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COUPLED HILFER AND HADAMARD FRACTIONAL DIFFERENTIAL SYSTEMS IN GENERALIZED BANACH SPACES

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Abstract. This article deals with some existence and uniqueness of solutions for some coupled systems of Hilfer and Hilfer-Hadamard fractional differential equations. Some applications are made of generalizations of classical fixed point theorems on generalized Banach spaces.
Key Words and Phrases: Fractional differential equation, left-sided mixed Riemann-Liouville integral of fractional order, left-sided mixed Hadamard integral of fractional order, Hilfer fractional derivative, Hadamard fractional derivative, coupled system, generalized Banach space, fixed point.
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1. Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [10, 23]. For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs of Abbas *et al.* [1, 3, 4], Samko *et al.* [21], Kilbas *et al.* [13] and Zhou *et al.* [28], and the papers [2, 6], and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative [7, 8, 10, 11, 24, 27], and the references therein.

In this paper we discuss the existence and uniqueness of solutions for the following coupled system of Hilfer fractional differential equations

$$\begin{cases}(D_0^{\alpha_1,\beta_1}u)(t) = f_1(t,u(t),v(t)) \\ (D_0^{\alpha_2,\beta_2}v)(t) = f_2(t,u(t),v(t))\end{cases} \; ; \; t \in I := [0,T], \tag{1.1}$$

with the following initial conditions

$$\begin{cases}(I_0^{1-\gamma_1}u)(0)=\phi_1\\(I_0^{1-\gamma_2}v)(0)=\phi_2,\end{cases}\tag{1.2}$$

where $T>0,\ \alpha_i\in(0,1),\ \beta_i\in[0,1],\ \gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i,\ \phi_i\in\mathbb{R}^m,\ f_i:I\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m;\ i=1,2,$ are given functions, $I_0^{1-\gamma_i}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma_i$, $\mathbb{R}^m;\ m\in\mathbb{N}^*$ is the Euclidian Banach space with a suitable norm $\|\cdot\|$, and $D_0^{\alpha_i,\beta_i}$ is the generalized Riemann-Liouville derivative (Hilfer) operator of order α_i and type $\beta_i:\ i=1,2$.

Next, we consider the following coupled system of Hilfer-Hadamard fractional differential equations

$$\begin{cases}({}^HD_1^{\alpha_1,\beta_1}u)(t)=g_1(t,u(t),v(t))\\({}^HD_1^{\alpha_2,\beta_2}v)(t)=g_2(t,u(t,),v(t))\end{cases}\ ;\ t\in[1,T],\tag{1.3}$$

with the following initial conditions

$$\begin{cases}({}^HI_1^{1-\gamma_1}u)(1)=\psi_1\\({}^HI_1^{1-\gamma_2}v)(1)=\psi_2,\end{cases}\tag{1.4}$$

where $T>1,\ \alpha_i\in(0,1),\ \beta_i\in[0,1],\ \gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i,\ \psi_i\in\mathbb{R}^m,\ g_i:[1,T]\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}^m;\ i=1,2$ are given functions, ${}^HI_1^{1-\gamma_i}$ is the left-sided mixed Hadamard integral of order $1-\gamma_i$, and ${}^HD_1^{\alpha_i,\beta_i}$ is the Hilfer-Hadamard fractional derivative of order α_i and type $\beta_i;\ i=1,2$.

2. Preliminaries

Let C be the Banach space of all continuous functions from I into $\mathbb{R}^m$ with the supremum (uniform) norm $\|\cdot\|_\infty$. As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}^m$. By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v:I\to\mathbb{R}^m$ with the norm

$$\|v\|_1=\int_0^T\|v(t)\|dt.$$

By $C_\gamma(I)$ and $C_\gamma^1(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I)=\{w:(0,T]\to\mathbb{R}^m:t^{1-\gamma}w(t)\in C\},$$

with the norm

$$\|w\|_{C_\gamma}:=\sup_{t\in I}\|t^{1-\gamma}w(t)\|,$$

and

$$C_\gamma^1(I)=\{w\in C:\frac{dw}{dt}\in C_\gamma\},$$

with the norm

$$\|w\|_{C_\gamma^1}:=\|w\|_\infty+\|w'\|_{C_\gamma}.$$

Also, by $\mathcal{C} := C_{\gamma_1} \times C_{\gamma_2}$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathcal{C}} = \|u\|_{C_{\gamma_1}} + \|v\|_{C_{\gamma_2}}.$$

Now, we give some results and properties of fractional calculus.

