Fixed Point Theory, 23(2022), No. 1, 75-84
DOI: 10.24193/fpt-ro.2022.1.05
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ON ASYMPTOTICALLY NONEXPANSIVE MAPPINGS WITH NON-CONVEX DOMAINS

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Abstract. In this paper, we prove the existence of periodic points of asymptotically nonexpansive mappings defined on non-convex domains in uniformly convex Banach spaces. We also introduce the notion of firmly asymptotically nonexpansive mappings and prove the existence of fixed points of such mappings with non-convex domains.
Key Words and Phrases: Asymptotically nonexpansive mapping, fixed point, periodic point.
2020 Mathematics Subject Classification: 46B20, 47E10, 47H10.

1. Introduction

The notion of asymptotically nonexpansive mapping, which is a natural extension of the idea of nonexpansive mapping, was first introduced by Goebel and Kirk [5] in 1972. There, the authors showed the existence of fixed points of any such mapping acting on a nonempty, closed, convex domain of a uniformly convex Banach space. Many researchers have extended this fixed point result and have proposed iteration schemes for this kind of mappings, see for instance [3, 8, 9, 10, 13, 12, 14]. Most recently, Alfuraidan and Khamsi [1] established an analog to the fixed point result of [5] for monotone asymptotically nonexpansive mappings. All these results concern mappings acting on a convex set. The classical fixed point result discussed in [5] does not hold on non-convex domain.

The discussion on fixed points of (firmly) nonexpansive mappings in a non-convex domain has been undertaken in many studies, see for instance [2, 6, 7, 11] and some references therein.

In the present work we deal with the non-convex situation and are, in fact, able to prove the existence of periodic points of asymptotically nonexpansive mapping acting on a non-convex subset of an arbitrary, uniformly convex Banach space. In particular, we prove the existence of fixed point of such mappings (with asymptotic regularity) in a non-convex domain. Furthermore, we introduce the notion of λ-firmly asymptotically nonexpansive mapping and prove the existence of fixed points of such mapping in a non-convex domain.

2. Preliminaries

We start with recalling the definition of asymptotically nonexpansive mapping introduced in [5].

Definition 2.1. [5] Let D be a nonempty subset of a Banach space $(X, \|.\|)$. A mapping $T : D \to D$ is said to be asymptotically nonexpansive if there exists a sequence of positive numbers $\{k_p\}$ such that $\lim\limits_{p\to\infty} k_p = 1$ and

$$\|T^p x - T^p y\| \leq k_p \|x - y\|,$$

for every $x, y \in D$. A point $x \in D$ is said to be a fixed point of T if $T(x) = x$; x is said to be a periodic point of T if x is a fixed point of an iterate of T, i.e., if $T^p(x) = x$, for some integer $p \geq 1$.

The concept of asymptotic regularity is due to Browder and Petryshyn [4].
Specifically,

Definition 2.2. [4] Led D be a nonempty subset of a Banach space $(X, \|.\|)$. A mapping $T : D \to D$ is said to be asymptotically regular if for each $x \in D$,

$$\lim_{p\to\infty} \left\|T^p x - T^{p+1} x\right\| \to 0. \tag{2.1}$$

We will use the following technical lemma in the proof of the main theorems.

Lemma 2.1. [1] *Let C be a nonempty, closed and convex subset of a uniformly convex Banach space $(X, \|.\|)$. Let $\tau : C \to [0, \infty)$ be a type function, i.e., assume there exists a bounded sequence $\{x_n\} \in X$ such that*

$$\tau(x) = \limsup_{n\to\infty} \|x_n - x\|,$$

for every $x \in C$. Then τ has a unique minimum point $z \in C$ such that

$$\tau(z) = \inf\{\tau(x);\ x \in C\}.$$

Moreover, if $\{z_n\}$ is a minimizing sequence in C, i.e., if $\lim\limits_{n\to\infty} \tau(z_n) = \tau(z)$, then $\{z_n\}$ converges strongly to z.

3. Periodic points of asymptotically nonexpansive mappings

As stated in the introduction, the classical fixed point theorem for asymptotically nonexpansive mappings [5] does not hold for mappings acting on non-convex domains. Nonetheless, the following theorem shows that such mappings do possess periodic points.

