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# HÖLDER-TYPE SPACES, SINGULAR OPERATORS, AND FIXED POINT THEOREMS

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**Abstract.** In this note, we give a sufficient condition for the existence of Hölder-type solutions to a class of fractional initial value problems involving Caputo derivatives. Since imposing (classical or general) global Lipschitz conditions on the nonlinear operators involved leads to degeneracy phenomena, the main emphasis is put on local Lipschitz conditions or fixed point principles of Schauder and Darbo type. To this end, we study continuity and boundedness conditions for linear Riemann-Liouville operators and nonlinear Nemytskij operators in Hölder spaces of integral type which have much better properties than classical Hölder spaces.
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## 1. Statement of the problem

There is a vast literature on boundary and initial value problems for nonlinear second order differential equations. However, replacing a second order differential operator by operators of fractional order have found less attention, although they frequently occur in applications. For example, in the recent paper [10], the following result has been proved.

**Theorem 1.1.** *Suppose that* $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ *is continuous and satisfies the Lipschitz condition*

$$|f(t,u) - f(t,v)| \leq k|u - v| \qquad (a \leq t \leq b, u, v \in \mathbb{R}). \tag{1.1}$$

*If*

$$b - a < \frac{\Gamma(\tau)^{\frac{1}{\tau}}\tau^{\frac{\tau+1}{\tau}}}{k^{\frac{1}{\tau}}(\tau-1)^{\frac{\tau-1}{\tau}}},$$

*where* $\Gamma$ *denotes the classical Euler Gamma function, then the boundary value problem*

$$\begin{cases} D^\tau x(t) = f(t, x(t)) \quad (a < t < b), \\ x(a) = 0, \ x(b) = B, \end{cases} \tag{1.2}$$

*where* $B \in \mathbb{R}$, $\tau > 0$, *and* $D^\tau$ *denotes the fractional Riemann-Liouville derivative of order* $\tau$, *has a unique continuous solution on* $[a,b]$.

Theorem 1.1 is proved by applying the classical Banach-Caccioppoli contraction mapping principle in the space $C[a,b]$. Motivated by these results, the authors of [3] have studied the same question for the general fractional initial value problem

$$\begin{cases} D_c^\tau x(t) = f(t, x(t)) + D_c^{\tau-1} g(t, x(t)) \quad (a < t < b), \\ x(a) = \theta_1, \ x'(a) = \theta_2, \end{cases} \tag{1.3}$$

where $\tau > 1$, $f, g : [a,b]\times\mathbb{R} \to \mathbb{R}$ are given continuous functions, $\theta_1, \theta_2 \in \mathbb{R}$, and $D_c^\tau$ denotes the Caputo fractional derivative. Recall that the *Caputo fractional derivative* of order $\tau \geq 0$ of a function $x : [a,b] \to \mathbb{R}$ is defined by

$$D_c^\tau x(t) = \begin{cases} x(t) & \text{for} \quad \tau = 0, \\ \dfrac{1}{\Gamma(n-\tau)} \displaystyle\int_a^t (t-s)^{n-\tau-1} x^{(n)}(s)\, ds & \text{for} \quad \tau > 0 \end{cases} \tag{1.4}$$

for any $t \in [a,b]$, where $n = [\tau] + 1$ and $[\tau]$ denotes the integer part of $\tau$, provided that the right side of (1.4) is pointwise defined.

In the existence and uniqueness proof for solutions of (1.3) it is not required that $f$ and $g$ satisfy the Lipschitz condition (1.1), but the more general condition

$$|f(t,u) - f(t,v)| \leq \phi(|u - v|),$$

and similarly for $g$, where $\phi$ is a *comparison function* in the terminology of the book [19]. This means that $\phi : [0,\infty) \to [0,\infty)$ is increasing and right-continuous and satisfies $\phi(u) < u$ for $u > 0$. The main tool in [3] is then a generalized contraction mapping principle due to Matkowski [16] which reads as follows: *Let* $X$ *be a Banach space,* $M \subseteq X$ *closed, and* $T : M \to M$ *an operator satisfying*

$$\|Tx - Ty\| \leq \phi(\|x - y\|) \qquad (x, y \in M) \tag{1.5}$$

*for some comparison function* $\phi$. *Then* $T$ *has a unique fixed point in* $M$.

Of course, the choice $\phi(t) = kt$ with $0 < k < 1$ gives the classical Banach-Caccioppoli contraction mapping principle. However, allowing more general choices for $\phi$ considerably enlarges the applicability of this fixed point theorem, as was shown in [3] by means of examples involving the comparison functions $\phi(u) = \arctan u$ and $\phi(u) = \log(1 + u)$.

Such results, however, have an essential flaw: the space $C[a, b]$ is not very suitable for fractional integral and differential operators. Instead, such operators are "better behaved" in Lebesgue, Sobolev, or Hölder spaces. For that reason, we will study (a variant of) problem (1.3) in Hölder-type spaces involving moduli of continuity in integral form.

However, if we try to apply Matkowski's fixed point theorem by imposing the generalized contraction condition (1.5) on simple nonlinear operators like the Nemytskij operator

$$Fx(t) = f(t, x(t)) \tag{1.6}$$

in Hölder spaces, we encounter another very unpleasant surprise: such a (global) contraction condition can be satisfied only if the underlying function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is *affine*. As far as we know, a degeneracy phenomenon of this kind has been proved first for the classical contraction condition by Matkowski [17] in the space of Lipschitz continuous functions, by Matkowska [15] in the space of Hölder continuous functions, and subsequently in many other function spaces. We will show below that the same degeneracy phenomenon holds for the operator (1.6) in Hölder spaces if we replace a classical norm-contraction condition by the more general condition (1.5).