Definition 2.1. [3, 13, 21] The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$(I_0^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s)ds; \ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt; \ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t); \ for\ a.e.\ t \in I.$$

Definition 2.2. [3, 13, 21] The Riemann-Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned} (D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s)ds; \ for\ a.e.\ t \in I. \end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r w)(t) = w(t); \ for\ all\ t \in (0,T].$$

Moreover, if $I_0^{1-r} w \in C_{1-\gamma}^1(I)$, then the following composition is proved in [21]

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r} w)(0^+)}{\Gamma(r)} t^{r-1}; \ for\ all\ t \in (0,T].$$

Definition 2.3. [3, 13, 21] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned} ({}^c D_0^r w)(t) &= \left(I_0^{1-r} \frac{d}{dt} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \int_0^t (t-s)^{-r} \frac{d}{ds} w(s)ds; \ for\ a.e.\ t \in I. \end{aligned}$$

In [10], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as specific cases (see also [11, 24].

Definition 2.4. (Hilfer derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, and $I_0^{(1-\alpha)(1-\beta)} w \in AC(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{(1-\alpha)(1-\beta)} w \right)(t); \ for\ a.e.\ t \in I. \tag{2.1}$$

Properties. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.

1. The operator $(D_0^{\alpha,\beta} w)(t)$ can be written as

$$(D_0^{\alpha,\beta} w)(t) = \left(I_0^{\beta(1-\alpha)} \frac{d}{dt} I_0^{1-\gamma} w \right)(t) = \left(I_0^{\beta(1-\alpha)} D_0^{\gamma} w \right)(t); \ for\ a.e.\ t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1], \ \gamma \geq \alpha, \ \gamma > \beta, \ 1-\gamma < 1 - \beta(1-\alpha).$$

2. The generalization (2.1) for $\beta = 0$, coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative.

$$D_0^{\alpha,0} = D_0^{\alpha}, \ and\ D_0^{\alpha,1} = {}^cD_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)} w$ exists and is in $L^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = (I_0^{\beta(1-\alpha)} D_0^{\beta(1-\alpha)} w)(t); \ for\ a.e.\ t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)} w \in C_\gamma^1(I)$, then

$$(D_0^{\alpha,\beta} I_0^{\alpha} w)(t) = w(t); \ for\ a.e.\ t \in I.$$

4. If $D_0^{\gamma} w$ exists and is in $L^1(I)$, then

$$(I_0^{\alpha} D_0^{\alpha,\beta} w)(t) = (I_0^{\gamma} D_0^{\gamma} w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1}; \ for\ a.e.\ t \in I.$$

Corollary 2.5. *Let $h \in C_\gamma(I)$. Then the Cauchy problem*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t); \ t \in I, \\ \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{cases}$$

has the following unique solution

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^{\alpha} h)(t).$$

Let $x, , y \in \mathbb{R}^m$ with $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_m)$.
By $x \leq y$ we mean $x_i \leq y_i$; $i = 1, \ldots, m$. Also

$$|x| = (|x_1|, |x_2|, \ldots, |x_m|),$$

$$\max(x,y) = (\max(x_1,y_1), \max(x_2,y_2), \ldots, \max(x_m,y_m)),$$

and

$$\mathbb{R}_+^m = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+, \ i = 1, \ldots, m\}.$$

If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$; $i = 1, \ldots, m$.

