^\infty A_k$. In this way $\lambda(A) = 0$ and by the Hahn-Banach Theorem, $w_j(x,\xi) = \Phi_j(u,v)(x,\xi)$ for all $(x,\xi) \in ([0,1]\times K)\setminus A$. $\square$

The following hypothesis will play a crucial role:

$$(\mathcal{A}_3) \qquad \begin{cases} \text{For } j = 1,2,\ \mathcal{N}_{\sigma_j} \text{ is weakly sequentially continuous} \\ \text{and acts from } \overline{B}_{r_1}\times\overline{B}_{r_2} \text{ into } \overline{B}_{r_j} \\ \big|\mathcal{N}_{\sigma_j}(\Psi_1,\Psi_2)(x,v)) - \mathcal{N}_{\sigma_j}(\Psi_1',\Psi_2')(x,v))\big| \\ \leqslant |\rho_{j,1}(x,v)||\Psi_1 - \Psi_1'| + |\rho_{j,2}(x,v)||\Psi_2 - \Psi_2'| \\ \text{where } \overline{B}_r = \{\Psi \in X \text{ such that } \|\Psi\| \leqslant r\} \\ \text{and } \rho_{j,1}(.,.), \rho_{j,2}(.,.) \in L^\infty(D, dxdv), \end{cases}$$

Let λ be a complex number such that $Re(\lambda) < \lambda_0$. Then due to Proposition 4.1, the mapping $T_{H^j} - \lambda I$, $j = 1,2$ is invertible and therefore, the problem (4.1)-(4.2) is equivalent to the following system:

$$\begin{cases} \Psi_1 = \mathcal{F}_1(\lambda_1)(\Psi_1,\Psi_2) + \mathcal{H}_1(\lambda_1)(\Psi_1,\Psi_2) \\ \Psi_2 = \mathcal{F}_2(\lambda_2)(\Psi_1,\Psi_2) + \mathcal{H}_2(\lambda_2)(\Psi_1,\Psi_2) \\ \Psi_1 \in D(T_{H^1}), \Psi_2 \in D(T_{H^2}),\ Re(\lambda_j) < \lambda_0 \end{cases} \tag{4.4}$$

where

$$\begin{cases} \mathcal{F}_j(\lambda_j) := (T_{H^j} - \lambda_j I)^{-1} F_j \mathcal{N}_{f_j} L_j \\ \mathcal{H}_j(\lambda_j) := (T_{H^j} - \lambda_j I)^{-1} \mathcal{N}_{-\sigma_j} \qquad j = 1,2 \end{cases}$$

Now, the system (4.4) is equivalent to the following fixed point problem:

$$\begin{cases} (\Psi_1,\Psi_2) = \mathcal{F}(\lambda_1,\lambda_2)(\Psi_1,\Psi_2) + \mathcal{H}(\lambda_1,\lambda_2)(\Psi_1,\Psi_2) \\ (\Psi_1,\Psi_2) \in D(T_{H^1}) \times D(T_{H^2}),\ Re(\lambda_j) < \lambda_0 \text{ for } j = 1,2 \end{cases} \tag{4.5}$$

where

$$\mathcal{F}(\lambda_1,\lambda_2) := \begin{pmatrix} \mathcal{F}_1(\lambda_1) \\ \mathcal{F}_2(\lambda_2) \end{pmatrix} = \begin{pmatrix} (T_{H^1} - \lambda_1 I)^{-1} F_1 \mathcal{N}_{f_1} L_1 \\ (T_{H^2} - \lambda_2 I)^{-1} F_2 \mathcal{N}_{f_2} L_2 \end{pmatrix},$$

$$\mathcal{H}(\lambda_1,\lambda_2) := \begin{pmatrix} \mathcal{H}_1(\lambda_1) \\ \mathcal{H}_2(\lambda_2) \end{pmatrix} = \begin{pmatrix} (T_{H^1} - \lambda_1 I)^{-1} \mathcal{N}_{-\sigma_1} \\ (T_{H^2} - \lambda_2 I)^{-1} \mathcal{N}_{-\sigma_2} \end{pmatrix}$$

Theorem 4.1. *Assume that $\mathcal{A}_1 - \mathcal{A}_3$ hold and that for $j = 1,2, F_j$ is a regular operator on X. Let $U_{r_1} \times U_{r_2}$ be a weakly open subset of $\overline{B}_{r_1} \times \overline{B}_{r_2}$ with $0 \in U_{r_1} \times U_{r_2}$. In addition, suppose that: for any solution $(\Psi_1,\Psi_2) \in X^2$ to*

$$(\Psi_1,\Psi_2) = \alpha \mathcal{F}(\lambda)(\Psi_1,\Psi_2) + \alpha \mathcal{H}(\lambda)\left(\frac{\Psi_1}{\alpha}, \frac{\Psi_2}{\alpha}\right)$$

a.e. $0 < \alpha < 1$, we have $(\Psi_1,\Psi_2) \notin \partial_{\overline{B}_{r_1}} U_{r_1} \times \partial_{\overline{B}_{r_2}} U_{r_2}$ (the weak boundary of U_{r_j} in $B_{r_j}, j = 1,2$) holds. Then there exists $\lambda^* < \lambda_0$, such that for $Re(\lambda_j) < \lambda^*, j = 1,2$ the problem $(4.1) - (4.2)$ has a solution in $\overline{U_{r_1}}^{\omega} \times \overline{U_{r_2}}^{\omega}$.

