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ESSENTIAL WEAKLY MÖNCH TYPE MAPS AND CONTINUATION THEORY

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Abstract. In this paper we present a variety of Leray–Schauder type continuation theorems for general classes of weakly Mönch type maps.
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1. Introduction

In this paper we introduce the notion of an essential map for a general class of Mönch type maps. Essential maps were originally introduced by Granas [2] and extended by many authors in the literature; see the book by O'Regan and Precup [11] and the references therein. In particular Leray–Schauder type alternatives for a variety of compact and noncompact (including contractive, condensing, 1–set contractive, countably condensing) classes of maps (single and multivalued) were discussed extensively in the literature; we refer the reader to the book of O'Regan and Precup [11] (see also the reference list therein) and the papers of Petryshyn and Fitzpatrick [12] and Väth [14]. The most general type of noncompact map seems to be Mönch [6] type maps and these maps were considered in detail for different classes of maps in Cardinali, O'Regan and Rubbioni [4], Precup [13] and O'Regan and Precup [10, 11]. Leray–Schauder type alternatives in the weak topology setting for different classes of maps (including weakly sequentially upper semicontinuous, weakly compact or more generally weakly condensing maps) were considered also in the literature; we refer the reader to the book by Ben Amar and O'Regan [3] (see also the reference list therein) and the papers of Ben Amar and Mnif [2] and O'Regan [1, 7]. In this paper we present new Leray–Schauder type alternatives for general weakly Mönch type maps (these maps generalize weakly compact, weakly condensing and weakly countable condensing maps).

2. Main results

Let Y be a Hausdorff locally convex topological vector space and U a weakly open subset of X where X is a closed convex subset of Y (alternatively, assume X is a weakly closed subset of Y). In this section we consider classes $\mathbf{A}$ and $\mathbf{D}$ of maps.

Definition 2.1. We say $F \in M(\overline{U^w}, X)$ (respectively $F \in MD(\overline{U^w}, X)$) if $F : \overline{U^w} \to 2^X$ and $F \in \mathbf{A}(\overline{U^w}, X)$ (respectively $F \in \mathbf{D}(\overline{U^w}, X)$); here $\overline{U^w}$ denotes the weak closure of U in X.

Definition 2.2. We say $F \in MA(\overline{U^w}, X)$ if $F \in M(\overline{U^w}, X)$ and there exists a selection $\Psi \in MD(\overline{U^w}, X)$ of F.

Remark 2.3. (i) Note in Definition 2.2, Ψ is a selection of F if $\Psi(x) \subseteq F(x)$ for $x \in \overline{U^w}$.

(ii) In many applications the spaces MA and MD are the same. For examples where the spaces MA and MD are different we refer the reader to [1] (for example, MA is the space of $WDKT$ maps [1] and MD is the space of weakly continuous single valued maps).

Definition 2.4. (i) We say $\Psi \in MD^M(\overline{U^w}, X)$ if $\Psi \in MD(\overline{U^w}, X)$ and if $D \subseteq \overline{U^w}$ and $D \subseteq \overline{co}\,(\{0\} \cup \Psi(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup \Psi(C))$ then $\overline{C^w}$ is weakly compact.

(ii) We say $F \in MA^M(\overline{U^w}, X)$ if $F \in MA(\overline{U^w}, X)$ and there exists a selection $\Psi \in MD^M(\overline{U^w}, X)$ of F.

(iii) We say $\Psi \in MD^{MM}(\Omega, X)$ (here $\Omega \subseteq X$) if $\Psi \in MD(\Omega, X)$ and if $D \subseteq \Omega$, $D = \overline{co}\,(\{0\} \cup \Psi(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup \Psi(C))$ (or $\overline{C^w} = \overline{co}\,(\{0\} \cup \Psi(C))$) then $\overline{C^w}$ is weakly compact.

(iv). We say $F \in MA^{MM}(\Omega, X)$ (here $\Omega \subseteq X$) if $F \in MA(\Omega, X)$ and there exists a selection $\Psi \in MD^{MM}(\Omega, X)$ of F.