A natural idea would therefore be to study the above problem by means of the Schauder fixed point principle. But here we have another problem: we need then a compactness criterion for closed bounded sets, and such a criterion is simply unknown in Hölder spaces.

It turns out that the integral-type Hölder spaces which we will use below for studying the initial value problem (1.3) do not have these drawbacks: nonlinear operators in general do not degenerate here if we impose a Lipschitz condition in norm, and compactness criteria can be easily formulated. This illustrates quite well the fact that using integral-type Hölder spaces, rather than classical Hölder spaces, is a very useful device.

This paper is organized as follows. In the next section we recall the definition and some properties of integral-type Hölder spaces. Afterwards we study linear operators (like Riemann-Liouville integral operators) and nonlinear operators (like the Nemytskij operator (1.6)) in such spaces. Subsequently, we formulate compactness conditions and study growth conditions which guarantee the existence of invariant closed balls for the linear and nonlinear operators involved. This leads to existence theorems for solutions of problem (1.3) which are not just continuous, but have better properties. In the last section we give an example of such a theorem.

## 2. Integral-type Hölder spaces

Since we are going to apply fixed point theorems for compact and condensing operators, we do not need, as in the paper [3], to impose restrictions on the existence interval of solutions to (1.3). So in this and the following sections we take without loss of generality $[a, b] = [0, 1]$.

Recall that the *modulus of continuity* of a continuous function $x : [0, 1] \to \mathbb{R}$ is defined by

$$\omega(x; \sigma) := \sup \{|x(s) - x(t)| : 0 \le s, t \le 1, |s - t| \le \sigma\}. \tag{2.1}$$

Occasionally we will also consider the modulus of continuity of a function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ of two variables which is analogously defined by

$$\begin{aligned}\omega(f; \sigma, \mu) := \sup \{|f(s, u) - f(t, v)| : 0 \le s, t \le 1, u, v \in \mathbb{R},\\ |s - t| \le \sigma, |u - v| \le \mu\}.\end{aligned} \tag{2.2}$$

Following [2] we put, for $0 < \alpha \le 1$, $\alpha \le \beta < \infty$, and $0 < s \le 1$,

$$j_{\alpha,\beta}(x; [0, s]) := \int_0^s \sigma^{-(\beta+1)} \omega(x; \sigma)^{\beta/\alpha} \, d\sigma, \tag{2.3}$$

and denote by $J_{\alpha,\beta}[0, 1]$ the linear space of all functions $x \in C[0, 1]$ for which $j_{\alpha,\beta}(x; [0, 1])$ is finite. Equipped with the norm

$$\|x\|_{\alpha,\beta} := |x(0)| + j_{\alpha,\beta}(x; [0, 1])^{\alpha/\beta}, \tag{2.4}$$

or the equivalent norm

$$|||x|||_{\alpha,\beta} := \|x\|_C + j_{\alpha,\beta}(x; [0, 1])^{\alpha/\beta}, \tag{2.5}$$

the set $J_{\alpha,\beta}[0, 1]$ is a Banach space. We will use both norms in the sequel. One may extend this definition to the case $\beta = \infty$ by putting

$$j_{\alpha,\infty}(x; [0, 1]) := \sup \{\sigma^{-\alpha} \omega(x; \sigma) : 0 < \sigma \le 1\}$$

and

$$\|x\|_{\alpha,\infty} := |x(0)| + j_{\alpha,\infty}(x; [0, 1]).$$

The corresponding set $J_{\alpha,\infty}[0, 1]$ is then nothing else but the classical Hölder space $C^\alpha[0, 1]$. We will refer to $J_{\alpha,\beta}[0, 1]$ as *integral-type Hölder space* in what follows.

Let us note that a similar space was introduced and studied by Gusejnov and Mukhtarov in Chapter 2 of the monograph [12]. Given $p \in [1, \infty)$ and a function $\varphi : [0, 1] \to \mathbb{R}$ with

$$\int_0^1 \frac{\varphi(t)^p}{t^{1+p}} \, dt = \infty, \qquad \int_0^1 \frac{\varphi(t)^p}{t} \, dt < \infty,$$

the authors of [12] denote by $I_{\varphi,p}[0, 1]$ the linear space of all continuous functions satisfying

$$\int_0^1 \sigma^{-(p+1)} \omega(x; \sigma)^p \varphi(\sigma)^p \, d\sigma < \infty.$$

A comparison with (2.3) shows that $I_{\varphi,p}[0, 1] = J_{\alpha,\beta}[0, 1]$ if we choose $p := \beta/\alpha$ and $\varphi(t) := t^{1-\alpha}$. The reader who is familiar with interpolation theory may also have noticed a similarity with the Lorentz space $L_{p,q}$ (in particular, $L_{p,p} = L_p$ for $q = p$). Indeed, if we replace the modulus of continuity (2.1) of a continuous function $x$ by the modulus of measurability

$$\omega(x; \sigma) := \text{meas}(\{s : 0 \le s \le 1, |x(s)| < 1/\sigma\})$$

of a measurable function $x$, then the condition

$$\int_0^1 \sigma^{-(q+1)} \omega(x;\sigma)^{q/p}\, d\sigma < \infty$$

is fulfilled if and only if $x \in L_{p,q}[0,1]$ (see, e.g., [5,14]). As the $L_{p,q}$-spaces, the spaces $J_{\alpha,\beta}$ are also "decreasing" in the first index and "increasing" in the second index. More precisely, the chain of (strict) inclusions

$$J_{\alpha,\beta}[0,1] \subset J_{\alpha,\infty}[0,1] = C^\alpha[0,1] \subset J_{\gamma,\delta}[0,1]$$

holds for $\gamma < \alpha$ and $\delta > \beta$. To see that these inclusions are strict it suffices to note that the function $x_\theta(t) := t^\theta$ belongs to $J_{\alpha,\beta}[0,1]$ for $\theta > \alpha$, and to $C^\alpha[0,1]$ for $\theta \geq \alpha$.