Proof. The proof will be given in several steps:
Step 1: The maps $\mathcal{F}(\lambda_1,\lambda_2)$ and $\mathcal{H}(\lambda_1,\lambda_2)$ are weakly sequentially continuous for suitable $\lambda = (\lambda_1,\lambda_2)$. Indeed, we have for $j = 1,2$, $\mathcal{N}_{\sigma_j}$ is weakly sequentially continuous and for $Re(\lambda_j) < \lambda_0$, the linear operator $(T_{H^j} - \lambda_j)^{-1}, j = 1,2$ is bounded, so the operator

$$\mathcal{H}(\lambda_1,\lambda_2) := ((T_{H^1} - \lambda_1 I)^{-1} \mathcal{N}_{-\sigma_1},\ (T_{H^2} - \lambda_2 I)^{-1} \mathcal{N}_{-\sigma_2})$$

is weakly sequentially continuous, for $Re(\lambda_j) < \lambda_0, j = 1,2$. Moreover, using ([6] page 89), we have

$$\mathcal{F}(\lambda_1,\lambda_2) := \big((T_{H^1} - \lambda_1 I)^{-1} F_1 \mathcal{N}_{f_1} L_1,\ (T_{H^2} - \lambda_2 I)^{-1} F_2 \mathcal{N}_{f_2} L_2\big)$$

is weakly sequentially continuous, for $Re(\lambda_j) < \lambda_0, j = 1,2$.
Step 2: $\mathcal{F}(\lambda)(\overline{U_{r_1}}^{\omega} \times \overline{U_{r_2}}^{\omega})$ is relatively weakly compact in $X \times X$. Using the hypothesis $(\mathcal{A}_2)$, we get $\mathcal{N}_{f_j} L_j(\overline{U_{r_1}}^{\omega} \times \overline{U_{r_2}}^{\omega})$ is a bounded subset of X. So from Lemma 4.1 we have $\mathcal{F}(\lambda)(\overline{U_{r_1}}^{\omega} \times \overline{U_{r_2}}^{\omega})$ is relatively weakly compact in $X \times X$.

Step 3: $\mathcal{H}(\lambda_1, \lambda_2)$ is a contraction mapping on $\overline{B}_{r_1} \times \overline{B}_{r_2}$.
Indeed, let $(\Psi_1, \Psi_2), (\Psi_1', \Psi_2') \in B_{r_1} \times B_{r_2}$. We have

$$\begin{aligned}
&\|\mathcal{H}(\lambda)(\Psi_1, \Psi_2) - \mathcal{H}(\lambda)(\Psi_1', \Psi_2')\| \\
&= \begin{pmatrix} \|(T_{H^1} - \lambda_1 I)^{-1}(\mathcal{N}_{-\sigma_1}(\Psi_1, \Psi_2) - \mathcal{N}_{-\sigma_1}(\Psi_1', \Psi_2'))\| \\ \|(T_{H^2} - \lambda_2 I)^{-1}(\mathcal{N}_{-\sigma_2}(\Psi_1, \Psi_2) - \mathcal{N}_{-\sigma_2}(\Psi_1', \Psi_2'))\| \end{pmatrix} \\
&\leqslant \begin{pmatrix} \|(T_{H^1} - \lambda_1 I)^{-1}\|\|\mathcal{N}_{-\sigma_1}(\Psi_1, \Psi_2) - \mathcal{N}_{-\sigma_1}(\Psi_1', \Psi_2')\| \\ \|(T_{H^2} - \lambda_2 I)^{-1}\|\|\mathcal{N}_{-\sigma_2}(\Psi_1, \Psi_2) - \mathcal{N}_{-\sigma_2}(\Psi_1', \Psi_2')\| \end{pmatrix} \\
&\leqslant \begin{pmatrix} \|(T_{H^1} - \lambda_1 I)^{-1}\|(\|\rho_{1,1}\|_\infty \|\Psi_1 - \Psi_1'\| + \|\rho_{1,2}\|_\infty \|\Psi_2 - \Psi_2'\|) \\ \|(T_{H^2} - \lambda_2 I)^{-1}\|(\|\rho_{2,1}\|_\infty \|\Psi_1 - \Psi_1'\| + \|\rho_{2,2}\|_\infty \|\Psi_2 - \Psi_2'\|) \end{pmatrix} \\
&\leqslant \max_{j \in \{1,2\}} (\|(T_{H^j} - \lambda_j I)^{-1}\|) \begin{pmatrix} \|\rho_{1,1}\|_\infty & \|\rho_{1,2}\|_\infty \\ \|\rho_{2,1}\|_\infty & \|\rho_{2,2}\|_\infty \end{pmatrix} \begin{pmatrix} \|\Psi_1 - \Psi_1'\| \\ \|\Psi_2 - \Psi_2'\| \end{pmatrix} \\
&\leqslant M\|(\Psi_1, \Psi_2) - (\Psi_1' - \Psi_2')\|
\end{aligned}$$

