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SOME VARIANTS OF FIBRE CONTRACTION PRINCIPLE AND APPLICATIONS: FROM EXISTENCE TO THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS

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Abstract. Let $(X_1, \to)$ and $(X_2, \hookrightarrow)$ be two L-spaces, U be a nonempty subset of $X_1 \times X_2$ such that $U_{x_1} := \{x_2 \in X_2 \mid (x_1, x_2) \in U\}$ is nonempty, for each $x_1 \in X_1$. Let $T_1 : X_1 \to X_1$, $T_2 : U \to X_2$ be two operators and $T : U \to X_1 \times X_2$ defined by $T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2))$. If we suppose that $T(U) \subset U$, $F_{T_1} \neq \emptyset$ and $F_{T_2(x_1,\cdot)} \neq \emptyset$ for each $x_1 \in X_1$, the problem is in which additional conditions T is a weakly Picard operator ? In this paper we study this problem in the case when the convergence structures on X_1 and X_2 are defined by metrics. Some applications to the fixed point equations on spaces of continuous functions are also given.
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1. Introduction

The purpose of this paper is to give some (partial) answers to the following general problems.

Problem 1.1. Let $(X_1, \to)$ and $(X_2, \hookrightarrow)$ be two L-spaces in the sense of Fréchet, see e.g. [32]. Given $T_1 : X_1 \to X_1$ and $T_2 : X_1 \times X_2 \to X_2$ we define the following triangular operator

$$T : X_1 \times X_2 \to X_1 \times X_2 \text{ given by } T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2)).$$

We consider the following problem:

We suppose that:

(i) the fixed point set $F_{T_1} := \{u \in X_1 : u = T_1(u)\}$ of T_1 is nonempty;

(ii) the fixed point set of the operator $T_2(x_1, \cdot)$ is nonempty, for each $x_1 \in X_1$.

The problem is in which conditions the operator T is weakly Picard.

Problem 1.2. Let $(X_1, \to)$ and $(X_2, \hookrightarrow)$ be two L-spaces and $U \subset X_1 \times X_2$ be a nonempty set, such that $U_{x_1} := \{x_2 \in X_2 : (x_1, x_2) \in U\}$ is nonempty, for each $x_1 \in X_1$. Given $T_1 : X_1 \to X_1$, $T_2 : U \to X_2$ and the triangular operator

$$T : U \to X_1 \times X_2 \text{ defined by } T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2)),$$

we suppose that:

(i) F_{T_1} is nonempty;

(ii) the fixed point set of the operator $T_2(x_1, \cdot)_{|U_{x_1}}$ is nonempty;

(iii) $T(U) \subset U$.

The problem is in which conditions T is a weakly Picard operator.

Problem 1.1 is well-known in the literature, see [10], [24], [40], [41], [42], [32], ... In this paper, we will study Problem 1.2 in the case when the convergence structures $\to$ and $\hookrightarrow$ are generated by metrics. Some applications of the fixed point equations on spaces of continuous functions are also given.

The structure of the paper is the following one:

1. Introduction
2. Preliminaries
   - 2.1. Weakly Picard operators
   - 2.2. Standard fibre contractions
   - 2.3. Applications of the standard fibre contraction
3. Fibre contractions on metric spaces
4. Fibre generalized contractions on metric spaces
5. Fibre generalized contractions on some generalized metric spaces
6. Technique of $\mathbb{R}^p_+$-valued metrics in the theory of fibre contractions
7. Applications.

Throughout the paper we shall use the notations and the terminology from [27], [32], [31].

2. Preliminaries

2.1. **Weakly Picard operators.** Let $(X, \to)$ be an L-space, where X is a nonempty set and $\to$ is a convergence structure defined on X. If $T : X \to X$ is an operator, then we denote by $F_T := \{x \in X : x = T(x)\}$ the fixed point set of T.

In the above context, $T : X \to X$ is called a weakly Picard operator (briefly WPO) if, for each $x \in X$, the sequence of Picard iterations $(T^n(x))_{n\in\mathbb{N}}$ converges with respect to $\to$ to a fixed point of T. In particular, if $F_T = \{x^*\}$, then T is called a Picard operator (briefly PO).

If $T : X \to X$ is a WPO, then we define a set retraction $T^\infty : X \to F_T$ by the formula

$$T^\infty(x) := \lim_{n\to\infty} T^n(x).$$

If T is PO with its unique fixed point x^*, then $T^\infty(X) = \{x^*\}$.

References: [27], [32], [26].

2.2. **Standard fibre contractions.** We recall now the following standard fibre contraction theorem.