## 3. Riemann-Liouville operators

In our study of the initial value problem we have to consider, apart from the nonlinear Nemytskij operator (1.6), a weakly singular linear integral operator whose definition we recall in the following

**Definition 3.1.** Let $\tau \geq 0$ and $f$ be a real function defined on the interval $[0,1]$. The *Riemann-Liouville fractional integral* of order $\tau$ of $f$ is defined by

$$I^\tau x(t) = \begin{cases} x(t) & \text{for} \quad \tau = 0, \\ \dfrac{1}{\Gamma(\tau)} \displaystyle\int_0^t (t-s)^{\tau-1} x(s)\, ds & \text{for} \quad \tau > 0 \end{cases} \tag{3.1}$$

for any $t \in [0,1]$.

The operator $I^\tau$ has many interesting properties. For instance, the *semigroup property*

$$(I^\sigma \circ I^\tau)x(t) = I^{\sigma+\tau}x(t) \qquad (\sigma, \tau \geq 0)$$

holds for any $x \in L_1[0,1]$. Moreover, for $\alpha + \tau < 1$, the operator $I^\tau$ is a bijection between $C_0^\alpha[0,1]$ and $C_0^{\alpha+\tau}[0,1]$ with inverse

$$D^\tau y(t) = \begin{cases} y(t) & \text{for} \quad \tau = 0, \\ \dfrac{1}{\Gamma(1-\tau)} \dfrac{d}{dt} \displaystyle\int_0^t (t-s)^{-\tau} y(s)\, ds & \text{for} \quad 0 < \tau < 1-\alpha, \end{cases} \tag{3.2}$$

where $C_0^\alpha[0,1]$ denotes the subspace of all $x \in C^\alpha[0,1]$ satisfying $x(0) = 0$.

The Caputo derivative occuring in (1.4), however, is *not* the inverse operator to (3.1). Instead, the equality

$$(I^\tau \circ D_c^\tau)x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k \tag{3.3}$$

holds true, whenever $x \in C^{n-1}[0,1]$ and $x^{(n)}$ exists almost everywhere on $[0,1]$. The equality (3.3) may be regarded as a fractional analogue to the Lagrange mean value theorem for differentiable functions in the classical sense. For more details about the theory and applications of fractional differential equations we refer to the monograph [13].

Since we will always consider functions over $[0,1]$ in the sequel, we will drop from now on the interval in the notation of function spaces. As pointed out in the first section, the space $C$ of continuous functions is not very suitable for studying the operator (3.1). Instead, this operator is much better behaved, e.g., in Lebesgue spaces. For instance, the classical *Hardy-Littlewood theorem* states that $I^\tau$ maps $L_p$ continuously into $L_{p/(1-p\tau)}$ provided that $1 < p < 1/\tau$. Our first and second examples show that this is false for $p = 1$ or $p = 1/\tau$.

**Example 3.2.** Let $p = 1$, i.e.,

$$\frac{p}{1-p\tau} = \frac{1}{1-\tau}.$$

The positive function

$$x(t) = \begin{cases} \dfrac{1}{t}\left(\log\dfrac{1}{t}\right)^{(\tau-3)/2} & \text{for} \quad 0 < t < \dfrac{1}{2}, \\ 0 & \text{for} \quad \dfrac{1}{2} \leq t < 1 \end{cases}$$

has the primitive

$$z(t) := \frac{2}{1-\tau}\left(\log\frac{1}{t}\right)^{-(1-\tau)/2}.$$

Therefore

$$\int_0^1 |x(t)|\,dt = \frac{2}{1-\tau}(\log 2)^{-(1-\tau)/2} < \infty,$$

i.e., $x \in L_1$. On the other hand, the substitution $\sigma := s/t$ yields

$$I^\tau x(t) = \int_0^t \frac{\left(\log\frac{1}{s}\right)^{(\tau-3)/2}}{s(t-s)^{1-\tau}}\,ds = \frac{1}{t^{1-\tau}}\int_0^1 \frac{\left(\log\frac{1}{t\sigma}\right)^{(\tau-3)/2}}{\sigma(1-\sigma)^{1-\tau}}\,d\sigma$$

$$\geq \frac{2}{(1-\tau)t^{1-\tau}}\left(\log\frac{1}{t}\right)^{-(1-\tau)/2},$$

hence

$$\|I^\tau x\|_{L_{1/(1-\tau)}}^{1/(1-\tau)} = \int_0^{1/2} |I^\tau x(t)|^{1/(1-\tau)}\,dt \geq \frac{2^{1/(1-\tau)}}{(1-\tau)^{1-\tau}}\int_0^{1/2}\frac{1}{t}\left(\log\frac{1}{t}\right)^{-1/2} dt = \infty,$$

i.e., $I^\tau x \notin L_{1/(1-\tau)}$ as claimed. □

**Example 3.3.** Let $p = 1/\tau$, i.e.,

$$\frac{p}{1-p\tau} = \infty.$$

Now we "reflect" the function $x$ of the preceding example at $1/2$, which means that we consider the function

$$\hat{x}(t) = x(1-t) = \begin{cases} \dfrac{1}{1-t}\left(\log\dfrac{1}{1-t}\right)^{(\tau-3)/2} & \text{for} \quad 0 < t < \dfrac{1}{2}, \\ 0 & \text{for} \quad \dfrac{1}{2} \le t < 1. \end{cases}$$