Theorem 3.1. *Let $(X, \|.\|)$ be a uniformly convex Banach space and $C = \bigcup_{i=1}^{m} C_i$ be a union of nonempty, closed, convex and bounded subsets of X. If $T : C \to C$ is an asymptotically nonexpansive mapping, then T has a periodic point.*

Proof. Let $\{k_p\}$ be the Lipschitz constants associated to the asymptotically nonexpansive mapping T. Without loss of generality, we may assume that $k_p \geq 1$ for every $p \in \mathbb{N}$. Let $x_0 \in C$ and consider the type function generated by $\{T^n(x_0)\}$, i.e.,

$$\tau(x) = \limsup_{n\to\infty} \|T^n(x_0) - x\| .$$

Lemma 2.1 implies the existence of a unique minimum point $z_i \in C_i$ for all $1 \leq i \leq m$, i.e.

$$\tau(z_i) = \inf\{\tau(x);\ x \in C_i\}.$$

We will split the proof in the following two cases:

Case 1. There exists some $i \in \{1, 2, \dots, m\}$ such that a subsequence $\{T^{\varphi_i(p)}(z_i)\}$ of $\{T^p(z_i)\}$ is in C_i, for all $p \in \mathbb{N}$.

The asymptotic nonexpansive behavior of T implies the following inequality:

$$\|T^n(x_0) - T^{\varphi_i(p)}(z_i)\| \leq k_{\varphi_i(p)} \|T^{n-\varphi_i(p)}(x_0) - z_i\|,$$

for every $n \geq \varphi_i(p)$. Letting $n \to \infty$, it follows that

$$\tau(z_i) \leq \tau(T^{\varphi_i(p)}(z_i)) \leq k_{\varphi_i(p)} \tau(z_i).$$

The condition $\lim\limits_{p\to\infty} k_{\varphi_i(p)} = 1$ yields

$$\lim_{p\to\infty} \tau(T^{\varphi_i(p)}(z_i)) = \tau(z_i),$$

which means that $\{T^{\varphi_i(p)}(z_i)\}$ is a minimizing sequence of τ in C_i. By virtue of Lemma 2.1, $\{T^{\varphi_i(p)}(z_i)\}$ converges to z_i. Let $q \geq 1$ be the smallest positive integer such that $T^q(z_i) \in C_i$. Since τ and T^q are continuous, it follows that

$$T^q(z_i) = T^q \left(\lim_{p\to\infty} T^{\varphi_i(p)}(z_i) \right) = \lim_{p\to\infty} T^{q+\varphi_i(p)}(z_i),$$

and therefore that $\tau(T^q(z_i)) = \lim\limits_{p\to\infty} \tau(T^{q+\varphi_i(p)}(z_i))$. The inequality

$$\tau\left(T^q(z_i)\right) \leq \limsup_{p\to\infty}\ k_{q+\varphi_i(p)}\ \tau(z_i) = \tau(z_i),$$

yields $\tau(z_i) = \tau(T^q(z_i))$, since $T^q(z_i) \in C_i$. On account of the uniqueness of the minimum point of τ in C_i it follows that $T^q(z_i) = z_i$. In all, z_i is a periodic point of T.

Case 2. For every $i \in \{1, 2, \ldots, m\}$, C_i contains only a finite number of elements of the orbit $\{T^p(z_i)\}$.

In this case, there exist integers $\{m_1, m_2, \ldots, m_r\}$ in $\{1, 2, \ldots, m\}$ such that a subsequence $\{T^{\varphi_i(p)}(z_{m_i})\}$ of $\{T^p(z_{m_i})\}$ lies in $C_{m_{i+1}}$, $i = 1, 2, \ldots, r-1$, and a subsequence $\{T^{\varphi_r(p)}(z_{m_r})\}$ of $\{T^p(z_{m_r})\}$ is contained in C_{m_1}. Without lost of generality, the sequence C_{m_i} can be arranged in such a way that $m_i = i$ for all $i = 1, 2, \ldots, r-1$. In other words, we have $\{T^{\varphi_i(p)}(z_i)\}$ is in C_{i+1}, for $i = 1, 2, \ldots, r-1$, and $\{T^{\varphi_r(p)}(z_r)\}$ is in C_1. Since T is asymptotically nonexpansive, the following inequalities hold

$$\begin{aligned}
\tau(z_1) \leq& \tau\left(T^{\varphi_r(p)}(z_r)\right) \leq k_{\varphi_r(p)}\tau(z_r) \\
\tau(z_r) \leq& \tau\left(T^{\varphi_{r-1}(p)}(z_{r-1})\right) \leq k_{\varphi_{r-1}(p)}\tau(z_{r-1}) \\
&\vdots \\
\tau(z_3) \leq& \tau\left(T^{\varphi_2(p)}(z_2)\right) \leq k_{\varphi_2(p)}\tau(z_2) \\
\tau(z_2) \leq& \tau\left(T^{\varphi_1(p)}(z_1)\right) \leq k_{\varphi_1(p)}\tau(z_1),
\end{aligned}$$