Definition 2.5. We say $F \in MA^M_{\partial U}(\overline{U^w}, X)$ (respectively $F \in MD^M_{\partial U}(\overline{U^w}, X)$) if $F \in MA^M(\overline{U^w}, X)$ (respectively $F \in MD^M(\overline{U^w}, X)$) with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the weak boundary of U in X.

Definition 2.6. Let $F \in MA^M_{\partial U}(\overline{U^w}, X)$. We say F is essential in $MA^M_{\partial U}(\overline{U^w}, X)$ if for any selection $\Psi \in MD^M(\overline{U^w}, X)$ of F and any map $J \in MD^M_{\partial U}(\overline{U^w}, X)$ with $J|_{\partial U} = \Psi|_{\partial U}$ there exists an $x \in U$ with $x \in J\,(x)$.

Remark 2.7. (i) Note if $F \in MA^M_{\partial U}(\overline{U^w}, X)$ is essential in $MA^M_{\partial U}(\overline{U^w}, X)$ and if $\Psi \in MD^M(\overline{U^w}, X)$ is any selection of F then there exists an $x \in U$ with $x \in \Psi\,(x)$ (take $J = \Psi$ in Definition 2.6; note if $x \in \partial U$ then $\Psi(x) \subseteq F(x)$ so $x \notin \Psi(x)$). Finally note if $x \in \Psi(x)$ for $x \in U$ then $x \in \Psi(x) \subseteq F(x)$.

(ii) In Definition 2.4 (and throughout the paper) we could replace $\{0\}$ with $\{x_0\}$ where $x_0 \in X$ is fixed.

(iii) We note here that an assumption was inadvertently left out in [1]. In [1, Definition 2.8] the weakly continuous selection Ψ of F should be required to satisfy

Property (A) (this was inadvertently left out) i.e. if F satisfies Property (A) then it should be assumed that any weakly continuous selection Ψ of F satisfies Property (A) (of course this assumption is automatically satisfied for the type of map considered in the literature i.e. Property (A) usually means that the map is weakly compact or weakly condensing).

We begin with a nonlinear alternative of Leray–Schauder type (a more general result will be presented in Theorem 2.14 and Theorem 2.15).

Theorem 2.8. *Let Y be a Hausdorff locally convex topological vector space, U a weakly open subset of X where X is a closed convex subset of Y and $F \in MA^M(\overline{U^w}, X)$. Also assume*

$$\begin{cases} \text{the zero map (denoted by } 0\text{) is in } MA^M_{\partial U}(\overline{U^w}, X) \\ \text{and in } MD(\overline{U^w}, X) \text{ and } 0 \text{ is essential} \\ \text{in } MA^M_{\partial U}(\overline{U^w}, X). \end{cases} \tag{2.1}$$

In addition for any selection $\Psi \in MD^M(\overline{U^w}, X)$ of F suppose

$$\begin{cases} \mu\,\Psi \in MD(\overline{U^w}, X) \text{ for any weakly continuous} \\ \text{map } \mu : \overline{U^w} \to [0,1] \text{ with } \mu(\partial U) = 0 \end{cases} \tag{2.2}$$

$$x \notin t\,\Psi(x) \text{ for every } x \in \partial U \text{ and } t \in (0,1) \tag{2.3}$$

and

$$\begin{cases} \Omega = \{x \in \overline{U^w} :\ x \in t\,\Psi(x) \text{ for some } t \in [0,1]\} \\ \text{is weakly compact.} \end{cases} \tag{2.4}$$

Then there exists an $x \in \overline{U^w}$ with $x \in F(x)$.

Remark 2.9. In (2.1) if $0 \in MA(\overline{U^w}, X)$ then $0 \in MA^M(\overline{U^w}, X)$ since if there exists a selection $\Lambda \in MD^M(\overline{U^w}, X)$ of 0 (note $\Lambda = 0$ since $\Lambda(x) \subseteq 0(x)$ for $x \in \overline{U^w}$ and in (2.1) we have $0 \in MD(\overline{U^w}, X)$) then if $D \subseteq \overline{U^w}$, $D \subseteq \overline{co}\,(\{0\} \cup \Lambda(D)) = \overline{co}\,(\{0\} \cup 0(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup 0(C))$ (note $0(x) = \{0\}$ for $x \in C$) so (trivially) $\overline{C^w}$ is weakly compact.