Theorem 2.1. *Let $(X_0, \to)$ be an L-space. For $m \in \mathbb{N}^*$, let (X_i, d_i), $i \in \{1, \cdots, m\}$ be complete metric spaces. Let $T_0 : X_0 \to X_0$ be an operator and, for $i \in \{1, \cdots, m\}$, let us consider $T_i : X_0 \times X_1 \times \cdots \times X_i \to X_i$. We suppose that:*

(1) *T_0 is a WPO;*
(2) *for each $i \in \{1, 2, \cdots, m\}$, the operators $T_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i$ are l_i-contractions;*
(3) *for each $i \in \{1, 2, \cdots, m\}$, the operators T_i are continuous.*

Then, the operator $T = (T_0, T_1, \cdots, T_m) : \prod_{i=0}^{m} X_i \to \prod_{i=0}^{m} X_i$, defined by

$$T(x_0, \ldots, x_m) := (T_0(x_0), T_1(x_0, x_1), \ldots, T_m(x_0, \ldots, x_m))$$

is a WPO. Moreover, when T_0 is a PO, then T is a PO too.

References:
(1) fibre contractions: [10], [24], [32], [26], [42], ...
(2) fibre generalized contractions: [23], [35], [40], [42], ...
(3) fibre generalized contractions on generalized metric spaces: [1], [2], [22], [25], [23], [33], [34], [39], ...

2.3. **Applications of standard fibre contractions.** The fibre contraction principles are essential tools in the study of regularity of solution of various equations, such as: operatorial, differential, integral, functional differential, functional integral. See [4], [5], [8], [13], [15], [14], [16], [17], [33], [37], [38], [43], [44], ...

There exists another direction of application of the fibre contraction principle, as the following example illustrates.

Example 2.2. Let us consider the following functional differential equation

$$x'(t) = f(t, x(t), x(t-h)), \ t \in [a, a+2h], \tag{2.1}$$

with Cauchy condition

$$x(t) = \varphi(t), \ t \in [a-h, a], \tag{2.2}$$

where $h > 0$, $f \in C([a, a+2h] \times \mathbb{R}^2)$, $\varphi \in C[a-h, a]$ and there exists $L > 0$ such that

$$|f(t, u, v) - f(t, \widetilde{u}, v)| \le L|u - \widetilde{u}|, \tag{2.3}$$

for all $t \in [a, a+2h]$, $u, \widetilde{u} \in \mathbb{R}$.

We are looking for solutions $x \in C[a-h, a+2h] \cap C^1[a, a+2h]$ of problem (2.1)-(2.2).

The problem (2.1)-(2.2) is equivalent with the following functional integral equation

$$x(t) = \begin{cases} \varphi(t), & t \in [a-h, a] \\ \varphi(a) + \displaystyle\int_a^t f(s, x(s), x(s-h))ds, & t \in [a, a+2h]. \end{cases} \tag{2.4}$$

Now we consider the operator $V : C[a-h, a+2h] \to C[a-h, a+2h]$ defined by

$$V(x)(t) := \text{second part of (2.4)}.$$

By step method one proves that $F_V = \{x^*\}$. The proof is as follow.

First we consider the operator $V_1 : C[a-h, a+h] \to C[a-h, a+h]$ defined by $V_1(x_1) := V(\widetilde{x})$, where $\widetilde{x} \in C[a-h, a+2h]$ is such that $\widetilde{x}\Big|_{[a-h,a+h]} = x_1$.

By the Contraction Principle we have that $F_{V_1} = \{x_1^*\}$ and $V_1^n(x_1^0) \to x_1^*$ for all $x_1^0 \in C[a-h, a+h]$.

Secondly, we consider the operator $V_2 : C[a+h, a+2h] \to C[a+h, a+2h]$ defined by

$$V_2(x_2)(t) := x_1^*(a+h) + \int_{a+h}^{t} f(s, x_2(s), x_1^*(s-h))ds,\ t \in [a+h, a+2h].$$

Again, by the Contraction Principle we have that $F_{V_2} = \{x_2^*\}$ and $V_2^n(x_2^0) \to x_2^*$ as $n \to \infty$, for all $x_2^* \in C[a+h, a+2h]$.

It is clear that

$$x^*(t) = \begin{cases} x_1^*(t), & t \in [a-h, a+h] \\ x_2^*(t), & t \in [a+h, a+2h]. \end{cases}$$

Let $x^n := V^n(x^0)$, $x^0 \in C[a-h, a+2h]$. The problem is in which conditions $x^n \to x^*$ as $n \to \infty$?