where

$$M = \max_{j \in \{1,2\}} (\big\|(T_{H^j} - \lambda_j I)^{-1}\big\|) \begin{pmatrix} \|\rho_{1,1}\|_\infty & \|\rho_{1,2}\|_\infty \\ \|\rho_{2,1}\|_\infty & \|\rho_{2,2}\|_\infty \end{pmatrix}$$

On the other hand, we have for $Re(\lambda_j) < \lambda_0, j = 1, 2,$

$$\|(T_{H^j} - \lambda_j)^{-1}\| \leqslant \frac{-1}{Re(\lambda_j)} \left(1 + \frac{\|H^j\|}{1 - e^{Re(\lambda_j)\|H^j\|}}\right).$$

(See [6], page 89). So, $\|(T_{H^j} - \lambda_j)^{-1}\| \leqslant \Upsilon(Re(\lambda_j))$ where

$$\Upsilon(t) = \frac{-1}{t} \left(1 + \frac{\|H^j\|}{1 - e^{t\|H^j\|}}\right)$$

Clearly, Υ is continuous and satisfies $\lim_{t \to -\infty} \Upsilon(t) = 0$. Hence there exists $\lambda' < 0$ such that for $Re(\lambda_j) < \min(\lambda_0, \lambda')$, we have

$$(\max_{j \in \{1,2\}} \|(T_{H^j} - \lambda_j)^{-1}\|\|\rho_{k,l}\|_\infty)_{1 \le k,l \le 2}$$

are small enough and so, M is a matrix convergent to zero. In conclusion, the operator $\mathcal{H}(\lambda_1, \lambda_2)$ is a contraction mapping on $\overline{B}_{r_1} \times \overline{B}_{r_2}$.
Step 4: We will show that for suitable $\lambda = (\lambda_1, \lambda_2)$, we have

$$\mathcal{F}(\lambda)(\overline{U_{r_1}}^\omega \times \overline{U_{r_2}}^\omega) + \mathcal{H}(\lambda)(\overline{B}_{r_1} \times \overline{B}_{r_2}) \subset \overline{B}_{r_1} \times \overline{B}_{r_2}.$$

To do so, let $(\Psi_1, \Psi_2) \in \overline{U_{r_1}}^\omega \times \overline{U_{r_2}}^\omega$ and $(\varphi_1, \varphi_2) \in \overline{B}_{r_1} \times \overline{B}_{r_2}$. Then we have

$$\begin{aligned}
&\|\mathcal{H}(\lambda)(\varphi_1, \varphi_2) + \mathcal{F}(\lambda)(\Psi_1, \Psi_2)\| \\
&= \left\| \begin{pmatrix} (T_{H^1} - \lambda_1 I)^{-1} \Big(\mathcal{N}_{-\sigma_1}(\varphi_1, \varphi_2) + F_1 \mathcal{N}_{f_1} L_1(\Psi_1, \Psi_2)\Big) \\ (T_{H^2} - \lambda_2 I)^{-1} \Big(\mathcal{N}_{-\sigma_2}(\varphi_1, \varphi_2) + F_2 \mathcal{N}_{f_2} L_2(\Psi_1, \Psi_2)\Big) \end{pmatrix} \right\| \\
&\leqslant \left(\left\| \begin{matrix} (T_{H^1} - \lambda_1 I)^{-1} \\ (T_{H^2} - \lambda_2 I)^{-1} \end{matrix} \right\| \begin{matrix} (\|\mathcal{N}_{-\sigma_1}(\varphi_1, \varphi_2)\| + \|F_1\|\|(\mathcal{N}_{f_1} L_1(\Psi_1, \Psi_2))\|) \\ (\|\mathcal{N}_{-\sigma_2}(\varphi_1, \varphi_2)\| + \|F_2\|\|(\mathcal{N}_{f_2} L_2(\Psi_1, \Psi_2))\|) \end{matrix} \right) \\
&\leqslant \left(\left\| \begin{matrix} (T_{H^1} - \lambda_1 I)^{-1} \\ (T_{H^2} - \lambda_2 I)^{-1} \end{matrix} \right\| \begin{matrix} (M_1(r_1, r_2) + \|F_1\|(\|a_1\|\|h_1\|_\infty) \\ (M_2(r_1, r_2) + \|F_2\|(\|a_2\|\|h_2\|_\infty) \end{matrix} \right)
\end{aligned}$$