Definition 2.6. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:

(i) $d(x, y) \geq 0$ for all $x, y \in X$, and if $d(x, y) = 0$, then $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We call the pair (X, d) a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ d_2(x,y) \\ \cdot \\ \cdot \\ \cdot \\ d_m(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1, \dots, m$ are metrics on X.

Definition 2.7. [5, 25] A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc, i.e., $|\lambda| < 1$; for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the unit matrix of $M_{m\times m}(\mathbb{R})$.

Example 2.8. The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0,\ a, d > 0$ and $\max\{a, d\} < 1$.
(2) $c = 0,\ a, d > 0,\ a + d < 1$ and $-1 < b < 0$.
(3) $a + b = c + d = 0,\ a > 1,\ c > 0$ and $|a - c| < 1$.

In the sequel we will make use of the following fixed point theorems in generalized Banach spaces:

Theorem 2.9. [19, 20] *Let (X, d) be a complete generalized metric space and $N : X \to X$ a contraction operator with a matrix M convergent to zero, i.e.,*

$$d(N(x), N(y)) \leq M d(x, y),\ \text{for every } x, y \in X.$$

Then N has a unique fixed point x^* and for each $x \in X$ we have

$$d(N^k(x), x^*) \leq M^k (I - M)^{-1} d(x, N(x));\ for\ all\ k \in \mathbb{N}.$$

For $n = 1$, we recover the classical Banach's contraction principle.

Theorem 2.10. [16, 26] *Let X be a generalized Banach space and $N : X \to X$ be a continuous and compact mapping. Then either:*

(a) *The set*

$$\mathcal{A} := \{x \in X : x = \lambda N(x) \ for \ some \ \lambda \in (0,1)\}$$

in unbounded, or

(b) *The operator N has a fixed point.*

Also, we will use the following Gronwall lemma:

Lemma 2.11. [9] *Let $u : I \to [0,\infty)$ be a real function and $u(\cdot)$ is a nonnegative, locally integrable function on I. Assume that there exist constants $c > 0$ and $r < 1$ such that*

$$u(t) \le v(t) + c\int_0^t \frac{u(s)}{(t-s)^r}ds,$$

then,there exists a constant $K := K(r)$ such that

$$u(t) \le v(t) + cK\int_0^t \frac{v(s)}{(t-s)^r}ds,$$

for every $t \in I$.

3. Coupled Hilfer fractional differential systems

In this section, we are concerned with the existence and uniqueness results of the system (1.1)-(1.2).

Definition 3.1. By a solution of the problem (1.1)-(1.2) we mean a coupled continuous functions $(u,v) \in C_{\gamma_1} \times C_{\gamma_2}$ those satisfy the equation (1.1) on I, and the conditions $(I_0^{1-\gamma_1}u)(0^+) = \phi_1$, and $(I_0^{1-\gamma_2}v)(0^+) = \phi_2$.

The following hypotheses will be used in the sequel.

(H_1) There exist continuous functions $p_i, q_i : I \to (0,\infty);\ i = 1,2$ such that

$$\|f_i(t,u_1,v_1) - f_i(t,u_2,v_2)\| \le p_i(t)\|u_1 - u_2\| + q_i(t)\|v_1 - v_2\|;$$

$for \ a.e. \ t \in I, \ and \ each \ u_i, v_i \in \mathbb{R}^m, \ i = 1,2.$

(H_2) There exist continuous functions $a_i, b_i : I \to (0,\infty);\ i = 1,2$ such that

$$\|f_i(t,u,v)\| \le a_i(t)\|u\| + b_i(t)\|v\|; \ for \ a.e. \ t \in I, \ and \ each \ u, v \in \mathbb{R}^m.$$