Let $X_1 := C[a-h, a+h]$, $X_2 := C[a+h, a+2h]$ and $X := X_1 \times X_2$ endowed with suitable Bielecki norms. The following relations:

$$V(x)(t) = \begin{cases} \varphi(t), & t \in [a-h, a], \\ \varphi(a) + \displaystyle\int_a^t f(s, x(s), \varphi(s-h))ds, & t \in [a, a+h] \end{cases}$$

and

$$\begin{aligned} V(x)(t) = \varphi(a) &+ \int_a^{a+h} f(s, x(s), \varphi(s-h))ds \\ &+ \int_{a+h}^{t} f(s, x(s), x(s-h))ds,\ t \in [a+h, a+2h] \end{aligned}$$

suggest us to consider the following operators:

$$T_1 : X_1 \to X_1,\ T_1(x_1) = V_1(x_1),$$
$$T_2 : X_1 \times X_2 \to X_2,$$

defined by

$$\begin{aligned} T_2(x_1, x_2)(t) = \varphi(a) &+ \int_a^{a+h} f(s, x_1(s), \varphi(s-h))ds \\ &+ \int_{a+h}^{t} f(s, x_2(s), x_1(s-h))ds,\ t \in [a+h, a+2h] \end{aligned}$$

and

$$T : X_1 \times X_2 \to X_1 \times X_2,\ T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2)).$$

By the standard fibre contraction principle (see Theorem 2.1), T is a PO.

Let $R : X \to X_1 \times X_2$ defined by

$$R(x) := \Big(x\big|_{[a-h,a+h]}, x\big|_{[a+h,a+2h]}\Big)$$

and

$$U := \{(x_1, x_2) \in X_1 \times X_2 \mid x_1(a+h) = x_2(a+h)\}.$$

We observe that $R(X) = U$ and $R : X \to U$ is an homeomorphism. Moreover we have $V = R^{-1}TR$ and $V^n = R^{-1}T^nR$. Since T is PO it follows that V is a PO.

For more considerations on this example, see [28]. For other applications of the fibre contraction principle to integro-differential equations with delays see [9], [12], [18], [36].

3. Fibre contractions on metric spaces

For $i \in \{1, 2\}$, let (X_i, d_i) be two metric spaces, $U \subset X_1 \times X_2$ be a nonempty subset, such that

$$U_{x_1} := \{x_2 \in X_2 \mid (x_1, x_2) \in U\} \neq \emptyset, \text{ for all } x_1 \in X_1.$$

For two operators $T_1 : X_1 \to X_1$, $T_2 : U \to X_2$, we consider the operator $T : U \to X_1 \times X_2$ defined by

$$T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2)).$$

For the problem formulated in Introduction, we have in this case the following result.

Theorem 3.1. *We suppose that:*
(1) *(X_2, d_2) is a complete metric space and U is a closed subset of $X_1 \times X_2$;*
(2) *$T(U) \subset U$;*
(3) *T_1 is a WPO;*
(4) *there exist $L > 0$ and $0 < l < 1$ such that*

$$d_2(T_2(x_1, x_2), T_2(\widetilde{x}_1, \widetilde{x}_2)) \le Ld_1(x_1, \widetilde{x}_1) + ld_2(x_2, \widetilde{x}_2),$$

for all $(x_1, x_2), (\widetilde{x}_1, \widetilde{x}_2) \in U$.
Then T is a WPO. If T_1 is a PO, then T is a PO too.

Proof. Let $(x_1^0, x_2^0) \in U$. Since T_1 is WPO, the sequence $x_1^n := T_1^n(x_1^0) \to x_1^* \in F_{T_1}$ as $n \to \infty$. From (1), $U_{x_1^*}$ is a closed subset of X_2. From (4), $T_2(x_1^*, \cdot) : U_{x_1^*} \to U_{x_1^*}$ is an l-contraction. Let x_2^* its unique fixed point. It is clear that $(x_1^*, x_2^*) \in F_T$.

Let $x_2^{n+1} := T_2(x_1^n, x_2^n)$. From (2) this sequence is well defined. We shall prove that $x_2^n \to x_2^*$ as $n \to \infty$.

From (2) and (4) we have:

$$\begin{aligned} d_2(x_2^{n+1}, x_2^*) &= d_2(T_2(x_1^n, x_2^n), T_2(x_1^*, x_2^*)) \le Ld_1(x_1^n, x_1^*) + ld_2(x_2^n, x_2^*) \\ &\le Ld_1(x_1^n, x_1^*) + l[Ld_1(x_1^{n-1}, x_1^*) + ld_2(x_2^{n-1}, x_2^*)] \\ &= Ld_1(x_1^n, x_1^*) + lLd_1(x_1^{n-1}, x_1^*) + l^2 d_2(x_2^{n-1}, x_2^*) \\ &\le \ldots \le Ld_1(x_1^n, x_1^*) + lLd_1(x_1^{n-1}, x_1^*) + \ldots + \\ &+ l^n Ld_1(x_1^0, x_1^*) + l^{n+1} d_2(x_2^0, x_2^*) \to 0 \text{ as } n \to \infty. \end{aligned}$$

This follows from a well known Cauchy lemma (see [35]). □

By a successive application of Theorem 3.1 we have the general following result.

Let (X_i, d_i) $(i \in \{1, ..., m\}$ where $m \geq 2)$ be metric spaces and $U_1 \subset X_1 \times X_2$, $U_2 \subset U_1 \times X_3, \ldots, U_{m-1} \subset U_{m-2} \times X_m$, be nonempty subsets.