where for $j = 1, 2, M_j(r_1, r_2)$ denotes respectively the upper bound of $\mathcal{N}_{-\sigma_j}$ on $\overline{B}_{r_1} \times \overline{B}_{r_2}$. So, for $Re\lambda_j < \min(\lambda', \lambda_1), \lambda_1 < 0$, we obtain

$$\|\mathcal{H}(\lambda)(\varphi_1, \varphi_2) + \mathcal{F}(\lambda)(\Psi_1, \Psi_2)\| \leq \max_{j \in \{1,2\}} \Upsilon(Re(\lambda_j)) \begin{pmatrix} M_1(r_1, r_2) + \|F_1\|(\|a_1\|\|h_1\|_\infty \\ M_2(r_1, r_2) + \|F_2\|(\|a_2\|\|h_2\|_\infty \end{pmatrix},$$

where Υ is defined in step 3.
Thus, there exists $\lambda'' < 0$ such that for $Re(\lambda_j) < \min(\lambda_0, \lambda', \lambda''), j = 1, 2$, we have

$$\mathcal{H}(\lambda_1, \lambda_2)(\varphi_1, \varphi_2) + \mathcal{F}(\lambda_1, \lambda_2)(\Psi_1, \Psi_2) \subset \overline{B}_{r_1} \times \overline{B}_{r_2}.$$

Consequently, for $Re(\lambda_j) < \lambda^* = \min(\lambda_0, \lambda', \lambda''), j = 1, 2$ the mappings $\mathcal{F}(\lambda_1, \lambda_2)$ and $\mathcal{H}(\lambda_1, \lambda_2)$ satisfy the assumptions of Corollary (3.1) on the nonempty bounded, closed and convex subset $\overline{B}_{r_1} \times \overline{B}_{r_2}$. Consequently the problem $(4.1 - 4.2)$ has a solution (φ, ψ) in $\overline{B}_{r_1} \times \overline{B}_{r_2}$ for all $\lambda = (\lambda_1, \lambda_2)$ such that $Re\lambda_j < \lambda^*, j = 1, 2$. □

5. Application II: Existence of weak solutions

We discuss the existence of weak solutions for a coupled system of mixed fractional differential equations

$$\begin{cases} D^{\alpha}_{1^-}(D^{\beta_1}_{0^+}u(t)) + f_1(t, u(t), v(t)) = 0, \\ D^{\alpha}_{1^-}(D^{\beta_2}_{0^+}v(t)) + f_2(t, u(t), v(t)) = 0; \ t \in I := [0, 1], \end{cases} \tag{5.1}$$

with the following initial conditions:

$$\begin{cases} D^{\beta_1}_{0^+}u(0) = D^{\beta_1}_{0^+}u(1) = D^{\beta_2}_{0^+}v(0) = D^{\beta_2}_{0^+}v(1) = 0, \\ u(0) = u'(1) = v(0) = v'(1) = 0; \end{cases} \tag{5.2}$$

where $\alpha > 1, \beta_i < 2$, for $i = \{1, 2\}$, $f_1, f_2 : I \times E \times E \to E$ are given continuous functions, E is a real (or complex) Banach space with norm $\|.\|_E$ and dual E^* such that E is the dual of a weakly compactly generated Banach space X. Let's remember that

$$D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds$$

and

$$D^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(n-\alpha)}(-\frac{d}{dt})^n \int_t^b (s-t)^{n-\alpha-1} f(s)ds$$

where $n = [\alpha] + 1$, are, respectively, the right and left Riemann-Liouville fractional derivatives of order α and

$$I^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds$$

and

$$I^{\alpha}_{b^-}f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)ds$$

are, respectively, the right and left Riemann-Liouville fractional integrals of order α. Let $C(I, E)$ be the Banach space of all continuous functions w from I into E with the supremum (uniform) norm. As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into E. Also by $C(I,E)^2 = C^2$, we denote the product space of continuous functions with the norm

$$\|(u,v)\|_{C^2} = \begin{pmatrix} \|u\|_C \\ \|v\|_C \end{pmatrix}.$$

Let $(E,w) = (E,\sigma(E,E^*))$ be the Banach space E with its weak topology.

Definition 5.1. A Banach space X is called weakly compactly generated (WCG for short) if it contains a weakly compact set whose linear span is dense in X.

Definition 5.2. ([18]) The function $u : I \to E$ is said to be Pettis integrable on I if and only if there is an element $u_J \in E$ corresponding to each $J \subset I$ such that

$$\phi(u_J) = \int_J \phi(u(s))ds$$

for all $\phi \in E^*$, where the integral on the right-hand side is assumed to exist in the sense of Lebesgue (by definition, $u_J = \int_J u(s)ds$)

Let $P(I,E)$ be the space of all E-valued Pettis integrable functions on I, and $L^1(I,E)$ be the Banach space of Lebesgue integrable functions $u : I \to E$. Define the class $P_1(I,E)$ by

$$P_1(I,E) = \Big\{ u \in P(I,E) : \phi(u) \in L^1(I,E) \text{ for every } \phi \in E^* \Big\}.$$

The space $P_1(I,E)$ is normed by

$$\|u\|_{P_1} = \sup_{\phi\in E^*, \|\phi\|\leqslant 1} \int_0^1 |\phi(u(x))| d\lambda x,$$

where λ stands for a Lebesgue measure on I. The following result is due to Pettis (see [18], Theorem 3.4 and Corollary 3.41).