First, we prove an existence and uniqueness result for the coupled system (1.1)- (1.2) by using Banach's fixed point theorem type in generalized Banach spaces. Set

$$p_i^* := \sup_{t\in I} p(t), \ q_i^* := \sup_{t\in I} q(t); \ i = 1,2.$$

Theorem 3.2. *Assume that the hypothesis (H_1) holds. If the matrix*

$$M := \begin{pmatrix} \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}p_1^* & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}q_1^* \\ \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)}p_2^* & \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)}q_2^* \end{pmatrix}$$

converges to 0*, then the coupled system (1.1)-(1.2) has a unique solution.*

Proof. Define the operators $N_i : \mathcal{C} \to C_{\gamma_i};\ i = 1, 2$ by

$$(N_1(u,v))(t) = \frac{\phi_1}{\Gamma(\gamma_1)} t^{\gamma_1 - 1} + \int_0^t (t-s)^{\alpha_1 - 1} \frac{f_1(s, u(s), v(s))}{\Gamma(\alpha_1)} ds, \tag{3.1}$$

and

$$(N_2(u,v))(t) = \frac{\phi_2}{\Gamma(\gamma_2)} t^{\gamma_2 - 1} + \int_0^t (t-s)^{\alpha_2 - 1} \frac{f_2(s, u(s), v(s))}{\Gamma(\alpha_2)} ds. \tag{3.2}$$

Consider the operator $N : \mathcal{C} \to \mathcal{C}$ defined by

$$(N(u,v))(t) = ((N_1(u,v))(t), (N_2(u,v))(t)). \tag{3.3}$$

Clearly, the fixed points of the operator N are solutions of the system (1.1)-(1.2). For any $i \in \{1, 2\}$ and each $(u_1, v_1), (u_2, v_2) \in \mathcal{C}$ and $t \in I$, we have

$$\begin{aligned}
& \|t^{1-\gamma_1}(N_1(u_1,v_1))(t) - t^{1-\gamma_1}(N_1(u_2,v_2))(t)\| \\
\leq\ & \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} \|f_1(s, u_1(s), v_1(s)) - f_1(s, u_2(s), v_2(s))\| ds \\
\leq\ & \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} (p_1(t)\|u_1(s) - v_1(s)\| \\
+\ & q_1(t)\|u_2(s) - v_2(s)\|) ds \\
\leq\ & \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1 - 1} (p_1(t) s^{1-\gamma_1}\|u_1(s) - v_1(s)\| \\
+\ & q_1(t) s^{1-\gamma_1}\|u_2(s) - v_2(s)\|) ds \\
\leq\ & \frac{p_1(t)\|u_1 - v_1\|_{C_{\gamma_1}} + q_1(t)\|u_2 - v_2\|_{C_{\gamma_2}}}{\Gamma(\alpha_1)} \\
\times\ & \int_0^t (t-s)^{\alpha_1 - 1} ds \\
\leq\ & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} (p_1(t)\|u_1 - v_1\|_{C_{\gamma_1}} + q_1(t)\|u_2 - v_2\|_{C_{\gamma_2}}).
\end{aligned}$$

Then,

$$\begin{aligned}
& \|N_1(u_1,v_1) - N_1(u_2,v_2)\|_{C_{\gamma_1}} \\
\leq\ & \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)} (p_1^*\|u_1 - v_1\|_{C_{\gamma_1}} + q_1(^*\|u_2 - v_2\|_{C_{\gamma_2}}).
\end{aligned}$$

Also, for each $(u_1, v_1), (u_2, v_2) \in \mathcal{C}$ and $t \in I$, we get

$$\begin{aligned}
& \|N_2(u_1,v_1) - N_2(u_2,v_2)\|_{C_{\gamma_2}} \\
\leq\ & \frac{T^{\alpha_2}}{\Gamma(1+\alpha_2)} (p_2^*\|u_1 - v_1\|_{C_{\gamma_1}} + q_2^*\|u_2 - v_2\|_{C_{\gamma_2}}).
\end{aligned}$$

Thus,

$$d(N(u_1,v_1), N(u_2,v_2)) \leq M d((u_1,v_1),(u_2,v_2)),$$

where

$$d((u_1,v_1),(u_2,v_2)) = \begin{pmatrix} \|u_1 - v_1\|_{C_{\gamma_1}} \\ \|u_2 - v_2\|_{C_{\gamma_2}} \end{pmatrix}.$$

Since the matrix M converges to zero, then Theorem 2.9 implies that system (1.1)-(1.2) has a unique solution.