$\forall\, p \in \mathbb{N}$. Since $\lim\limits_{p\to\infty} k_p = 1$, we obtain

$$\tau(z_1) \leq \tau(z_r) \leq \tau(z_{r-1}) \leq \cdots \leq \tau(z_2) \leq \tau(z_1),$$

which implies $\tau(z_1) = \tau(z_2) = \cdots = \tau(z_{r-1}) = \tau(z_r)$. Moreover, one has

$$\tau(z_{i+1}) \leq \lim_{p\to\infty} \tau\left(T^{\varphi_i(p)}(z_i)\right) \leq \tau(z_i),$$

$\forall\; i = 1, 2, \ldots, r$, where $z_{r+1} = z_1$. Using Lemma 2.1, we conclude that $\{T^{\varphi_i(p)}(z_i)\}$ converges to z_{i+1}, $\forall\; i = 1, 2, \ldots, r$. Let $q_i \geq 1$ be the smallest integer such that $T^{q_i}(z_i) \in C_{i+1}$, for $i = 1, 2, \ldots, r$. As in Case 1, the continuity of both τ and T^p, $p \in \mathbb{N}$, allows us to conclude that

$$\begin{aligned}
\tau(z_{i+1}) \leq \tau\left(T^{q_i}(z_i)\right) &= \lim_{p\to\infty} \tau\left(T^{q_i+\varphi_{i-1}(p)}(z_{i-1})\right), \\
&\leq \lim_{p\to\infty} k_{q_i+\varphi_{i-1}(p)}\tau(z_{i-1}) \\
&\leq \tau(z_{i-1}) = \tau(z_{i+1}),
\end{aligned}$$

for $i = 2, \ldots, r$, and that

$$\begin{aligned}
\tau(z_2) \leq \tau\left(T^{q_1}(z_1)\right) &= \lim_{p\to\infty} \tau\left(T^{q_1+\varphi_r(p)}(z_r)\right) \\
&\leq \lim_{p\to\infty} k_{q_1+\varphi_r(p)}\tau(z_r) \\
&\leq \tau(z_r) = \tau(z_2).
\end{aligned}$$

Hence $\tau\left(T^{q_i}(z_i)\right) = \tau(z_{i+1})\ \forall\ i = 1, 2, \ldots, r$. An application of Lemma 2.1, yields $T^{q_i}(z_i) = z_{i+1}$, $\forall i = 1, 2, \ldots, r$. Let $q = \sum\limits_{i=1}^{r} q_i$, we have

$$\begin{aligned}
T^q(z_1) &= T^{q_r+q_{r-1}+\cdots+q_2+q_1}(z_1)\\
&= T^{q_r+q_{r-1}+\cdots+q_2}(T^{q_1}(z_1))\\
&= T^{q_r+q_{r-1}+\cdots+q_2}(z_2)\\
&= \ldots\\
&= T^{q_r}(T^{q_{r-1}}(z_{r-1}))\\
&= T^{q_r}(z_r) = z_{r+1} = z_1.
\end{aligned}$$

Hence, z_1 is a periodic point of T. □

Next, we investigate the conditions under which a periodic point is in fact a fixed point. This is not true in general, as the next example shows:

Example 3.1. Let $X = \mathbb{R}^n$, endowed with the Euclidean norm and set

$$\{e_i;\ i = 1, \cdots, n\}$$

to be the canonical basis of X. Then, $\|e_i - e_j\| = \sqrt{2}$, for any $i \neq j$ in $\{1, \cdots, n\}$. Let $C = \{e_i;\ i = 1, \cdots, n\} = \bigcup\limits_{i=1}^{n} \{e_i\}$ and $T : C \to C$ defined by

$$T(e_i) = e_{i+1},\ \ i = 1, \cdots, n-1,\ \ and\ \ T(e_n) = e_1.$$

It is easy to check that T is an isometry which obviously implies that T is asymptotically nonexpansive. However, when $n \geq 2$, T doesn't have any fixed point.

This example shows that one cannot expect the existence of a fixed point under the mere assumptions of Theorem 3.1. In order to discuss additional conditions guaranteeing the existence of fixed points, we first observe that there is nothing special about the mapping defined in Example 3.1. Let start with the following technical lemma.