For $x \in X_1$, we define

$$U_{1x} := \{x_2 \in X_2 \mid (x, x_2) \in U_1\},$$

for $x \in U_1$, we define

$$U_{2x} := \{x_3 \in X_3 \mid (x, x_3) \in U_2\}, \ldots,$$

and for $x \in U_{m-2}$, we define

$$U_{m-1x} := \{x_m \in X_m \mid (x, x_m) \in U_{m-1}\}.$$

We suppose that $U_{1x}, U_{2x}, \ldots, U_{m-1x}$ are nonempty.

If $T_1 : X_1 \to X_1$, $T_2 : U_1 \to X_2, \ldots, T_m : U_{m-1} \to X_m$, then we consider the operator

$$T : U_{m-1} \to X_1 \times X_2 \times \ldots \times X_m,$$

defined by

$$T(x_1, \ldots, x_m) := (T_1(x_1), T_2(x_1, x_2), \ldots, T_m(x_1, x_2, \ldots, x_m)).$$

The result is the following.

Theorem 3.2. *We suppose that:*

(1) *for $m \in \mathbb{N}$, $m \geq 2$ and for $i \in \{2, ..., m\}$, the pairs (X_i, d_i) are complete metric spaces;*

(2) *for $i \in \{1, ..., m-1\}$, the sets U_i are closed;*

(3) *$(T_1, T_2, \ldots, T_{i+1})(U_i) \subset U_i$, $i \in \{1, ..., m-1\}$;*

(4) *T_1 is a WPO;*

(5) *there exist $L_i > 0$ and $0 < l_i < 1$, $i \in \{1, ..., m-1\}$ such that*

$$d_{i+1}(T_{i+1}(x, y,), T_{i+1}(\widetilde{x}, \widetilde{y})) \leq L_i \widetilde{d_i}(x, \widetilde{x}) + l_i d_{i+1}(y, \widetilde{y}),$$

for all $(x, y), (\widetilde{x}, \widetilde{y}) \in U_i$, $i \in \{1, ..., m-1\}$, where $\widetilde{d_i}$ is a metric induced by $d_1, \ldots, d_i$ on $X_1 \times \ldots \times X_i$, defined by $\tilde{d_i} := \max\{d_1, \cdots, d_i\}$.

Then T is WPO. If T_1 is PO, then T is a PO too.

Remark 3.3. Notice that the completeness of the metric space (X_1, d_1) is not required in Theorem 3.1 and Theorem 3.2.

4. Fibre generalized contractions on metric spaces

Let us look to condition (4) in Theorem 3.1:

$$\text{there exist } \ L > 0 \text{ and } 0 < l < 1 \text{ such that :}$$

$$d_2(T_2(x_1, x_2), T_2(\widetilde{x}_1, \widetilde{x}_2)) \leq L d_1(x_1, \widetilde{x}_1) + l d_2(x_2, \widetilde{x}_2), \text{ for all } (x_1, x_2), (\widetilde{x}_1, \widetilde{x}_2) \in U.$$

This condition suggest us to consider similar conditions coming from generalized contractions. Here are some of such conditions:

(4′) there exist $L > 0$ and $0 < l < \dfrac{1}{2}$ such that

$$d_2(T_2(x_1, x_2), T_2(\widetilde{x}_1, \widetilde{x}_2)) \leq Ld_1(x_1, \widetilde{x}_1) + l[d_2(T_2(x_1, x_2), x_2) + d_2(T_2(\widetilde{x}_1, \widetilde{x}_2), \widetilde{x}_2)],$$

$$\forall\ (x_1, x_2), (\widetilde{x}_1, \widetilde{x}_2) \in U;$$

(4′′) there exist $L > 0$ and a comparison function, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d_2(T_2(x_1, x_2), T_2(\widetilde{x}_1, \widetilde{x}_2)) \leq Ld_1(x_1, \widetilde{x}_1) + \varphi(d_2(x_2, \widetilde{x}_2)),\ \forall\ (x_1, x_2), (\widetilde{x}_1, \widetilde{x}_2) \in U.$$

As an example, in this section we shall study the problem in the case of condition (4′) of Kannan type. We have:

Theorem 4.1. *We suppose that we are in the conditions of Theorem 3.1, where instead of condition* (4) *we consider condition* (4′)*. Then the operator T is WPO. If T_1 is PO then T is PO.*

Proof. The proof is similar with that of Theorem 3.1. Let us prove that $x_2^n \to x_2^*$ as $n \to \infty$. We have that:

$$\begin{aligned} d_2(x_2^{n+1}, x_2^*) &= d_2(T_2(x_1^n, x_2^n), T_2(x_1^*, x_2^*)) \\ &\leq Ld_1(x_1^n, x_1^*) + ld_2(T_2(x_1^n, x_2^n), x_2^n) \\ &\leq Ld_1(x_1^n, x_1^*) + ld_2(x_2^{n+1}, x_2^*) + ld(x_2^n, x_2^*). \end{aligned}$$