Proposition 5.1. ([18]) *If $u \in P_1(I,E)$ and h is a measurable and essentially bounded E-valued function, then $uh \in P_1(J,E)$.*

For all that follows, the symbol $\int$ denotes the Pettis integral.

Proposition 5.2. *Let E be a normed space, and $x_0 \in E$ with $x_0 \neq 0$. Then there exists $\phi \in E^*$ with $\|\phi\| = 1$ and $\phi(x_0) = \|x_0\|$.*

Let us start by defining what we mean by a weak solution of the coupled system (5.1) – (5.2).

Definition 5.3. A coupled function $(u,v) \in C^2$ is said to be a weak solution of the system (5.1) – (5.2) if (u,v) satisfies equations (5.1) and conditions (5.2) on I.

The following hypotheses will be used in the sequel:

(H_1) For a.e. $t \in I$, the functions $u \mapsto f_i(t,u,.)$ and $v \mapsto f_i(t,.,v); i = 1,2$ are weakly sequentially continuous.

(H_2) For a.e. $u,v \in C(I,E)$, the functions $t \mapsto f_i(t,u,v), i = 1,2$ are Pettis integrable a.e. on I.

(H_3) There exist $p_{ij} \in C(I, [0, \infty)), i = 1, 2$, such that

$\|f_i(t, u_1(t), u_2(t)) - f_i(t, v_1(t), v_2(t)\|_E \leq p_{i1}(t)\|u_1(t) - v_1(t)\|_E + p_{i2}(t)\|u_2(t) - v_2(t)\|_E$

for a.e. $t \in I$ and each $u_1, u_2, v_1, v_2 \in C$.

Let

$$p_{ij}^* = \sup_{t\in I} p_{i,j}(t), i, j = 1, 2.$$

We shall transform the system (5.1) – (5.2) to an equivalent system of integral equations. Consider the corresponding linear system:

$$D_{1^-}^{\alpha}(D_{0^+}^{\beta_i} u_i(t)) = -y_i(t), \ 0 < t < 1,$$

$$D_{0^+}^{\beta_i} u_i(0) = D_{0^+}^{\beta_i} u_i(1) = 0, \ u_i(0) = u_i'(1) = 0,$$

here $i \in \{1, 2\}$.

Lemma 5.1. [11] *Assume that $y_i \in C(0,1) \cap L_1(0,1)$, for $i \in \{1, 2\}$, then the boundary value problem* (5.1) – (5.2), *has a unique solution given by*

$$u_i(t) = \int_0^1 G_i(t, r) y_i(r) dr + g_i(t) \int_0^1 s^{\alpha-1} y_i(s) ds,$$

where

$$G_i(t,r) = \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)} \begin{cases} \displaystyle\int_0^r (t^{\beta_i - 1}(1-s)^{\beta_i - 2} - (t-s)^{\beta i - 1})(r-s)^{\alpha-1} ds, \\ \qquad 0 \leq r \leq t \leq 1, \\ \displaystyle t^{\beta_i - 1} \int_0^r (1-s)^{\beta_i - 2}(r-s)^{\alpha-1} ds - \int_0^t (t-s)^{\beta_i - 1}(r-s)^{\alpha-1} ds, \\ \qquad 0 \leq t \leq r \leq 1. \end{cases}$$

$$g_i(t) = \frac{1}{\Gamma(\beta_i)\Gamma(\alpha)} \left(\int_0^t (t-s)^{\beta_i - 1}(1-s)^{\alpha-1} ds - \frac{t^{\beta_i - 1}}{\alpha + \beta_i - 2} \right).$$

Lemma 5.2. [11] *The functions g_i and G_i, for all $i \in \{1, 2\}$ are continuous and satisfy the following properties:*

$$0 \leq G_i(t, r) \leq \frac{1}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)}, 0 \leq t, r \leq 1$$

$$g_i(t) \leq 0, \ |g_i(t)| \leq \frac{1}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)}, 0 \leq t \leq 1.$$