Of course, the function $\hat{x}$ also belongs to $L_1$. Putting

$$y(t) := \int_t^1 \frac{\hat{x}(s)}{|t-s|^{1-\tau}}\, ds \quad (0 \le t \le 1),$$

an easy calculations shows that $y(t) = I^\tau x(1-t)$, so $y \notin L_{1/(1-\tau)}$, by Example 3.2. On the other hand, if the operator (3.1) would map $L_{1/\tau}$ into $L_\infty$, integration by parts would imply

$$\int_0^1 z(t)y(t)\, dt \le \|\hat{x}\|_{L_1}\|I^\tau z\|_{L_\infty} < \infty$$

for all $z \in L_{1/\tau}$, a contradiction. □

Another version of the Hardy-Littlewood theorem states that $I^\tau$ also maps the Lebesgue space $L_p$ continuously into the Hölder space $C_0^\alpha$ provided that

$$\frac{1}{\tau} < p < \infty, \quad 0 < \alpha \le \tau - \frac{1}{p},$$

as well as the Hölder space $C_0^\alpha$ continuously into the Hölder space $C_0^{\alpha+\tau}$ provided that $0 < \alpha + \tau < 1$.

So the boundedness and continuity of the operator (3.1) is well studied in Hölder spaces. However, if we want to involve nonlinear operators and apply Schauder's theorem, we also need some compactness assumption, and compactness criteria in these spaces are either quite clumsy or simply unknown, see Section 6 below. That is the reason why we study the operators (3.1) and (1.6) in the integral-type Hölder spaces $J_{\alpha,\beta}$.

We start by proving a sufficient acting and boundedness condition for the Riemann-Liouville operator (3.1) in the space $J_{\alpha,\beta}$. It turns out that, under this condition, the operator (3.1) is also bounded in the norm (2.5). Since

$$|I^\tau x(t)| \le \frac{\|x\|_C}{\Gamma(\tau)}\left|\int_0^t (t-s)^{\tau-1}\, ds\right| \le \frac{\|x\|_C}{\Gamma(\tau+1)}, \tag{3.4}$$

the first term in the norm (2.5) does not provide any difficulty. It is the second term which requires a careful analysis.

**Lemma 3.4.** *Let $0 < \tau < 1$, $0 < \alpha < \tau$, and $\beta > \alpha$. Then the estimate*

$$\omega(I^\tau x; \sigma) \le \frac{\omega(x;\sigma) + \sigma^\tau\|x\|_C}{\Gamma(\tau+1)} \qquad (0 \le \sigma \le 1) \tag{3.5}$$

*holds for $x \in J_{\alpha,\beta}$ and $I^\tau$ given by* (3.1).

*Proof.* For $0 \leq s, t \leq 1$, the change of variables $\sigma := su$ (resp. $\sigma := tu$) yields

$$\Gamma(\tau)\,[I^\tau x(s) - I^\tau x(t)] = \int_0^s (s-\sigma)^{\tau-1} x(\sigma)\,d\sigma - \int_0^t (t-\sigma)^{\tau-1} x(\sigma)\,d\sigma$$

$$= \int_0^1 (s-su)^{\tau-1} x(su) s\,du - \int_0^s (t-tu)^{\tau-1} x(tu) t\,du$$

$$= \int_0^1 [s^\tau x(su)) - t^\tau x(tu)]\,(1-u)^{\tau-1}\,du.$$

Now, for $s, t, u \in [0,1]$ with $|s-t| \leq \sigma$ we have

$$|s^\tau x(su) - t^\tau x(tu)| \leq |s^\tau x(su) - s^\tau x(tu)| + |s^\tau x(tu) - t^\tau x(tu)|$$

$$\leq s^\tau |x(su) - x(tu)| + |s^\tau - t^\tau|\,|x(tu)| \leq \omega(x; s) + \sigma^\tau \|x\|_C.$$

Consequently,

$$|I^\tau x(s) - I^\tau x(t)| \leq \frac{\omega(x;\sigma) + \sigma^\tau \|x\|_C}{\tau\Gamma(\tau)},$$

and (3.5) follows by taking the supremum over $|s-t| \leq \sigma$. □

From Lemma 3.4 we immediately deduce the following acting and boundedness theorem which we will use several times below.

**Theorem 3.5.** *Given $\alpha$, $\beta$, and $\tau$ as in Lemma 3.4, the operator $I^\tau$ maps $J_{\alpha,\beta}$ into itself and is bounded in the norm* (2.5).

*Proof.* The estimate (3.5) shows that

$$\omega(I^\tau x;\sigma)^{\beta/\alpha} \leq c_1 \left[\omega(x;\sigma)^{\beta/\alpha} + \sigma^{\tau\beta/\alpha} \|x\|_C^{\beta/\alpha}\right] \qquad (0 \leq \sigma \leq 1)$$

with

$$c_1 := \frac{2^{1-\alpha/\beta}}{\Gamma(\tau+1)},$$

where we have used the fact that $\beta > \alpha$. From this we conclude that

$$j_{\alpha,\beta}(I^\tau x; [0,1])^{\alpha/\beta} = \left(\int_0^1 \sigma^{-(\beta+1)} \omega(I^\tau x;\sigma)^{\beta/\alpha}\,d\sigma\right)^{\alpha/\beta}$$

$$\leq c_1^{\alpha/\beta} \left(\int_0^1 \sigma^{-(\beta+1)} \omega(x;\sigma)^{\beta/\alpha}\,d\sigma + \|x\|_C^{\beta/\alpha} \int_0^1 \sigma^{-(\beta+1)+\tau\beta/\alpha}\,d\sigma\right)^{\alpha/\beta}$$

$$\leq c_2 j_{\alpha,\beta}(x; [0,1])^{\alpha/\beta} + c_2 \|x\|_C,$$

where

$$c_2 := c_1^{\alpha/\beta} \left(\int_0^1 \sigma^{-(\beta+1)+\tau\beta/\alpha}\,d\sigma\right)^{\alpha/\beta} = \left(\frac{c_1 \alpha}{\beta(\tau-\alpha)}\right)^{\alpha/\beta}.$$