This implies that

$$\begin{aligned} d_2(x_2^{n+1}, x_2^*) &\leq \frac{L}{1-l} d_1(x_1^n, x_1^*) + \frac{l}{1-l} d_2(x_2^n, x_2^*) \\ &\leq \frac{L}{1-l} d_1(x_1^n, x_1^*) + \frac{l}{1-l}\left[\frac{L}{1-l} d_1(x_1^{n-1}, x_1^*) + \frac{l}{1-l} d_2(x_2^{n-1}, x_2^*)\right] \\ &\leq \frac{L}{1-l} d_1(x_1^n, x_1^*) + \frac{l}{1-l} \cdot \frac{L}{1-l} d_1(x_1^{n-1}, x_1^*) + \dots \\ &+ \left(\frac{l}{1-l}\right)^n \frac{L}{1-l} d_1(x_1^0, x_1^*) + \left(\frac{l}{1-l}\right)^{n+1} d_2(x_2^0, x_2^*) \to 0 \end{aligned}$$

as $n \to \infty$, as above, by the Cauchy lemma.

5. Fibre generalized contractions on some generalized metric spaces

The problem is to study the fixed points of operator which are fibre generalized contractions on generalized metric spaces. Here, by a generalized metric we understand a distance (dislocated metric, partial metric, quasimetric, pseudometric,...), a vector-valued metric, cone-valued metric,... (see [7], [26], [31], [33], [42], ...).

As an example we shall study the problem in the case $m = 2$ and the metrics having values in $\mathbb{R}_+^p$, $p \in \mathbb{N}, p \geq 2$.

Let (X_1, d_1) and (X_2, d_2) be two generalized metric spaces with $d_1 : X_1 \times X_1 \to \mathbb{R}_+^p$ and $d_2 : X_2 \times X_2 \to \mathbb{R}_+^p$, $U \subset X_1 \times X_2$ be a nonempty subset and, for $x_1 \in X_1$, the set

$$U_{x_1} := \{x_2 \in X_2 \mid (x_1, x_2) \in U\} \neq \emptyset.$$

For $T_1 : X_1 \to X_1$ and $T_2 : U \to X_2$ we consider the triangular operator

$$T : U \to X_1 \times X_2,$$

defined by

$$T(x_1, x_2) := (T_1(x_1), T_2(x_1, x_2)).$$

For our problem we have the following result:

Theorem 5.1. *We suppose that:*

(1) (X_2, d_2) *is a complete metric space and* U *is a closed subset;*

(2) $T(U) \subset U$*;*

(3) T_1 *is WPO;*

(4) *there exist matrices* $L \in \mathbb{R}_+^{p\times p}$ *and* $S \in \mathbb{R}_+^{p\times p}$ *(where* S *has its spectral radius* $\rho(S) < 1$*), such that:*

$$d_2(T_2(x_1, x_2), T_2(\widetilde{x}_1, \widetilde{x}_2)) \le Ld_1(x_1, \widetilde{x}_1) + Sd_2(\widetilde{x}_1, \widetilde{x}_2), \ \forall \ (x_1, x_2), (\widetilde{x}_1, \widetilde{x}_2) \in U.$$

Then T *is WPO. If* T_1 *is a PO, then* T *is a PO.*

Proof. Let $(x_1^0, x_2^0) \in U$. Since T_1 is WPO the sequence $x_1^n := T_1^n(x_1^0)$ converges to $x_1^* \in F_{T_1}$. From (1), $U_{x_1^*} \subset X_2$ is a closed subset. From (4), $T_2(x_1^*, \cdot) : U_{x_1^*} \to U_{x_1^*}$ is a S-contraction. Let x_2^* its unique fixed point. We have that $(x_1^*, x_2^*) \in F_T$.

Let $(x_1^{n+1}, x_2^{n+2}) := (T_1(x_1^n), T_2(x_1^n, x_2^n))$. From (2), this sequence is well defined. For to prove that T is a WPO it is necessary to prove that $x_2^n \to x_2^*$ as $n \to \infty$.

From (2) and (4) we have:

$$\begin{aligned}
d_2(x_2^{n+1}, x_2^*) &= d_2(T_2(x_1^n, x_2^n), T_2(x_1^*, x_2^*)) \\
&\le Ld_1(x_1^n, x_1^*) + Sd_2(x_2^n, x_2^*) \\
&\le Ld_1(x_1, x_1^*) + SLd_1(x_1^{n-1}, x_1^*) + S^2 d_2(x_2^{n-1}, x_2^*) \le \dots \\
&\le Ld_1(x_1^n, x_1^*) + SLd_1(x_1^{n-1}, x_1^*) + \dots \\
&+ S^n Ld_1(x_1^0, x_1^*) + S^{n+1} d_2(x_2^0, x_2^*) \to 0
\end{aligned}$$

as $n \to \infty$, by a generalized Cauchy lemma (see [23], [35]).