Proof. Suppose $x \notin F(x)$ for $x \in \partial U$ (otherwise we are finished).
Let $\Psi \in MD^M(\overline{U^w}, X)$ be any selection of F and let Ω be as in the statement of Theorem 2.8. Note $\Omega \neq \emptyset$ since 0 is essential in $MA^M_{\partial U}(\overline{U^w}, X)$ (see Remark 2.7 (i)). Next note $\Omega \cap \partial U = \emptyset$ (see (2.3), $x \notin F(x)$ for $x \in \partial U$ is assumed at the beginning of the proof, and $0 \in MA^M_{\partial U}(\overline{U^w}, X)$). Now recall $Y = (Y, w)$, the space Y endowed with the weak topology, is completely regular. Thus there exists a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$ (note ∂U is weakly closed in X and X is weakly closed in Y so ∂U is weakly closed in Y). Define a map R by $R(x) = \mu(x)\,\Psi(x)$ and note (2.2) guarantees that $R \in MD(\overline{U^w}, X)$. We now show $R \in MD^M(\overline{U^w}, X)$. To see this let $D \subseteq \overline{U^w}$ and $D \subseteq \overline{co}\,(\{0\} \cup R(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup R(C))$. Note $R(C) \subseteq co\,(\{0\} \cup \Psi(C))$, $R(D) \subseteq co\,(\{0\} \cup \Psi(D))$ so

$$\overline{co}\,(\{0\} \cup R(D)) \subseteq \overline{co}\,(\{0\} \cup co\,(\{0\} \cup \Psi(D))) = \overline{co}\,(co\,(\{0\} \cup \Psi(D))) = \overline{co}\,(\{0\} \cup \Psi(D))$$

and

$$\overline{co}\,(\{0\} \cup R(C)) \subseteq \overline{co}\,(\{0\} \cup \Psi(C)).$$

Thus

$$D \subseteq \overline{co}\,(\{0\} \cup \Psi(D)) \ \text{ and } \ C \subseteq \overline{co}\,(\{0\} \cup \Psi(C)).$$

Then since $\Psi \in MD^M(\overline{U^w}, X)$ we have that $\overline{C^w}$ is weakly compact. Thus $R \in MD^M(\overline{U^w}, X)$. Next notice if $x \in \partial U$ then $R(x) = \{0\}$ (note $\mu(\partial U) = 0$) and since $0 \in MA^M_{\partial U}(\overline{U^w}, X)$ then $x \notin R(x)$. Thus $R \in MD^M_{\partial U}(\overline{U^w}, X))$ with $R|_{\partial U} = 0|_{\partial U}$ and since 0 is essential in $MA^M_{\partial U}(\overline{U^w}, X)$ (note any selection $\Lambda \in MD^M(\overline{U^w}, X)$ of 0 is 0 since $0 \in MD(\overline{U^w}, X)$ from (2.1) and $\Lambda(x) \subseteq 0(x)$ for $x \in \overline{U^w}$) then there exists a $x \in U$ with $x \in R(x) = \mu(x)\,\Psi(x)$. Thus $x \in \Omega$ so $\mu(x) = 1$ and as a result $x \in \Psi(x) \subseteq F(x)$. □

Remark 2.10. Note (2.3) in Theorem 2.3 could be replaced by $x \notin t\,F(x)$ for every $x \in \partial U$ and $t \in (0, 1)$.

We now present a result which guarantees (2.1).

Theorem 2.9. *Let Y be a Hausdorff locally convex topological vector space, U a weakly open subset of X where X is a closed convex subset of Y, $0 \in U$ and assume the following conditions hold:*

$$0 \in MA(\overline{U^w}, X) \quad \textit{and} \quad 0 \in MD(\overline{U^w}, X) \tag{2.5}$$

$$\begin{cases} \textit{for any map } J \in MD^M_{\partial U}(\overline{U^w}, X) \ \textit{ with } \ J|_{\partial U} = 0|_{\partial U} \ \textit{ and} \\ R(x) = \begin{cases} J(x), & x \in \overline{U^w} \\ \{0\}, & x \in X \setminus \overline{U^w}, \end{cases} \\ \textit{we have that } \ R \in MD(X, X) \end{cases} \tag{2.6}$$