Now, we prove an existence result for the coupled system (1.1)- (1.2) by using the nonlinear alternative of Leray-Schauder type in generalized Banach space.

Theorem 3.3. *Assume that the hypothesis* (H_2) *holds. Then the coupled system (1.1)-(1.2) has at least one solution.*

Proof. We show that the operator $N : \mathcal{C} \to \mathcal{C}$ defined in (3.3) satisfies all conditions of Theorem 2.10. The proof will be given in four steps.

Step 1. *N is continuous.*
Let $(u_n, v_n)_n$ be a sequence such that $(u_n, v_n) \to (u, v) \in \mathcal{C}$ as $n \to \infty$. For any $i \in \{1, 2\}$ and each $t \in I$, we have

$$\begin{aligned}
& \|t^{1-\gamma_i}(N_i(u_n,v_n))(t) - t^{1-\gamma_i}(N_i(u,v))(t)\| \\
\leq\ & \frac{t^{1-\gamma_i}}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}\|f_i(s,u_n(s),v_n(s)) - f_i(s,u(s),v(s))\|ds \\
\leq\ & \frac{1}{\Gamma(\alpha_i)}\int_0^t (t-s)^{\alpha_i-1}s^{1-\gamma_i}\|f_i(s,u_n(s),v_n(s)) - f_i(s,u(s),v(s))\|ds \\
\leq\ & \frac{T^{\alpha_i}}{\Gamma(1+\alpha_i)}\|f_i(\cdot,u_n(\cdot),v_n(\cdot)) - f_i(\cdot,u(\cdot),v(\cdot))\|_{C_{\gamma_1}}.
\end{aligned}$$

Since f_i is continuous, then by the Lebesgue dominated convergence theorem, we get

$$\|N_i(u_n,v_n) - N_i(u,v)\|_{C_{\gamma_1}} \to 0 \ as \ n \to \infty.$$

Hence N is continuous.

Step 2. *N maps bounded sets into bounded sets in $\mathcal{C}$.*
Set

$$a_i^* := \sup_{t\in I} a(t),\ \ b_i^* := \sup_{t\in I} b(t) :\ \ i = 1, 2.$$

Let $R > 0$ and set

$$B_R := \{(\mu,\nu) \in \mathcal{C} : \|\mu\|_{C_{\gamma_1}} \leq R, \|\nu\|_{C_{\gamma_2}} \leq R\}.$$

For each $(u, v) \in B_R$ and $t \in I$, we have

$$\begin{aligned}
\|t^{1-\gamma_1}(N_1(u,v))(t)\| \ \leq\ & \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u(s),v(s))\|ds \\
\leq\ & \frac{\|\phi_1\|}{\Gamma(\gamma_1)} \\
& + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}(a_1(s)\|u(s\| + b_1(s)\|v(s)\|)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)}+\frac{R}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}(a_1(s)+b_1(s))ds\\
&\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)}+\frac{(a_1^*+b_1^*T^{\alpha_1}}{\Gamma(1+\alpha_1)}\\
&:= \ell_1.
\end{aligned}$$

Thus,

$$\|N_1(u,v)\|_{C_{\gamma_1}}\leq \ell_1.$$

Also, for each $(u,v)\in B_R$ and $t\in I$, we get

$$\begin{aligned}
\|N_2(u,v)\|_{C_{\gamma_2}} &\leq \frac{\|\phi_2\|}{\Gamma(\gamma_2)}+\frac{(a_2^*+b_2^*T^{\alpha_2}}{\Gamma(1+\alpha_)}\\
&:= \ell_2.
\end{aligned}$$