6. Technique of $\mathbb{R}_+^p$-valued metrics in the theory of fibre contractions

The basic tool in the proofs of various fibre contraction-type principles is the Cauchy-Toeplitz Lemma (see [10], [23], [40], [42], [35], ...). In this section, we will present a variant of the fibre contraction principle which is a consequence of Perov's fixed point theorem (see [26], [32], [20], [19], [42], ...) in complete $\mathbb{R}_+^p$-valued metric spaces. We think that this approach open a new door for the use of vector-valued metrics in the theory of fibre contractions.

Let (X_i, d_i) ($i \in \{1, ..., m\}$ with $m \ge 2$) be metric spaces and $U_1 \subset X_1 \times X_2$, $U_2 \subset U_1 \times X_3, \dots, U_{m-1} \subset U_{m-2} \times X_m$, be nonempty and closed subsets.

If $T_1 : X_1 \to X_1$, $T_2 : U_1 \to X_2, \dots, T_m : U_{m-1} \to X_m$ are given operators, then we consider the operator $T : U_{m-1} \to X_1 \times X_2 \times \dots \times X_m$, defined by

$$T(x_1, \dots, x_m) := (T_1(x_1), T_2(x_1, x_2), \dots, T_m(x_1, x_2, \dots, x_m)).$$

In the framework of the Section 3's notation we have the following result.

Theorem 6.1. *We suppose that:*

(1) for $i \in \{1, ..., m\}$ (where $m \geq 2$) the pairs (X_i, d_i) are complete metric spaces;

(2) $(T_1, \cdots, T_{i+1})(U_i) \subset U_i$, for each $i \in \{1, ..., m-1\}$;

(3) there exists $l_1 \in]0,1[$ such that $d_1(T_1(x_1), T_1(\tilde{x}_1)) \leq l_1 d_1(x_1, \tilde{x}_1)$, for each $x_1, \tilde{x}_1 \in X_1$;

(4) there exist $L_{ij} > 0$ and $l_{i+1} \in]0,1[$, for $i \in \{1, \cdots, m-1\}$ and $j \in \{1, \cdots, i\}$ such that

$$d_{i+1}(T_{i+1}(x,y), T_{i+1}(\tilde{x},\tilde{y})) \leq \sum_{j=1}^{i} L_{ij} d_j(x_j, \tilde{x}_j) + l_{i+1} d_{i+1}(y, \tilde{y}),$$

for each $(x,y), (\tilde{x},\tilde{y}) \in U_i$, $i \in \{1, ..., m-1\}$.

Then, the triangular operator $T : U_{m-1} \to U_{m-1}$, $T := (T_1, T_2, \ldots, T_m)$ is a Picard operator with respect to the coordinatewise convergence on U_{m-1}.

Proof. On $X = \prod_{i=1}^{m} X_i$, we consider the $\mathbb{R}_+^m$-valued metric d_V defined by

$$d_V(x,y) := \begin{pmatrix} d_1(x_1, y_1) \\ \vdots \\ d_m(x_m, y_m) \end{pmatrix}.$$

We notice first that d_V induces on X the coordinatewise convergence. Secondly, from (3) and (4) we obtain that

$$d_V(T(x), T(y)) \leq S d_V(x,y), \text{ for every } x, y \in U_{m-1},$$

where

$$S = \begin{pmatrix} l_1 & 0 & 0 & \cdots & 0 & 0 \\ L_{11} & l_2 & 0 & \cdots & 0 & 0 \\ L_{21} & L_{22} & l_3 & \cdots & 0 & 0 \\ \vdots & & & & & \\ L_{m-1\ 1} & L_{m-1\ 2} & L_{m-1\ 3} & \cdots & L_{m-1\ m} & l_m \end{pmatrix}.$$

Since the spectral radius of S is $\rho(S) = \max\{l_1, \cdots l_m\} < 1$, the matrix S is convergent to 0 and Perov's theorem (see [32], [26], [19], [20]) applies. Thus, T is a PO. □

Since T is an S-contraction, we also have the following saturated variant of the above theorem, see [30].

Theorem 6.2. *In the conditions of Theorem 6.1, we also have the following conclusions:*

(i) $F_T = F_{T^m} = \{x^*\}$;

(ii) $T^n(x) \to x^*$ as $n \to \infty$, for every $x \in U_{m-1}$ (i.e., T is a PO);

(iii) $d_V(x, x^*) \leq (I_m - S)^{-1} d_V(x, T(x))$, for every $x \in U_{m-1}$;

(iv) $x^n \in U_{m-1}$ and $d_V(x^n, T(x^n)) \to 0$ as $n \to \infty$ implies that $x^n \to x^*$ as $n \to \infty$ (i.e., the fixed point problem for T is well-posed);

(v) $x^n \in U_{m-1}$ and $d_V(x^{n+1}, T(x^n)) \to 0$ as $n \to \infty$ implies that $x^n \to x^*$ as $n \to \infty$ (i.e., the operator T satisfies the Ostrowski property);

(vi) for each $\varepsilon \in (\mathbb{R}_+^*)^m$ and each y_ε^* satisfying the inequation $d_V(y, T(y)) \leq \varepsilon$ we have that $d_V(y_\varepsilon^*, x^*) \leq (I_m - S)^{-1}\varepsilon$ (i.e., the fixed point problem for T is Ulam-Hyers stable).