Combining this with the estimate (3.4) in the norm $\|\cdot\|_C$ we obtain

$$|||I^\tau x|||_{\alpha,\beta} = \|x\|_C + j_{\alpha,\beta}(x; [0,1])^{\alpha/\beta}$$

$$\leq c_1^{\alpha/\beta} j_{\alpha,\beta}(x; [0,1])^{\alpha/\beta} + (c_2 + \tau^{-1}) \|x\|_C \leq c_3 |||x|||_{\alpha,\beta}$$

with

$$c_3 := \max\{c_1^{\alpha/\beta}, c_2 + \tau^{-1}\}, \tag{3.6}$$

which proves the claim. □

## 4. Nemytskij operators

After analyzing the linear operator (3.1), let us now briefly recall the properties of the nonlinear operator (1.6) in Lebesgue and Hölder spaces. It is well-known that, whenever the Nemytskij operator $F$ generated by a Carathéodory function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ maps the space $L_p$ into itself, it is automatically continuous and bounded. So $F$ behaves in $L_p$ in rather the same way as in the space $C$ of continuous functions. However, this operator is never compact in $L_p$, except for the trivial case when it is constant (which means that $f = f(t)$).

In the Hölder space $C^\alpha$, the behavior of the operator (1.6) is quite pathological. The condition $F(C^\alpha) \subseteq C^\alpha$ does *not* imply the continuity or boundedness of $F$ in the norm of $C^\alpha$, see [6,7] for counterexamples. Even more surprising is the fact that $F$ may satisfy the condition $F(C^\alpha) \subseteq C^\alpha$ when the underlying function $f$ is *discontinuous* (and so $F(C) \not\subseteq C$), see [8]. In the autonomous case when the function $f = f(u)$ does not depend on $t$, however, the condition $F(C^\alpha) \subseteq C^\alpha$ implies the boundedness of $F$ in norm, but $F$ may still be discontinuous in norm [4,7].

In contrast to the strange behavior of the Nemytskij operator $F$ in the classical Hölder space $C^\alpha$, we show now that $F$ has more natural properties in the integral-type Hölder spaces $J_{\alpha,\beta}$. So let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous, and denote for $r > 0$, similarly as in (2.2), by

$$\begin{aligned} \omega_r(f;\sigma,\mu) := \sup\{|f(s,u) - f(t,v)| : 0 \le s,t \le 1, |u|,|v| \le r, \\ |s-t| \le \sigma, |u-v| \le \mu\} \end{aligned} \tag{4.1}$$

the modulus of continuity of $f$ on the rectangle $[0,1] \times [-r,r]$ for $r > 0$. In the following theorem we give a sufficient condition under which the Nemytskij operator (1.6) maps the space $J_{\alpha,\beta}[0,1]$ into itself and is bounded in the norm (2.5), and so also in the norm (2.4). To this end, we denote for $r > 0$ by

$$c_{0,r} := \max\left\{|f(t,u)| : 0 \le t \le 1, |u| \le r\right\} \tag{4.2}$$

the norm of $f$ in the space of continuous functions $f : [0,1] \times [-r,r] \to \mathbb{R}$. Moreover, we write

$$B_r(J_{\alpha,\beta}) := \{x \in J_{\alpha,\beta} : |||x|||_{\alpha,\beta} \le r\}$$

for the closed ball of radius $r > 0$ in the space $J_{\alpha,\beta}$ with norm (2.5).

**Theorem 4.1.** *Let $r > 0$, $b_r > 0$, $a_r \in L_1$, $0 < \alpha < 1$, and $\beta > \alpha$. Suppose that the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the estimate*

$$\omega_r(f;\sigma,\mu)^{\beta/\alpha} \le a_r(\sigma)\sigma^{\beta+1} + b_r\mu^{\beta/\alpha} \tag{4.3}$$

*for $\sigma, \mu \ge 0$. Then $F$ maps the ball $B_r(J_{\alpha,\beta})$ into the ball $B_R(J_{\alpha,\beta})$, where*

$$R := c_{0,r} + \left(\|a_r\|_{L_1} + b_r r^{\beta/\alpha}\right)^{\alpha/\beta}. \tag{4.4}$$