$$\begin{cases} \textit{for any map } J \in MD^M_{\partial U}(\overline{U^w}, X) \ \textit{ with } \ J|_{\partial U} = 0|_{\partial U} \ \textit{ and for any} \\ \textit{countable set } \ P \subseteq X \ \textit{ with } \ P \cap \overline{U^w} \ \textit{ relatively weakly compact} \\ \textit{we have that the set } \ \overline{co}\,(\{0\} \cup J(P \cap \overline{U^w})\,) \ \textit{ is weakly compact} \end{cases} \tag{2.7}$$

and

$$\begin{cases} \textit{for any map } \ H \in MD^{MM}(X, X) \ \textit{ there exists} \\ x \in X \ \textit{ with } \ x \in H(x). \end{cases} \tag{2.8}$$

Then the zero map is essential in $MA^M_{\partial U}(\overline{U^w}, X)$.

Remark 2.12. Note conditions to guarantee (2.8) can be found in [8, 9].

Remark 2.13. In the proof below we will in fact show R in (2.6) is in $MD^{MM}(X, X)$ so one could replace (2.8) with: there exists $x \in X$ with $x \in R(x)$.

Proof. Let $\Psi \in MD^M(\overline{U^w}, X)$ be any selection of 0; note $\Psi = 0$ (note $\Psi(x) \subseteq 0(x)$ for $x \in \overline{U^w}$ and $0 \in MD(\overline{U^w}, X)$). Now consider the map $J \in MD^M_{\partial U}(\overline{U^w}, X)$ with $J|_{\partial U} = \Psi|_{\partial U} = 0|_{\partial U}$. We must show there exists a $x \in U$ with $x \in J(x)$. Let R be as in (2.6) and note $R \in MD(X, X)$. We claim $R \in MD^{MM}(X, X)$. To see this let $D \subseteq X$ and $D = \overline{co}\,(\{0\} \cup R(D))$ with $C \subseteq D$ countable and $C \subseteq \overline{co}\,(\{0\} \cup R(C))$ (or $\overline{C^w} = \overline{co}\,(\{0\} \cup R(C))$). First note $\overline{co}\,(\{0\} \cup R(D)) \subseteq \overline{co}\,(\{0\} \cup J(D \cap \overline{U^w}))$ so

$$D = \overline{co}\,(\{0\} \cup R(D)) \subseteq \overline{co}\,(\{0\} \cup J(D \cap \overline{U^w}))$$

and

$$C \subseteq \overline{co}\,(\{0\} \cup J(C \cap \overline{U^w})).$$

As a result

$$D \cap \overline{U^w} \subseteq \overline{co}\,(\{0\} \cup J(D \cap \overline{U^w})) \text{ and } C \cap \overline{U^w} \subseteq \overline{co}\,(\{0\} \cup J(C \cap \overline{U^w})); \qquad (2.9)$$

note $C \cap \overline{U^w}$ is countable. Now since $J \in MD^M(\overline{U^w}, X)$ we have (see (2.9)) that $\overline{C \cap \overline{U^w}}^w$ is weakly compact. Now (2.7) guarantees that $\overline{C^w}$ is weakly compact (recall $C \subseteq \overline{co}\,(\{0\} \cup J(C \cap \overline{U^w}))$. Thus $R \in MD^{MM}(X, X)$.

Now (2.8) guarantees that there exists a $x \in X$ with $x \in R(x)$. There are two cases to consider, namely $x \in U$ and $x \in X \setminus U$. If $x \in U$ then $x \in J(x)$, and we are finished. If $x \in X \setminus U$ then since $R(x) = \{0\}$ (note also $J|_{\partial U} = 0|_{\partial U}$) we have $0 \in X \setminus U$, and this contradicts $0 \in U$. □

Our final two results are generalizations of Theorem 2.8.