For other types of saturated fibre contraction principle see [41].

7. Applications

Let us consider the following Cauchy problem:

$$x'(t) = f(t, \phi(x)(t)),\ t \in [a, b], \tag{7.1}$$

$$x(a) = \alpha, \tag{7.2}$$

where $(\mathbb{B}, |\cdot|)$ is a (real or complex) Banach space, $f \in C([a, b] \times \mathbb{B}, \mathbb{B})$, $\alpha \in \mathbb{B}$, $\phi : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ is a given operator.

We suppose that:
(C_1) $\exists\ L_f > 0$: $|f(t, u) - f(t, v)| \leq L_f |u - v|,\ \forall\ t \in [a, b],\ u, v \in \mathbb{B}$;
(C_2) $\exists\ L_\phi > 0$: $|\phi(y)(t) - \phi(z)(t)| \leq L_\phi \max\limits_{a \leq \tau \leq t} |y(\tau) - z(\tau)|$, for all $t \in [a, b]$.

For a better understanding of condition (C_2) we consider the following examples:
(E_1) $\phi(x) := x$;
(E_2) $\phi(x)(t) := x(g(t))$, $t \in [a, b]$, where $g \in C([a, b], [a, b])$, $g(t) \leq t$, $t \in [a, b]$;
(E_3) $\phi(x)(t) := \max\limits_{a \leq \tau \leq t} |x(\tau)|$, $t \in [a, b]$;
(E_4) $\mathbb{B} := \mathbb{R}$, $\phi(x)(t) := \max\limits_{a \leq \tau \leq t} x(\tau)$, $t \in [a, b]$.

In all these cases, $L_\phi = 1$.

The problem $(7.1) - (7.2)$ is equivalent with the following functional integral equation:

$$x(t) = \alpha + \int_a^t f(s, \phi(x)(s))ds,\ t \in [a, b]. \tag{7.3}$$

Let $V : C([a, b], \mathbb{B}) \to C([a, b], \mathbb{B})$ be defined by

$$V(x)(t) := \text{second part of (7.3)}.$$

For problem $(7.1) - (7.2)$ we have the following result.

Theorem 7.1. *In the condition $(C_1) - (C_2)$ we have that:*

(i) the problem $(7.1) - (7.2)$ has in $C^1([a, b], \mathbb{B})$ a unique solution denoted by x^*;

(ii) the sequence $x_n := V^n(x_0)$, $n \in \mathbb{N}$, converges in $\left(C([a, b], \mathbb{B}) \stackrel{unif}{\longrightarrow}\right)$ to x^*, for each $x_0 \in \mathbb{B}$.

Proof. For $m \in \mathbb{N}^*$ we consider

$$t_k := a + \frac{k(b - a)}{m},\ k =\in \{0, ..., m\}.$$

Let $X_i = C[t_{i-1}, t_i]$, $i \in \{1, ..., m\}$, $X = \prod_{i=1}^{m} X_i$, endowed with the max-norm. We consider the following subsets:

$$U_1 := \{(x_1, x_2) \in X_1 \times X_2 \mid x_1(t_1) = x_2(t_1)\},$$
$$U_2 := \{(x_1, x_2, x_3) \in U_1 \times X_3 \mid x_2(t_2) = x_3(t_2)\},$$
$$\vdots$$
$$U_{m-1} := \{(x_1, x_2, \dots, x_m) \in U_{m-2} \times X_m \mid x_{m-1}(t_{m-1}) = x_m(t_{m-1})\},$$

and

$$U_{1x} := \{x_2 \in X_2 \mid (x, x_2) \in U_1\},\ \text{for } x \in X_1,$$
$$U_{2x} := \{x_3 \in X_3 \mid (x, x_3) \in U_2\},\ \text{for } x \in U_1, \dots,$$
$$U_{m-1x} := \{x_m \in X_m \mid (x, x_m) \in U_{m-1}\},\ \text{for } x \in U_{m-2}.$$

We remark that $U_i \neq \emptyset$, $U_{ix} \neq \emptyset$ and closed sets, $i \in \{1, ..., m-1\}$.