Lemma 3.1. *Let C be a subset of a Banach space $(X, \|.\|)$ and $T : C \to C$ be an asymptotically nonexpansive mapping. If z_1 is a periodic point of T, then T is an isometry on the orbit of z_1.*

Proof. Let q be the period of z_1, i.e., $T^q(z_1) = z_1$. Set $z_i = T^{i-1}(z_1),\ i = 1, 2, \ldots, q$. Clearly, T^q is the identity mapping when restricted to $\{z_1, \cdots, z_q\}$.
Fix $i, j \in \{1, 2, \ldots, q\}$. Since T is asymptotically nonexpansive, we get

$$\begin{aligned}
\|T(z_i) - T(z_j)\| &= \|T^{pq}T(z_i) - T^{pq}T(z_j)\|\\
&\leq k_{pq+1}\|z_i - z_j\|,
\end{aligned}$$

and

$$\begin{aligned}
\|z_i - z_j\| &= \|T^{pq}(z_i) - T^{pq}(z_j)\|\\
&\leq k_{pq-1}\|T(z_i) - T(z_j)\|,
\end{aligned}$$

for all $p \geq 1$. Moreover, it follows from $\lim\limits_{n\to\infty} k_n = 1$, that

$$\|T(z_i) - T(z_j)\| \leq \|z_i - z_j\| \leq \|T(z_i) - T(z_j)\|.$$

The last inequality completes the proof of Lemma 3.1. $\square$

To the best of our knowledge, the following result on the existence of fixed points of asymptotically nonexpansive mappings defined on non-convex domains is new.

Theorem 3.2. *Let $(X, \|.\|)$ be a uniformly convex Banach space. Let $C = \bigcup\limits_{i=1}^{m} C_i$ be the union of nonempty, closed, convex and bounded subsets of X. If $T : C \to C$ is asymptotically nonexpansive and asymptotically regular, then any periodic point of T is a fixed point. In particular, T has a fixed point in C.*

Proof. Let z_1 be a periodic point of T with period q, that is, assume that $T^q(z_1) = z_1$. On account of Lemma 3.1, it is clear that

$$\begin{aligned}\|T(z_1) - z_1\| &= \|T^{q+1}(z_1) - T^q(z_1)\| \\ &= \|T^{pq+1}(z_1) - T^{pq}(z_1)\|,\end{aligned}$$

for any $p \geq 1$. The asymptotic regularity of T implies that

$$\|T(z_1) - z_1\| = \lim_{p\to\infty} \|T^{pq+1}(z_1) - T^{pq}(z_1)\| = 0.$$

Hence, z_1 is a fixed point of T. Theorem 3.1 implies such periodic points do exist. The existence of fixed points of T has thus been proved. $\square$

Another class of mappings for which the conclusion of Theorem 3.2 holds is given in the next section.

4. Firmly asymptotically nonexpansive mappings

In this section, a new concept, referred to as λ-firmly asymptotic nonexpansiveness is introduced and a fixed point theorem for such mappings acting on non-convex domains in uniformly convex Banach spaces is proved.

Definition 4.1. Let C be a subset of a Banach space $(X, \|.\|)$. The mapping $T : C \to C$ is said to be λ-firmly asymptotically nonexpansive, for some $\lambda \in (0, 1)$, if there exists a sequence of positive numbers $\{k_p\}$ such that $\lim\limits_{p\to\infty} k_p = 1$ and

$$\|T^p(x) - T^p(y)\| \leq k_p \|(1 - \lambda)(x - y) + \lambda(T^p(x) - T^p(y))\|,$$

for every $x, y \in C$.

We next present a remarkable result about λ-firmly asymptotically nonexpansive mappings, as it connects periodic points and fixed points.