*Proof.* From

$$a_r(\sigma)\sigma^{\beta+1} \geq \omega_r(f;\sigma,0) \geq 0 \qquad (\sigma \geq 0)$$

it follows that $a_r(\sigma) \geq 0$ for all $\sigma \in [0,1]$. Moreover, $f$ is uniformly continuous on $[0,1] \times [-r,r]$, because

$$\lim_{\delta\to 0+} \omega_r(f;\delta,\delta) = \liminf_{\delta\to 0+} \omega_r(f;\delta,\delta) \leq \liminf_{\delta\to 0+} \left(a_r(\delta)\delta^{\beta+1} + b_r\delta^{\beta/\alpha}\right) = 0,$$

where we have used the monotonicity of $\omega_r(f;\cdot,\cdot)$ and the fact that $a_r \in L_1$. The estimate

$$\begin{aligned} j_{\alpha,\beta}(Fx;[0,1]) &= \int_0^1 \sigma^{-\beta-1}\omega(Fx;\sigma)^{\beta/\alpha}\,d\sigma \\ &\leq \int_0^1 \sigma^{-\beta-1}\omega_r\big(f;\sigma,\omega(x;\sigma)\big)^{\beta/\alpha}\,d\sigma \\ &\leq \int_0^1 \sigma^{-\beta-1}\big(a_r(\sigma)\sigma^{\beta+1} + b_r\omega(x;\sigma)^{\beta/\alpha}\big)\,d\sigma \end{aligned} \tag{4.5}$$

$$= \int_0^1 a_r(\sigma)\,d\sigma + b_r\int_0^1 \sigma^{-1-\beta}\omega(x;\sigma)^{\beta/\alpha}\,d\sigma = \|a_r\|_{L_1} + b_r j_{\alpha,\beta}(x;[0,1])$$

holds for our choice of $\alpha$ and $\beta$. Moreover, the continuity of $f$ implies that $c_{0,r} < \infty$, see (4.2), and so we may take $R$ as in (4.4) and obtain

$$|||Fx|||_{\alpha,\beta} = ||Fx||_C + j_{\alpha,\beta}(Fx;[0,1])^{\alpha/\beta} \leq R$$

as claimed. □

A particularly important class of nonlinearities $f$ which satisfies (4.3) for every $r > 0$ is given by functions which are uniformly Lipschitz continuous with respect to both variables. We state this as

**Example 4.2.** Let $r > 0$, and suppose that $f : [0,1] \times [-r,r] \to \mathbb{R}$ satisfies the following two conditions.

(a) There is some $c_{1,r} > 0$ such that

$$|f(s,u) - f(t,u)| \leq c_{1,r}|s-t| \tag{4.6}$$

for all $s,t \in [0,1]$ and $u \in [-r,r]$.

(b) There is some $c_{2,r} > 0$ such that

$$|f(s,u) - f(s,v)| \leq c_{2,r}|u-v| \tag{4.7}$$

for all $s \in [0,1]$ and $u,v \in [-r,r]$.

For $s,t \in [0,1]$ and $u,v \in [-r,r]$ we then have

$$|f(s,u) - f(t,v)| \leq |f(s,u) - f(t,u)| + |f(t,u) - f(t,v)| \leq c_{1,r}|s-t| + c_{2,r}|u-v|.$$

Taking the supremum over $|s-t| \leq \sigma$ and $|u-v| \leq \mu$ yields

$$\omega_r(f;\sigma,\mu) \leq c_{1,r}\sigma + c_{2,r}\mu.$$

So for $0 < \alpha < 1$ and $\beta > \alpha$ we obtain

$$\omega_r(f;\sigma,\mu)^{\beta/\alpha} \leq 2^{\beta/\alpha-1}\left(c_{1,r}^{\beta/\alpha}\sigma^{\beta/\alpha} + c_{2,r}^{\beta/\alpha}\mu^{\beta/\alpha}\right) = a_r(\sigma)\sigma^{\beta+1} + b_r\mu^{\beta/\alpha}$$

with

$$a_r(\sigma) := 2^{\beta/\alpha-1}c_{1,r}^{\beta/\alpha}\sigma^{\beta/\alpha-\beta-1}, \qquad b_r := 2^{\beta/\alpha-1}c_{2,r}^{\beta/\alpha},$$

which is precisely (4.3). According to Theorem 4.1, the operator $F$ maps the ball $B_r(J_{\alpha,\beta})$ into the ball $B_R(J_{\alpha,\beta})$, where the relation between $R$ and $r$ is given by

$$R = c_{0,r} + 2^{1-\alpha/\beta}\left(c_{1,r}^{\beta/\alpha}\frac{\alpha}{\beta(1-\alpha)} + c_{2,r}^{\beta/\alpha}r^{\beta/\alpha}\right)^{\alpha/\beta}. \tag{4.8}$$

We conclude that, for functions satisfying (a) and (b) for each $r > 0$, the operator $F : J_{\alpha,\beta} \to J_{\alpha,\beta}$ is bounded on every ball. □

Now we come to the problem of finding sufficient conditions for the continuity of the operator $F$ in $J_{\alpha,\beta}$. This turns out to be more difficult than proving boundedness. First we need two technical lemmas.

**Lemma 4.3.** *Let $r > 0$ and $0 \leq \theta \leq 1$, and suppose that $f : [0,1]\times[-r,r] \to \mathbb{R}$ satisfies* (a) *and* (b) *of Example 4.2. Then there exists a constant $C_{\theta,r} > 0$ such that*

$$|f(s,u) - f(s,v) - f(t,u) + f(t,v)| \leq C_{\theta,r}|s-t|^{\theta}|u-v|^{1-\theta}$$

*for all $s,t \in [0,1]$ and $u,v \in [-r,r]$.*

*Proof.* From (a) of Example 4.2 we obtain

$$|f(s,u) - f(s,v) - f(t,u) + f(t,v)|$$
$$\leq |f(s,u) - f(t,u)| + |f(s,v) - f(t,v)| \leq 2c_{1,r}|s-t|,$$

and from (b) we get

$$|f(s,u) - f(s,v) - f(t,u) + f(t,v)|$$
$$\leq |f(s,u) - f(s,v)| + |f(t,u) - f(t,v)| \leq 2c_{2,r}|u-v|.$$