Theorem 2.14. *Let Y be a Hausdorff locally convex topological vector space, U a weakly open subset of X where X is a closed convex subset of Y (alternatively, assume X is a weakly closed subset of Y), $F \in MA^M(\overline{U^w}, X)$ and $G \in MA^M_{\partial U}(\overline{U^w}, X)$ is essential in $MA^M_{\partial U}(\overline{U^w}, X)$. For any selector $\Psi \in MD^M(\overline{U^w}, X)$ (respectively $\Lambda \in MD^M(\overline{U^w}, X)$) of F (respectively G) suppose there exists a map $H^{\Lambda,\Psi} : \overline{U^w} \times [0,1] \to 2^X$ with $H^{\Lambda,\Psi}(\,.\,,\eta(\,.\,)) \in MD^M(\overline{U^w}, X)$ for any weakly continuous function $\eta : \overline{U^w} \to [0,1]$ with $\eta(\partial U) = 0$, $x \notin H^{\Lambda,\Psi}_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H^{\Lambda,\Psi}_t(x) = H^{\Lambda,\Psi}(x,t)$), $H^{\Lambda,\Psi}_1 = \Psi$, $H^{\Lambda,\Psi}_0 = \Lambda$ and $\Omega = \{x \in \overline{U^w} : x \in H^{\Lambda,\Psi}(x,t) \text{ for some } t \in [0,1]\}$ is weakly compact. Then there exists a $x \in \overline{U^w}$ with $x \in F(x)$.*

Proof. Suppose $x \notin F(x)$ for $x \in \partial U$ (otherwise we are finished).
Let $\Psi \in MD^M(\overline{U^w}, X)$ (respectively $\Lambda \in MD^M(\overline{U^w}, X)$) be any selector of F (respectively G) and let Ω and $H^{\Lambda,\Psi}$ be as in the statement of Theorem 2.14. Note $\Omega \neq \emptyset$ (note G is essential in $MA^M_{\partial U}(\overline{U^w}, X)$ and $H^{\Lambda,\Psi}_0 = \Lambda$) and $\Omega \cap \partial U = \emptyset$. Note $Y = (Y, w)$ is completely regular so there exists a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H^{\Lambda,\Psi}(x, \mu(x))$ and note $R \in MD^M_{\partial U}(\overline{U^w}, X)$ with $R|_{\partial U} = \Lambda|_{\partial U}$ (note if $x \in \partial U$ then $R(x) = H^{\Lambda,\Psi}(x,0) = \Lambda(x)$). Since G is essential in $MA^M_{\partial U}(\overline{U^w}, X)$ there an exists $x \in U$ with $x \in R(x) = H^{\Lambda,\Psi}_{\mu(x)}(x)$. Thus $x \in \Omega$ so $\mu(x) = 1$. As a result $x \in H^{\Lambda,\Psi}_1(x) = \Psi(x)$. □

It is also possible to generalize slightly the result in Theorem 2.14 if one modifies slightly the assumptions.

Theorem 2.15. *Let Y be a Hausdorff locally convex topological vector space, U a weakly open subset of X where X is a closed convex subset of Y (alternatively, assume X is a weakly closed subset of Y), $F \in MA^M_{\partial U}(\overline{U^w}, X)$ and $G \in MA^M_{\partial U}(\overline{U^w}, X)$ is essential in $MA^M_{\partial U}(\overline{U^w}, X)$.*
For any selector $\Psi \in MD^M(\overline{U^w}, X)$ (respectively $\Lambda \in MD^M(\overline{U^w}, X)$) of F (respectively G) and any map $J \in MD^M_{\partial U}(\overline{U^w}, X)$ with $J|_{\partial U} = \Psi|_{\partial U}$ there exists a

map $H^{J,\Lambda,\Psi} : \overline{U^w} \times [0,1] \to 2^X$ *with* $H^{J,\Lambda,\Psi}(\,.\,, \eta(\,.\,)) \in MD^M(\overline{U^w}, X)$ *for any weakly continuous function* $\eta : \overline{U^w} \to [0,1]$ *with* $\eta(\partial U) = 0$, $x \notin H_t^{J,\Lambda,\Psi}(x)$ *for any* $x \in \partial U$ *and* $t \in (0,1)$ *(here* $H_t^{J,\Lambda,\Psi}(x) = H^{J,\Lambda,\Psi}(x,t)$*),* $H_1^{J,\Lambda,\Psi} = J$, $H_0^{J,\Lambda,\Psi} = \Lambda$ *and* $\Omega = \{x \in \overline{U^w} : \ x \in H^{J,\Lambda,\Psi}(x,t) \ \text{ for some } \ t \in [0,1]\}$ *is weakly compact. Then* F *is essential in* $MA^M_{\partial U}(\overline{U^w}, X)$*.*