Define the integral operators A and B on C^2 by

$$A(u_1, u_2)(t) = \begin{pmatrix} A_1(u_1, u_2)(t) \\ A_2(u_1, u_2)(t) \end{pmatrix}, \text{ and } B(u_1, u_2)(t) = \begin{pmatrix} B_1(u_1, u_2)(t) \\ B_2(u_1, u_2)(t) \end{pmatrix},$$

where

$$A_i(u_1, u_2)(t) = \int_0^1 G_i(t, r) f_i(r, u_1(r), u_2(r)) dr,$$

$$B_i(u_1, u_2)(t) = g_i(t) \int_0^1 s^{\alpha-1} f_i(s, u_1(s), u_2(s)) ds.$$

First notice that the hypotheses H_1 and H_2 imply that the operators A and B are well defined. By [11], The function $u = (u_1, u_2) \in C^2$ is a solution of the system $(5.1) - (5.2)$ if, and only if, $Au(t) + Bu(t) = u(t)$ for all $t \in I$. Let $R > 0$ be such that

$$R > \sup\left\{\frac{4L}{(\alpha+\beta_1-2)\Gamma(\beta_1)\Gamma(\alpha)}, \frac{4L}{(\alpha+\beta_2-2)\Gamma(\beta_2)\Gamma(\alpha)}\right\}$$

where $L = \sup\{|f_i(t,0,0)|, 0 \le t \le 1, i = 1,2\}$, and consider the closed subset of $(C(I,E))^2$ defined by:

$$\mathcal{B}_R = \left\{(u,v) \in (C(I,E))^2; \|(u,v)\|_{C^2} \le \begin{pmatrix} R \\ R \end{pmatrix}\right\}.$$

Theorem 5.1. *Assume that hypotheses* $(H_1 - H_3)$ *hold. Let* U *be a weakly open subset of* $\mathcal{B}_R$. *If*

$$\frac{p_{i1}^* + p_{i2}^*}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} < \frac{1}{4} \tag{5.3}$$

for $i \in \{1,2\}$ *and if for any solution* (u,v) *of* $(u,v) = \lambda A(u,v) + \lambda B(\frac{u}{\lambda}, \frac{v}{\lambda})$ *with* $\lambda \in (0,1)$, *we have* $(u,v) \notin \partial_{\mathcal{B}_R} U$,*then the coupled system* $(5.1) - (5.2)$ *has at least one weak solution defined on* I.

Proof. We shall show that the operators A and B satisfies all the assumptions of Corollary 3.1. The proof will be given in several steps.
Step 1: A and B are relatively weakly compact. Let (u_n, v_n) be a sequence in $\mathcal{B}_R$ and let $(u_n(t), v_n(t)) \rightharpoonup (u(t), v(t))$ in $(E \times E, \omega)$ for each $t \in I$. Fix $t \in I$, since the functions $f_i, i = 1,2$ satisfy the assumption (H_1), we have $f_i(t, u_n(t), v_n(t))$ converge weakly uniformly to $f_i(t, u(t), v(t))$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies that $A(u_n, v_n)(t)$ (respectively $B(u_n, v_n)(t)$) converges weakly uniformly to $A(u,v)(t)$ (respectively $B(u,v)(t)$) in $(E \times E, \omega)$, for each $t \in I$. Thus, $A(u_n, v_n) \rightharpoonup A(u,v)$ and $B(u_n, v_n) \rightharpoonup B(u,v)$. Hence, A and B are weakly sequentially continuous.
Step 2: The operator A is relatively weakly compact. Let U be a weakly open subset of B_R such that $0 \in U$. Let $(u,v) \in \overline{U}^{\omega}$ be an arbitrary point. We shall prove $A(u,v) \in \mathcal{B}_R$. Fix $t \in I$ and consider $A(u,v)(t)$. Without loss of generality, we may assume that $A_i(u,v)(t) \neq 0$. By the Hahn-Banach Theorem there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ such that $\|A_i(u,v)(t)\|_E = \varphi(A_i(u,v)(t))$. Thus,

$$\begin{aligned}
\|A_i(u,v)(t)\|_E &\le \int_0^1 G_i(t,r)\varphi(f_i(r,u(r),v(r)))dr \\
&\le \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} \int_0^1 \varphi(f_i(r,u(r),v(r)) - f_i(r,0,0)) + f_i(r,0,0)dr \\
&\le \frac{1}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)}(p_{i1}^*\|u\|_E + p_{i2}^*\|v\|_E + L) \\
&\le \frac{p_{i1}^* R + p_{i2}^* R + L}{(\alpha+\beta_i-2)\Gamma(\beta_i)\Gamma(\alpha)} \\
&\le \frac{R}{2}
\end{aligned}$$

Let $(A_i(u_n,v_n))$ be any sequence in $A_i(\overline{U}^{\omega})$. Notice that $\overline{U}^{\omega}$ is bounded.
By reflexiveness, for each $t \in I$ the set $\{A_i(u_n,v_n)(t), n \in \mathbb{N}\}$ is relatively weakly compact. Let $(u,v) \in \overline{U}^{\omega}, 0 \le t \le s \le 1$, we have

$$\|A_i(u,v)(t) - A_i(u,v)(s)\|_E \le \int_0^t |G_i(t,r) - G_i(s,r)|\varphi(f_i(r,u(r),v(r)))dr$$

$$+ \int_t^s |G_i(t,r) - G_i(s,r)|\varphi(f_i(r,u(r),v(r)))dr$$

$$+ \int_s^1 |G_i(t,r) - G_i(s,r)|\varphi(f_i(r,u(r),v(r)))dr$$

$$\le \frac{L}{\Gamma(\beta_i)\Gamma(\alpha)}\left(\frac{3(s^{\beta_i-1} - t^{\beta_i-1})}{\beta_i - 1} + \frac{2((s^{\beta_i} - t^{\beta_i}) - (s-t)^{\beta_i})}{\beta_i} + 3(s-t)\right).$$