Theorem 4.1. *Let C be a nonempty subset of a Banach space $(X, \|.\|)$. Let $T : C \to C$ be a λ-firmly asymptotically nonexpansive mapping, for some $\lambda \in (0, 1)$. Assume that $x \in C$ is a periodic point of T, i.e., $T^q(x) = x$, for some $q \geq 1$. Then x is a fixed point of T.*

Proof. Since T is λ-firmly asymptotically nonexpansive, there exists a sequence of positive numbers $\{k_p\}$ such that $\lim\limits_{p\to\infty} k_p = 1$ and for which

$$\|T^p(x) - T^p(y)\| \leq k_p \; \|(1-\lambda)(x-y) + \lambda(T^p(x) - T^p(y))\|$$

for every $x, y \in C$. Assume that $x \in C$ is a periodic point of T of order q. Set $x_1 = x$, $x_i = T^{i-1}(x)$, for $i \geq 2$. Then, $x_{i+q} = x_i$, for $i \geq 1$, which implies

$$T^{pq}(x_i) = x_{i+pq} = x_i,$$

for $i \geq 1$ and $p \in \mathbb{N}$. Hence for any $i, j \geq 1$ and $p \in \mathbb{N}$, we have

$$\begin{aligned} \|T(x_i) - T(x_j)\| &= \|T^{pq+1}(x_i) - T^{pq+1}(x_j)\| \\ &\leq k_{pq+1} \; \|(1-\lambda)(x_i - x_j) + \lambda(T^{pq+1}(x_i) - T^{pq+1}(x_j))\| \\ &\leq k_{pq+1} \; \|(1-\lambda)(x_i - x_j) + \lambda(T(x_i) - T(x_j))\|. \end{aligned}$$

Letting $p \to \infty$ and observing that $\lim\limits_{p\to\infty} k_p = 1$, it is easy to see that

$$\|T(x_i) - T(x_j)\| \leq \|(1-\lambda)(x_i - x_j) + \lambda(T(x_i) - T(x_j))\|, \tag{FN}$$

which implies $\|T(x_i) - T(x_j)\| \leq \|x_i - x_j\|$, for any $i, j \geq 1$. It follows that

$$\|x_i - x_j\| = \|x_{i+q} - x_{j+q}\| \leq \|x_{i+1} - x_{j+1}\| = \|T(x_i) - T(x_j)\|,$$

and consequently

$$\|x_{i+1} - x_{j+1}\| = \|T(x_i) - T(x_j)\| = \|x_i - x_j\|,$$

for any $i, j \geq 1$. In other words, the restriction of T to the orbit $\{x_i\}$ of x is an isometry. Define $T_\lambda = (1-\lambda)\, I + \lambda\, T$, where I is the identity map. Inequality (FN) yields

$$\|x_i - x_j\| = \|T(x_i) - T(x_j)\| \leq \|T_\lambda(x_i) - T_\lambda(x_j)\| \leq \|x_i - x_j\|,$$

for any $i, j \geq 1$. Hence, the restriction of T_λ to $\{x_i\}$ is also an isometry. Set

$$R_k = \|x_1 - x_{k+1}\| = \|x_1 - T^k(x_1)\|,$$

for any $k \geq 1$. We claim that $R_k = k\, R_1$, for any $k \geq 1$. Indeed, the relation is true for $k = 1$. Assume that it is true for all $1 \leq k \leq n$. Then

$$\begin{aligned} R_n &= \|x_1 - x_{n+1}\| \\ &= \|T_\lambda x_1 - T_\lambda x_{n+1}\| \\ &\leq (1-\lambda)\|x_1 - T_\lambda x_{n+1}\| + \lambda\|x_2 - T_\lambda x_{n+1}\| \\ &\leq (1-\lambda)^2\|x_1 - x_{n+1}\| + \lambda(1-\lambda)\|x_1 - x_{n+2}\| \\ &\qquad + \lambda(1-\lambda)\|x_2 - x_{n+1}\| + \lambda^2\|x_2 - x_{n+2}\| \\ &= (1-\lambda)^2 R_n + \lambda(1-\lambda)R_{n+1} + \lambda(1-\lambda)R_{n-1} + \lambda^2 R_n, \end{aligned}$$

which implies

$$\Big(n - (1-\lambda)^2 n - \lambda(1-\lambda)(n-1) - \lambda^2 n\Big)\; R_1 \leq \lambda(1-\lambda)R_{n+1}.$$

In other words,

$$\lambda(1-\lambda)(n+1)R_1 \leq \lambda(1-\lambda)R_{n+1}.$$

Since $\lambda \in (0,1)$,

$$\begin{aligned}(n+1)R_1 &\leq R_{n+1}\\ &= \|x_1 - x_{n+2}\|\\ &\leq \|x_1 - x_{n+1}\| + \|x_{n+1} - x_{n+2}\|\\ &\leq \|x_1 - x_{n+1}\| + \|x_1 - x_2\|\\ &= R_n + R_1 = nR_1 + R_1 = (n+1)R_1.\end{aligned}$$

Hence, $R_{n+1} = (n+1)R_1$. The induction argument completes the proof of the claim, that is, $R_n = n\ R_1$ for any $n \geq 1$. It follows thus

$$R_q = \|x_1 - x_{q+1}\| = \|x_1 - x_1\| = q\ \|x_1 - x_2\|,$$

which implies $x_1 = x_2 = T(x_1)$, i.e., $x_1 = x$ is a fixed point of T. □

Next we discuss an analogue to Theorem 3.1 for λ-firmly asymptotically nonexpansive mappings.