Raising the first inequality to the power $\theta$ and the second inequality to the power $1-\theta$, and multiplying the resulting inequalities we end up with

$$|f(s,u) - f(s,v) - f(t,u) + f(t,v)| \leq 2c_{1,r}^{\theta}c_{2,r}^{1-\theta}|s-t|^{\theta}|u-v|^{1-\theta}.$$

So choosing $C_{\theta,r} := 2c_{1,r}^{\theta}c_{2,r}^{1-\theta}$ proves the claim. □

**Lemma 4.4.** *Let $r > 0$, and suppose that $f : [0,1]\times[-r,r] \to \mathbb{R}$ satisfies the following two conditions.*

(a) *The function $f(t,\cdot)$ is differentiable for each $t \in [0,1]$, and*

$$|\partial_2 f(t,u) - \partial_2 f(t,v)| \leq A_r|u-v| \tag{4.9}$$

*for some constant $A_r > 0$ and each $u,v \in [-r,r]$.*

(b) *The norm $\|\partial_2 f(\cdot,0)\|_C$ is finite.*

*Then there exists a constant* $B_r > 0$ *such that*

$$\begin{aligned}|f(t,u_1) - f(t,v_1) - f(t,u_2) + f(t,v_2)| \\ \leq A_r\big(|u_1-u_2| + |v_1-v_2|\big)\big(|u_1-v_1| + |u_2-v_2|\big) + B_r|u_1-v_1-u_2+v_2|\end{aligned} \tag{4.10}$$

*for all* $t \in [0,1]$ *and* $u_1, u_2, v_1, v_2 \in [-r,r]$.

*Proof.* For $t \in [0,1]$ and $u \in [-r,r]$ we deduce from (a) and (b) that

$$\begin{aligned}|\partial_2 f(t,u)| \leq |\partial_2 f(t,u) - \partial_2 f(t,0)| + |\partial_2 f(t,0)| \\ \leq A_r|u| + \|\partial_2 f(\cdot,0)\|_C \leq A_r r + \|\partial_2 f(\cdot,0)\|_C =: B_r < \infty.\end{aligned}$$

Similarly as in the proof of [1, Lemma 5.48] one may then show that (4.10) holds for $u_1, u_2, v_1, v_2 \in [-r,r]$, which proves the assertion. □

Observe that condition (b) of Lemma 4.4 is trivially satisfied if $f$ has a continuous partial derivative $\partial_2 f(\cdot,\cdot)$. However, condition (a) is an additional requirement, namely Lipschitz continuity of $\partial_2 f(t,\cdot)$, uniformly with respect to $t$.

Now we come to a first continuity condition for the operator $F$ in the space $J_{\alpha,\beta}$. The following Theorem 4.5 shows even more.

**Theorem 4.5.** *Let* $0 < \alpha < \theta < 1$, $\beta > \alpha$, *and* $r > 0$, *and suppose that* $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *satisfies condition* (a) *of Example 4.2 and conditions* (a) *and* (b) *of Lemma 4.4 on* $[0,1]\times[-r,r]$. *Then* $F$ *maps the ball* $B_r(J_{\alpha,\beta})$ *into the ball* $B_R(J_{\alpha,\beta})$, *with* $R$ *given by* (4.8), *and is locally Hölder continuous on this ball with exponent* $1-\theta$, *where the Hölder constant of* $F$ *depends on the constants* $A_r$ *and* $B_r$ *from Lemma 4.4, as well as on the constant* $C_{\theta,r}$ *from Lemma 4.3.*

*Proof.* We use again the norm (2.5). Fix $x, y \in J_{\alpha,\beta}$ with

$$|||x-y|||_{\alpha,\beta} \leq 1, \qquad |||x|||_{\alpha,\beta}, |||y|||_{\alpha,\beta} \leq r;$$

in particular, $\|x\|_C, \|y\|_C \leq r$. As was shown in Lemma 4.4 we have

$$|\partial_2 f(s,u)| \leq B_r \qquad (s \in [0,1],\ u \in [-r,r]).$$

From the Mean Value Theorem it then follows that

$$|f(s,u) - f(s,v)| \leq B_r|u-v| \qquad (s \in [0,1],\ u,v \in [-r,r]),$$

so condition (b) of Example 4.2 is also satisfied with $c_{2,r} := B_r$. Applying Lemma 4.3 and Lemma 4.4 to $s,t \in [0,1]$ and $u_1 := x(s)$, $v_1 := y(s)$, $u_2 := x(t)$, and $v_2 := y(t)$

we get

$$\begin{aligned}
&|f(s,x(s)) - f(s,y(s)) - f(t,x(t)) + f(t,y(t))| \\
&\quad = |f(s,u_1) - f(s,v_1) - f(t,u_2) + f(t,v_2)| \\
&\quad \le |f(s,u_1) - f(s,v_1) - f(s,u_2) + f(s,v_2)| \\
&\qquad + |f(s,u_2) - f(s,v_2) - f(t,u_2) + f(t,v_2)| \\
&\quad \le A_r\big(|u_1-u_2| + |v_1-v_2|\big)\big(|u_1-v_1| + |u_2-v_2|\big) \\
&\qquad + B_r|u_1 - v_1 - u_2 + v_2| + C_{\theta,r}|s-t|^\theta |u_2-v_2|^{1-\theta},
\end{aligned}$$

where $A_r$, $B_r$, and $C_{\theta,r}$ are the constants from Lemma 4.3 and Lemma 4.4 which depend only on $r$ and $\theta$. Taking the supremum over all $s,t \in [0,1]$ with $|s-t| \le \sigma$ yields