Hence,

$$\|N(u,v)\|_{\mathcal{C}}\leq (\ell_1,\ell_2):=\ell.$$

Step 3. *N maps bounded sets into equicontinuous sets in $\mathcal{C}$.*
Let B_R be the ball defined in Step 2. For each $t_1,t_2\in I$ with $t_1\leq t_2$ and $(u,v)\in B_R$, we have

$$\begin{aligned}
&\|t_1^{1-\gamma_1}(N_1(u,v))(t_1)-t_2^{1-\gamma_1}(N_1(u,v))(t_2)\|\\
&\leq \frac{t_2^{1-\gamma_1}}{\Gamma(\alpha_1)}\int_{t-1}^{t_2}(t_2-s)^{\alpha_1-1}\|f_1(s,u(s),v(s))\|ds\\
&\leq \frac{T^{\alpha_1}}{\Gamma(1+\alpha_1)}(t_2-t_1)^{\alpha_1}(a_1^*\|u\|_{C_{\gamma_1}}+b_1(^*\|v\|_{C_{\gamma_2}})\\
&\leq \frac{RT^{\alpha_1}(a_1^*+b_1^*)}{\Gamma(1+\alpha_1)}(t_2-t_1)^{\alpha_1}\\
&\to 0 \ as\ t_1\to t_2.
\end{aligned}$$

Also, we get

$$\begin{aligned}
&\|t_1^{1-\gamma_2}(N_2(u,v))(t_1)-t_2^{1-\gamma_2}(N_2(u,v))(t_2)\|\\
&\leq \frac{RT^{\alpha_1 2}(a_2^*+b_2^*))}{\Gamma(1+\alpha_2)}(t_2-t_1)^{\alpha_2}\\
&\to 0 \ as\ t_1\to t_2.
\end{aligned}$$

As a consequence of Steps 1 to 3, with the Arzela-Ascoli theorem, we conclude that N maps B_R into a precompact set in $\mathcal{C}$.

Step 4. The set E consisting of $(u,v)\in\mathcal{C}$ such that $(u,v)=\lambda N(u,v)$ for some $\lambda\in(0,1)$ is bounded in $\mathcal{C}$.
Let $(u,v)\in\mathcal{C}$ such that $(u,v)=\lambda N(u,v)$. Then $u=\lambda N_1(u,v)$ and $v=\lambda N_2(u,v)$.

Thus, for each $t \in I$, we have

$$\begin{aligned} \|t^{1-\gamma_1}u(t)\| &\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{t^{1-\gamma_1}}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1}\|f_1(s,u(s),v(s))\|ds \\ &\leq \frac{\|\phi_1\|}{\Gamma(\gamma_1)} \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1}s^{1-\gamma_1}(a_1^*\|u(s\| + b_1^*\|v(s)\|)ds. \end{aligned}$$

Also, we get

$$\|t^{1-\gamma_2}v(t)\| \leq \frac{\|\phi_2\|}{\Gamma(\gamma_2)} + \frac{1}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1}s^{1-\gamma_2}(a_2^*\|u(s)\| + b_2^*\|v(s)\|)ds.$$

Hence, we obtain

$$\|t^{1-\gamma_1}u(t)\| + \|t^{1-\gamma_2}v(t)\| \leq a + bc\int_0^t (t-s)^{\alpha-1}(\|s^{1-\gamma_1}u(s)\| + \|s^{1-\gamma_2}v(s)\|)ds,$$

where

$$a := \frac{\|\phi_1\|}{\Gamma(\gamma_1)} + \frac{\|\phi_2\|}{\Gamma(\gamma_2)},\ b := \frac{1}{\Gamma(\alpha_1)} + \frac{1}{\Gamma(\alpha_2)},$$
$$c := \max\{a_1^* + a_2^*, b_1^* + b_2^*\},\ \alpha := \max\{\alpha_1, \alpha_2\}.$$

Lemma 2.11 implies that there exists $\rho := \rho(\alpha) > 0$ such that

$$\begin{aligned} \|t^{1-\gamma_1}u(t)\| + \|t^{1-\gamma_2}v(t)\| &\leq a + abc\rho\int_0^t (t-s)^{\alpha-1}ds \\ &\leq \frac{a + abc\rho T^\alpha}{\alpha} \\ &= L. \end{aligned}$$

This gives

$$\|u\|_{C_{\gamma_1}} + \|v\|_{C_{\gamma_2}} \leq L.$$

Hence

$$\|(u,v)\|_{\mathcal{C}} \leq L.$$

This shows that the set E is bounded.