Proof. Let $\Psi \in MD^M(\overline{U^w}, X)$ (respectively $\Lambda \in MD^M(\overline{U^w}, X)$) be any selector of F (respectively G). Consider any map $J \in MD^M_{\partial U}(\overline{U^w}, X)$ with $J|_{\partial U} = \Psi|_{\partial U}$. We must show there exists a $x \in U$ with $x \in J(x)$. Now let $H^{J,\Lambda,\Psi}$ and Ω be as in the statement of Theorem 2.15. Note $\Omega \neq \emptyset$ (note G is essential in $MA^M_{\partial U}(\overline{U^w}, X)$ and $H_0^{J,\Lambda,\Psi} = \Lambda$) and $\Omega \cap \partial U = \emptyset$. There exists a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define the map R by $R(x) = H^{J,\Lambda,\Psi}(x, \mu(x))$ and note $R \in MD^M_{\partial U}(\overline{U^w}, X)$ with $R|_{\partial U} = H_0^{J,\Lambda,\Psi}|_{\partial U} = \Lambda|_{\partial U}$ so since G is essential in $MA^M_{\partial U}(\overline{U^w}, X)$ there exists an $x \in U$ with $x \in R(x) = H^{J,\Lambda,\Psi}_{\mu(x)}(x)$. Thus $x \in \Omega$ so $\mu(x) = 1$. As a result $x \in H_1^{J,\Lambda,\Psi}(x) = J(x)$. □

References

[1] R.P. Agarwal, D. O'Regan, *Continuation methods for closed, weakly closed, DKT and WDKT maps*, Computers and Mathematics with Applications, **38**(1999), 81-88.

[2] A. Ben Amar, M. Mnif, *Leray-Schauder alternatives for weakly sequentially continuous mappings and applications to transport equations*, Math. Methods Appl. Sci., **33**(2010), 80-90.

[3] A. Ben Amar, D. O'Regan, *Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications*, Springer, Cham, 2016.

[4] T. Cardinali, D. O'Regan, P. Rubbioni, *Mönch sets and fixed point theorems for multimaps in locally convex topological spaces*, Fixed Point Theory, **18**(2017), 147-153.

[5] A. Granas, *Sur la méthode de continuité de Poincaré*, C.R. Acad. Sci. Paris, **282**(1976), 983-985.

[6] H. Mönch, *Boundary value problems for nonlinear ordinary differential equations in Banach spaces*, Nonlinear Anal., **4**(1980), 985-999.

[7] D. O'Regan, *Mönch type results for maps with weakly sequentially closed graphs*, Dynamic Systems and Applications, **24**(2015), 129-134.

[8] D. O'Regan, *Maps with weakly sequentially closed graphs satisfying compactness conditions on countable sets*, Pure and Applied Functional Analysis, 6(2021), 367-372.

[9] D. O'Regan, *Coincidence results for compositions of multivalued maps based on countable compactness principles*, Journal of Nonlinear and Convex Analysis, 21(2020), 543-549.

[10] D. O'Regan, R. Precup, *Fixed point theorems for set-valued maps and existence principles for integral inclusions*, J. Math. Anal. Appl., **245**(2000), 594-612.

[11] D. O'Regan, R. Precup, *Theorems of Leray-Schauder Type and Applications*, Gordon and Breach Science Publishers, Amsterdam, 2001.

[12] W.V. Petryshyn, P.M. Fitzpatrick, *A degree theory, fixed point theorems and mapping theorems for multivalued noncompact mappings*, Trans. Amer. Math. Soc., **194**(1974), 1-25.

[13] R. Precup, *On the topological transversality principle*, Nonlinear Anal., **20**(1993), 1-9.

[14] M. Väth, *Fixed point theorems and fixed point index for countable condensing maps*, Topological Methods in Nonlinear Anal., **13**(1999), 341-363.

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