In our considerations we need the following operators:

$$R_i : C([a, t_{i+1}], \mathbb{B}) \to X_1 \times \ldots \times X_{i+1}$$

defined by

$$R_i(x) := \left(x\big|_{[a,t_1]}, x\big|_{[t_1,t_2]}, \dots, x\big|_{[t_i,t_{i+1}]} \right),\ i \in \{1, ..., m-1\}.$$

We observe that $R_i(C([a, t_{i+1}], \mathbb{B})) = U_i$ and $R_i : C([a, t_{i+1}], \mathbb{B}) \to U_i$ is an homeomorphism, $i \in \{1, ..., m-1\}$.

From the definition of the operator V, we have the following relations:

$$V(x)(t) = \alpha + \int_a^t f(s, \phi(x)(s))ds,\ t \in [t_0, t_1],$$
$$V(x)(t) = \alpha + \int_a^{t_1} f(s, \phi(x)(s))ds + \int_{t_1}^t f(s, \phi(x)(s))ds,\ t \in [t_1, t_2],$$
$$\vdots$$
$$V(x)(t) = \alpha + \int_a^{t_1} f(s, \phi(x)(s))ds + \int_{t_1}^{t_2} f(s, \phi(x)(s))ds + \dots$$
$$+ \int_{t_{m-2}}^{t_{m-1}} f(s, \phi(x)(s))ds + \int_{t_{m-1}}^t f(s, \phi(x)(s))ds,\ t \in [t_{m-1}, b].$$

In the conditions $(C_1) - (C_2)$ the operators ϕ and V are Volterra operators, i.e.,

$$x, y \in C([a, b], \mathbb{B}),\ x\big|_{[a,t]} = y\big|_{[a,t]} \Rightarrow$$

$$\phi(x)\big|_{[a,t]} = \phi(y)\big|_{[a,t]} \text{ and } V(x)\big|_{[a,t]} = V(y)\big|_{[a,t]}.$$

The above relations suggest us to consider the following operators induced by the operator V:

$$
\begin{aligned}
&T_1 : X_1 \to X_1, && T_1(x_1)(t) := \alpha + \int_a^t f(s, \phi(x_1)(s))ds,\\
&T_2 : U_1 \to X_2, && T_2(x_1, x_2)(t) := \alpha + \int_a^{t_1} f(s, \phi(x_1)(s))ds\\
& && \qquad + \int_{t_1}^t f(s, \phi(R_1^{-1}(x_1, x_2))(s))ds,\\
&\vdots\\
&T_m : U_{m-1} \to X_m, && T_m(x_1, x_2, \ldots, x_m)(t) := \alpha + \int_a^{t_1} f(s, \phi(x_1)(s))ds\\
& && + \int_{t_1}^{t_2} f(s, \phi(R_1^{-1}(x_1, x_2)(s)))ds + \ldots\\
& && + \int_{t_{m-1}}^{t} f(s, \phi(R_{m-1}^{-1}(x_1, x_2, \ldots, x_m)(s)ds,\ t \in [t_{m-1}, b].
\end{aligned}
$$

If we choose on $X_1 \times \ldots \times X_i$, $i \in \{2, \cdots, m\}$, the norm $\max(\|x_1\|, \ldots, \|x_i\|)$, then

$$R_i : C([a, t_{i+1}], \mathbb{B}) \to U_i$$

is an isometry, $i \in \{1, ..., m-1\}$.

From the conditions $(C_1) - (C_2)$, for a suitable choice of m, the operator

$$T := (T_1, T_2, \ldots, T_m)$$

is in the conditions of Theorem 3.2. From this theorem, T is a PO.

Since $V = R_{m-1}^{-1} T R_{m-1}$ and $V^n = R_{m-1}^{-1} T^n R_{m-1}$, the operator V is PO.

Remark 7.2. If $\mathbb{B} := \mathbb{R}^m$ or $\mathbb{C}^m$, then the problem (7.1) – (7.2) take the following form:

$$
\begin{aligned}
&x_k'(t) = f_k(t, \phi(x_1, \ldots, x_m)(t)), && t \in [a, b],\\
&x_k(a) = \alpha_k, && k \in \{1, ..., m\},
\end{aligned}
$$

where $f_k \in C\left([a, b] \times \begin{array}{c}\mathbb{R}^m\\ \mathbb{C}^m\end{array}\right)$, $\phi : C\left([a, b],\ \begin{array}{c}\mathbb{R}^m\\ \mathbb{C}^m\end{array}\right) \to C\left([a, b],\ \begin{array}{c}\mathbb{R}^m\\ \mathbb{C}^m\end{array}\right)$.

Remark 7.3. If $\mathbb{B} := l^p(\mathbb{R})$ or $\mathbb{B} := l^p(\mathbb{C})$, $1 \le p \le +\infty$, or other Banach spaces of sequences, then the problem (7.1) – (7.2) is a Cauchy problem for an infinite system of functional differential equations.

Remark 7.4. For other applications of the abstract results of this paper to functional integral equations see [11], [21].

Remark 7.5. For functional differential and integral equations see [3], [6], [29], [13], [14], [17], [18], [28], [31], [37], [42], [43].

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