As a consequence of steps 1 to 4 together with Theorem 2.10, we can conclude that N has at least one fixed point in B_R which is a solution of the system (1.1)- (1.2).

4. Coupled Hilfer-Hadamard fractional differential systems

Now, we are concerned with the coupled system (1.3)-(1.4).
Set $C := C([1,T])$, and denote the weighted space of continuous functions defined by

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma}w(t) \in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]} |(\ln t)^{1-r}w(t)|.$$

Also, by $\mathcal{C}_{\gamma_1,\gamma_2,\ln}([1,T]) := C_{\gamma_1,\ln}([1,T]) \times C_{\gamma_2,\ln}([1,T])$ we denote the product weighted space with the norm

$$\|(u,v)\|_{\mathcal{C}_{\gamma_1,\gamma_2,\ln}([1,T])} = \|u\|_{C_{\gamma_1,\ln}} + \|v\|_{C_{\gamma_2,\ln}}.$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [13] for a more detailed analysis.

Definition 4.1. [13] (Hadamard fractional integral). The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,T])$, is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

Example 4.2. Let $0 < q < 1$. Let $g(x) = \ln x$, $x \in [0,e]$. Then

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(2+q)} (\ln x)^{1+q}; \ for \ a.e. \ x \in [0,e].$$

Set

$$\delta = x\frac{d}{dx}, \ q > 0, \ n = [q] + 1,$$

and

$$AC_\delta^n := \{u : [1,T] \to E : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Analogous to the Riemann-Liouville fractional derivative, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

Definition 4.3. [13] (Hadamard fractional derivative). The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^H D_1^q w)(x) = \delta^n ({}^H I_1^{n-q} w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^H D_1^q w)(x) = \delta({}^H I_1^{1-q} w)(x).$$

Example 4.4. Let $0 < q < 1$. Let $w(x) = \ln x$, $x \in [0,e]$. Then

$$({}^H D_1^q w)(x) = \frac{1}{\Gamma(2-q)} (\ln x)^{1-q}, \ for \ a.e. \ x \in [0,e].$$

It has been proved (see e.g. Kilbas [[12], Theorem 4.8]) that in the space $L^1(I)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [13], we have

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)} (\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo-Hadamard fractional derivative is defined in the following way:

Definition 4.5. (Caputo-Hadamard fractional derivative) The Caputo-Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^{Hc}D_1^q w)(x) = ({}^H I_1^{n-q} \delta^n w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^{Hc}D_1^q w)(x) = ({}^H I_1^{1-q} \delta w)(x).$$

From the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative (introduced for the first time in [17]) is defined in the following way:

Definition 4.6. (Hilfer-Hadamard fractional derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $w \in L^1(I)$, and ${}^H I_1^{(1-\alpha)(1-\beta)} w \in AC(I)$. The Hilfer-Hadamard fractional derivative of order α and type β applied to the function w is defined as

$$\begin{aligned}({}^H D_1^{\alpha,\beta} w)(t) &= \left({}^H I_1^{\beta(1-\alpha)}({}^H D_1^{\gamma} w)\right)(t) \\ &= \left({}^H I_1^{\beta(1-\alpha)} \delta({}^H I_1^{1-\gamma} w)\right)(t); \ for\ a.e.\ t \in [1,T].\end{aligned} \tag{4.1}$$

This new fractional derivative (4.1) may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative and when $\beta = 1$, we recover the Caputo-Hadamard fractional derivative.

$${}^H D_1^{\alpha,0} = {}^H D_1^{\alpha}, \ and \ {}^H D_1^{\alpha,1} = {}^{Hc} D_1^{\alpha}.$$

From Theorem 21 in [18], we concluded the following lemma.