Consequently, $\|A_i(u,v)(t) - A_i(u,v)(s)\|_E \to 0$, when $t \mapsto s$, for all $i \in \{1,2\}$. One shows that $\{A(u_n,v_n); n \in \mathbb{N}\}$ is a weakly equicontinuous subset of C^2. It follows now from the Ascoli-Arzela Theorem that $(A(u_n,v_n))$ is relatively weakly compact.
Step 3: B is $M-$contraction and $B(\mathcal{B}_R)$ is bounded. Indeed, let $(u,v) \in \overline{U}^{\omega}$, then by using hypothesis (H_3) it yields

$$\|B_i(u_1,u_2)(t) - B_i(v_1,v_2)(t)\|_E$$

$$\le |g_i(t)| \int_0^1 s^{\alpha-1}\varphi(f_i(s,u_1(s),u_2(s)) - f_i(s,v_1(s),v_2(s)))ds$$

$$\le \frac{p_{i1}^*\|u_1 - v_1\|_C + p_{i2}^*\|u_2 - v_2\|_C}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)}$$

Then

$$\|B(u_1,u_2) - B(v_1,v_2)\|_{C^2} \le M\|u - v\|_{C^2}$$

where

$$M = \begin{pmatrix} \frac{p_{11}^*}{(\alpha+\beta_1-2)\Gamma(\beta_1)\Gamma(\alpha)} & \frac{p_{12}^*}{(\alpha+\beta_1-2)\Gamma(\beta_1)\Gamma(\alpha)} \\ \frac{p_{21}^*}{(\alpha+\beta_2-2)\Gamma(\beta_2)\Gamma(\alpha)} & \frac{p_{22}^*}{(\alpha+\beta_2-2)\Gamma(\beta_2)\Gamma(\alpha)} \end{pmatrix}$$

Also as in step 2, we have

$$\|B_i(u,v)(t)\|_E \le \frac{R}{2}$$

Step 4: Let $(u_1,u_2) \in \overline{U}^{\omega}$ and $(v_1,v_2) \in \mathcal{B}_R$.
It follows that $A(u_1,u_2) + B(v_1,v_2) \in \mathcal{B}_R$. Hence, the result follows.

Example 5.1. Let

$$E = l^1 = \{u = (u_1,u_2,\cdots,u_n,\cdots), \sum_{n=1}^{\infty}|u_n| < \infty\}$$

be the Banach space with the norm

$$\|u\|_E = \sum_{n=1}^{\infty}|u_n|.$$

We consider the following coupled fractional order system

$$\begin{cases} D_{1^-}^{1.2}(D_{0^+}^{1.9}u_n(t)) = f_n(t, u(t), v(t)) \\ D_{1^-}^{1.2}(D_{0^+}^{1.9}v_n(t)) = g_n(t, u(t), v(t)) \\ D_{0^+}^{1.9}u_n(0) = D_{0^+}^{1.9}u_n(1) = 0 \\ D_{0^+}^{1.9}v_n(0) = D_{0^+}^{1.9}v_n(1) = 0 \\ u_n'(1) = u_n(0) = 0,\ v_n'(1) = v_n(0) = 0, \end{cases} \tag{5.4}$$

$(\alpha = 1.2, \beta_1 = \beta_2 = 1.9)$, where

$$f_n(t, u(t), v(t)) = \frac{c}{n^2}\left(te^{-7}u_n(t) + \frac{e^{-(t+5)}}{1 + v_n(t)}\right),$$

and

$$g_n(t, u(t), v(t)) = \frac{c}{n^2}\left(\frac{te^{-6}}{1 + v_n(t)}\right),\ t \in I$$

with

$$u = (u_1, u_2, \cdots, u_n, \cdots),\ v = (v_1, v_2, \cdots, v_n, \cdots),\ c := \frac{0.1e^4}{4}\Gamma(1.2)\Gamma(1.9).$$

Set

$$f = (f_1, f_2, \cdots, f_n, \cdots) \text{ and } g = (g_1, g_2, \cdots, g_n, \cdots).$$

Clearly the functions f and g are continuous. For each $u, v \in E$ and $t \in I$, we have

$$\|f(t, u_1(t), u_2(t)) - f(t, v_1(t), v_2(t))\|_E \le c(e^{-7}\|u_1(t) - v_1(t)\| + e^{-(t+5)}\|u_2(t) - v_2(t)\|),$$

$$\|g(t, u_1(t), u_2(t)) - g(t, v_1(t), v_2(t))\|_E \le cte^{-6}\|u_2(t) - v_2(t)\|$$

and

$$L = \frac{c\pi^2}{6}e^{-5}.$$

Hence, the hypothesis (H_3) is satisfied with $p_{11}^* = ce^{-7}$, $p_{12}^* = ce^{-5}$, $p_{21}^* = 0$ and $p_{22}^* = ce^{-6}$. We shall show that condition (5.3) holds. Indeed:

$$\sup_{i=1,2}\left\{\frac{p_{i1}^* + p_{i2}^*}{(\alpha + \beta_i - 2)\Gamma(\beta_i)\Gamma(\alpha)}\right\} < \frac{1}{8}$$