Theorem 4.2. *Let $(X, \|.\|)$ be a uniformly convex Banach space and $C = \bigcup_{i=1}^{m} C_i$ be the union of nonempty, closed, convex and bounded subsets of X. If $T : C \to C$ is a λ-firmly asymptotically nonexpansive mapping, for some $\lambda \in (0,1)$, then T has a periodic point.*

Proof. Since T is λ-firmly asymptotically nonexpansive, there exists a sequence of positive numbers $\{k_p\}$ such that $\lim\limits_{p\to\infty}\ k_p = 1$ and

$$\|T^p(x) - T^p(y)\| \leq k_p\ \|(1-\lambda)(x-y) + \lambda(T^p(x) - T^p(y))\|$$

for every $x, y \in C$. In view of the fact that $\lim\limits_{p\to\infty}\ \lambda\ k_p = \lambda < 1$, there exists $p_0 \geq 1$ such that $\lambda\ k_p < 1$, for any $p \geq p_0$. In this case, we have

$$\|T^p(x) - T^p(y)\| \leq \kappa_p \|x - y\|,$$

where

$$\kappa_p = \frac{k_p(1-\lambda)}{1-\lambda k_p},$$

for $p \geq p_0$. Note that $\lim\limits_{p\to\infty}\ \kappa_p = 1$. This implies that T^p is asymptotically nonexpansive, for any $p \geq p_0$. Using Theorem 3.1, we conclude that T^p has a periodic point, for any $p \geq p_0$, which is also a periodic point of T. □

A combination of the preceding two theorems in tandem yield the following fixed point result:

Theorem 4.3. *Let $(X, \|.\|)$ be a uniformly convex Banach space and $C = \bigcup_{i=1}^{m} C_i$ be the union of nonempty, closed, convex and bounded subsets of X. If $T : C \to C$ is a λ-firmly asymptotically nonexpansive mapping, for some $\lambda \in (0,1)$, then T has a fixed point.*

References

[1] M.R. Alfuraidan, M.A. Khamsi, *A fixed point theorem for monotone asymptotic nonexpansive mapping*, Proc. Amer. Math. Soc., **146**(2018), 2451-2456.

[2] D. Ariza-Ruiz, L. Leustean, G. Lopez-Acedo, *Firmly nonexpansive mappings in classes of geodesic spaces*, Trans. Amer. Math. Soc., **366**(2014), no. 8, 4299-4322.

[3] S.C. Bose, *Weak convergence to the fixed point of an asymptotically nonexpansive map*, Proc. Amer. Math. Soc., **68**(1978), 305-308.

[4] F.E. Browder, W.V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc., **72**(1966), 571-575.

[5] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**(1972), 171-174.

[6] Y.M. Hong, Y.Y. Huang, *On λ-firmly nonexpansive mappings in non-convex sets*, Bull. Inst. Math. Acad. Sinica, **21**(1993), no. 1, 35-42.

[7] W. Kaczor, *Fixed points of λ-firmly nonexpansive mappings on non-convex sets*, Nonlinear Anal., **47**(2001), 2787-2792.

[8] B. Nanjaras, B. Panyanak, *Demiclosed principle for asymptotically nonexpansive mappings in CAT spaces*, Fixed Point Theory Appl., **2010**(2010).

[9] G.B. Passty, *Construction of fixed points for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **84**(1982), 213-216.

[10] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., **43**(1991), no. 1, 153-159.

[11] R. Smarzewski, *On firmly nonexpansive mappings*, Proc. Amer. Math. Soc., **113**(1991), no. 3, 723-725.

[12] K.K. Tan, H.K. Xu, *Fixed point iteration processes for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **122**(1994), 733-739.

[13] H.K. Xu, *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Anal.: Theory, Methods & Applications, **12**(16)(1991), 1139-1146.

[14] B.L. Xu, M.A. Noor, *Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **267**(2002), no. 2, 444-453.

Received: July 30, 2020; Accepted: October 22, 2020.