Lemma 4.7. *Let $g : [1,T] \times \mathbb{R}^m \to \mathbb{R}^m$ be such that $g(\cdot, u(\cdot)) \in C_{\gamma,\ln}([1,T])$ for any $u \in C_{\gamma,\ln}([1,T])$. Then problem (1.3) is equivalent to the following Volterra integral equation*

$$u(t) = \frac{\phi_0}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^{\alpha} g(\cdot, u(\cdot)))(t).$$

Definition 4.8. By a solution of the coupled system (1.3)-(1.4) we mean a coupled continuous functions $(u,v) \in C_{\gamma_1,\ln} \times C_{\gamma_2,\ln}$ those satisfy the conditions (1.4) and the equations (1.3) on $[1,T]$.

Now we give (without proof) similar existence and uniqueness results for the system (1.3)-(1.4). Let us introduce the following hypotheses:

(H_1') There exist continuous functions $p_i, q_i : [1,T] \to (0,\infty)$; $i = 1,2$ such that

$$\|g_i(t,u_1,v_1) - g_i(t,u_2,v_2)\| \le p_i(t)\|u_1 - u_2\| + q_i(t)\|v_1 - v_2\|;$$

$for\ a.e.\ t \in [1,T],\ and\ each\ u_i, v_i \in \mathbb{R}^m,\ i = 1,2.$

(H_2') There exist continuous functions $a_i, b_i : [1,T] \to (0,\infty)$; $i = 1,2$ such that

$$\|g_i(t,u,v)\| \le a_i(t)\|u\| + b_i(t)\|v\|; \ for\ a.e.\ t \in [1,T],\ and\ each\ u,v \in \mathbb{R}^m.$$

Theorem 4.9. *Assume that the hypothesis* (H_1') *holds. If the matrix*

$$\begin{pmatrix} \frac{(\ln T)^{\alpha_1}}{\Gamma(1+\alpha_1)}p_1^* & \frac{(\ln T)^{\alpha_1}}{\Gamma(1+\alpha_1)}q_1^* \\ \frac{(\ln T)^{\alpha_2}}{\Gamma(1+\alpha_2)}p_2^* & \frac{(\ln T)^{\alpha_2}}{\Gamma(1+\alpha_2)}q_2^* \end{pmatrix}$$

converges to 0, *then the coupled system (1.3)-(1.4) has a unique solution defined on* $[1,T]$.

Theorem 4.10. *Assume that the hypothesis* (H_2') *holds. Then the coupled system (1.3)-(1.4) has at least one solution defined on* $[1,T]$.

5. An example

Consider the following coupled system of Hilfer fractional differential equations

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}}u)(t) = f(t,u(t),v(t)); \\ (D_0^{\frac{1}{2},\frac{1}{2}}v)(t) = g(t,u(t),v(t)); \\ (I_0^{\frac{1}{4}}u)(0) = 1, \\ (I_0^{\frac{1}{4}}v_n)(0) = 0, \end{cases} \quad : \ t \in [0,1], \tag{5.1}$$

where

$$f(t,u,v) = \frac{t^{\frac{-1}{4}}(u(t)+v(t))\sin t}{64(1+\sqrt{t})(1+|u|+|v|)}; \ t \in [0,1],$$

$$g(t,u,v) = \frac{(u(t)+v(t))\cos t}{64(1+|u|+|v|)}; \ t \in [0,1].$$

Set $\alpha_i = \beta_i = \frac{1}{2}; \ i = 1,2$, then $\gamma_i = \frac{3}{4}; \ i = 1,2$. The hypothesis (H_1) is satisfied with

$$p_1(t) = p_2(t) = q_1(t) = q_2(t) = \frac{1}{64}.$$

Also the matrix

$$\frac{1}{64\sqrt{\pi}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

converges to 0. Hence, Theorem 3.2 implies that the system (5.1) has a unique solution defined on $[0,1]$.

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