So, all conditions of Theorem 5.1 are satisfied. Let now U be a weakly subset of $\mathcal{B}_R$, $(R > \frac{\pi^2}{6e})$. Then the coupled system (5.4) has at least one solution (u, v) in $\mathcal{B}_R$ or for any solution (u, v) of $(u, v) = \lambda A(u, v) + \lambda B(\frac{u}{\lambda}, \frac{v}{\lambda})$ with $\lambda \in (0, 1)$, we have $(u, v) \notin \partial_{\mathcal{B}_R}U$. $\square$

References

[1] J. Appell, E. De Pascale, *Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi di funzioni misurabili*, Boll. Un. Mat. Ital., **B(6)**(1984), no. 3, 497-515.

[2] O. Arino, S. Gautier, J.P. Penot, *A fixed point theorem for sequentially continuous mappings with applications to ordinary differential equations*, Funkc. Ekvacial, **27**(1984), 273-279.

[3] C.S. Barroso, E.V. Teixeira, *A topological and geometric approach to fixed point results for sum of operators and applications*, Nonlinear Anal., **60**(2005), no. 4, 625-660.

[4] A. Ben Amar, I. Feki, A. Jerbi, *Leray-Shauder and Furi-Pera types fixed point theorems for the sum of two weakly sequentially continuous mappings and application to transport equation*, Afrika Mat., **25**(2014), 707-722.

[5] A. Ben Amar, A. Jeribi, M. Mnif, *On a generalization of the Schauder and Krasnoselskii fixed point theorems on Dunford-Pettis spaces and applications*, Math. Methods. Appl. Sci., **28**(2005), 1737-1756.

[6] A. Ben Amar, M. Mnif, *Leray-Schauder alternatives for weakly sequentially continuous mappings and application to transport equation*, Math. Methods Appl. Sci., **33**(2010), 80-90.

[7] T. Cardinali, R. Precup, P. Rubbioni, *Two abstract approaches in vectorial fixed point theory*, Quaest. Math., **41**(2018), 173-188.

[8] F.S. De Blasi, *On a property of the unit sphere in Banach spaces*, Bull. Math. Soc. Sci. Math. Roumanie, **21**(1977), 259-262.

[9] N. Dunford, J.T. Schwartz, *Linear Operators, Part I: General Theory*, Interscience Publishers, New York, 1958.

[10] R.E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Reinhart and Winston, New York-Toronto-London, 1965.

[11] A. Guezane-Lakoud, S. Ramdane, *Existence of solutions for a system of mixed fractional differential equations. Journal of Taibah University for Science*, **12**(2018), no. 4, 421-426.

[12] I.M. James, *Topological Vector Spaces*, Springer, New York, 1987.

[13] M.A. Krasnosel'skii, *Two remarks on the method of successive approximations*, (Russian), Uspehi Mat. Nauk, **10**(1955), 123-127.

[14] K. Latrach, M.A. Taoudi, *Existence results for a generalized nonlinear Hammerstein equation on L_1 spaces*, Nonlinear Anal., **66**(2007), 2325-2333.

[15] K. Latrach, M.A. Taoudi, A. Zeghal, *Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations*, J. Diff. Equ., **221**(2006), 256-271.

[16] A.I. Perov, *On the Cauchy problem for a system of ordinary differential equations*, (Russian), Priblizhen. Metody Reshen. Differ. Uravn., **2**(1964), 115-134.

[17] I.-R. Petre, A. Petrusel, *Krasnoselskii's theorem in generalized Banach spaces and applications*, Electron. J. Qual. Theory Differ. Equ., **85**(2012), 20 pp.

[18] B.J. Pettis, *On integration in vector spaces*, Trans. Am. Math. Soc., **44**(1938), 277-304.

[19] R. Precup, *Methods in Nonlinear Integral Equations*, Kluwer, Dordrecht, 2002.

[20] R. Precup, *The role of matrices that are convergent to zero in the study of semilinear operator systems*, Mathematical and Computer Modelling, **49**(2009), no. 3-4, 703-708.

[21] H.H. Schaefer, *Neue existenzsä tze in der theorie nichtlinearer integragleichungen*, Sitzungsdh. Sächs. Akademie Wiss. Leipzig Matematisch-Naturwissenchaftliche, 1955, 101:7-40.

[22] A. Viorel, *Contributions to the study of nonlinear evolution equations*, Ph.D. Thesis, Babeş-Bolyai University, Cluj-Napoca, 2011.

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