$$\begin{aligned}
\omega(Fx-Fy;\sigma) &\le 2A_r\big[\omega(x;\sigma)+\omega(y;\sigma)\big]\|x-y\|_C + B_r\omega(x-y;\sigma) + C_{\theta,r}\sigma^\theta\|x-y\|_C^{1-\theta} \\
&\le D_{\theta,r}|||x-y|||_{\alpha,\beta}^{1-\theta}\big[\omega(x;\sigma)+\omega(y;\sigma)+\sigma^\theta\big] + B_r\omega(x-y;\sigma),
\end{aligned}$$

where $D_{\theta,r} := \max\{2A_r, C_{\theta,r}\}$ and we have used $\theta > 0$ and $|||x-y|||_{\alpha,\beta} \le 1$. By our assumption $\beta > \alpha$ we conclude that

$$\begin{aligned}
\omega(Fx-Fy;\sigma)^{\beta/\alpha} \le 4^{\beta/\alpha-1}\Big[&D_{\theta,r}^{\beta/\alpha}|||x-y|||_{\alpha,\beta}^{(1-\theta)\beta/\alpha}\big(\omega(x;\sigma)^{\beta/\alpha}+\omega(y;\sigma)^{\beta/\alpha}+\sigma^{\theta\beta/\alpha}\big) \\
&+ B_r^{\beta/\alpha}\omega(x-y;\sigma)^{\beta/\alpha}\Big].
\end{aligned}$$

Consequently, using $\alpha < \theta$, $\theta > 0$, and once more $|||x-y|||_{\alpha,\beta} \le 1$ we obtain the estimate

$$\begin{aligned}
j_{\alpha,\beta}(Fx-Fy;[0,1])^{\alpha/\beta} &= \left(\int_0^1 \sigma^{-(\beta+1)}\omega(Fx-Fy;\sigma)^{\beta/\alpha}\,d\sigma\right)^{\alpha/\beta} \\
&\le 4^{1-\alpha/\beta}\Big[D_{\theta,r}|||x-y|||_{\alpha,\beta}^{1-\theta}\Big(j_{\alpha,\beta}(x;[0,1])^{\alpha/\beta}+j_{\alpha,\beta}(y;[0,1])^{\alpha/\beta}+\gamma_\theta^{\alpha/\beta}\Big) \\
&\qquad +B_r j_{\alpha,\beta}(x-y;[0,1])^{\alpha/\beta}\Big] \\
&\le 4^{1-\alpha/\beta}\Big[D_{\theta,r}|||x-y|||_{\alpha,\beta}^{1-\theta}\Big(|||x|||_{\alpha,\beta}+|||y|||_{\alpha,\beta}+\gamma_\theta^{\alpha/\beta}\Big)+B_r|||x-y|||_{\alpha,\beta}\Big] \\
&\le 4^{1-\alpha/\beta}|||x-y|||_{\alpha,\beta}^{1-\theta}\Big[D_{\theta,r}\Big(|||x|||_{\alpha,\beta}+|||y|||_{\alpha,\beta}+\gamma_\theta^{\alpha/\beta}\Big)+B_r\Big] \\
&\le 4^{1-\alpha/\beta}|||x-y|||_{\alpha,\beta}^{1-\theta}\Big[D_{\theta,r}\Big(2r+\gamma_\theta^{\alpha/\beta}\Big)+B_r\Big]
\end{aligned}$$

with

$$\gamma_\theta := \int_0^1 \sigma^{-\beta-1}\sigma^{\theta\beta/\alpha}\,d\sigma = \frac{\alpha}{\beta(\theta-\alpha)}.$$

It remains to estimate the first term in the norm (2.5) of $Fx - Fy$, i.e., $\|Fx - Fy\|_C$. Taking $u_1 = u_2 = v_1 = u$ and $v_2 = v$ in Lemma 4.4 gives

$$|f(t,u) - f(t,v)| \leq A_r|u-v|^2 + B_r|u-v| \qquad (u, v \in [-r,r],\, t \in [0,1]).$$

So for $u = x(t)$ and $v = y(t)$ we obtain

$$\|Fx - Fy\|_C = \max_{0 \leq t \leq 1} |f(t,x(t)) - f(t,y(t))|$$

$$\leq \|x-y\|_C \big(A_r\|x-y\|_C + B_r\big) \leq |||x-y|||_{\alpha,\beta}^{1-\theta}(2rA_r + B_r),$$

where we again used our hypotheses $\theta > 0$ and $|||x-y|||_{\alpha,\beta} \leq 1$. Summarizing we obtain

$$|||Fx - Fy|||_{\alpha,\beta} \leq L_r|||x-y|||_{\alpha,\beta}^{1-\theta}$$

with

$$L_r := (2rA_r + B_r) + 4^{1-\alpha/\beta}\left[D_{\theta,r}\big(2r + \gamma_\theta^{\alpha/\beta}\big) + B_r\right],$$

and this is exactly our claim. □

Theorem 4.5 gives a first answer to the question, when the operator $F$ is continuous in $J_{\alpha,\beta}$. However, we can do better. In the follows we will prove that we can drop the Lipschitz continuity of the partial derivative $\partial_2 f(\cdot,\cdot)$ with respect to the second variable and replace it by ordinary continuity of $\partial_2 f(\cdot,\cdot)$. This will require some approximation procedure which is given in the following

**Lemma 4.6.** *Let $f : [0,1] \times [-r,r] \to \mathbb{R}$ be a function which satisfies* (a) *of Example 4.2 and has a continuous partial derivative $\partial_2 f(\cdot,\cdot)$ everywhere in $[0,1] \times [-r,r]$. Then there exists a sequence $(f_n)_n$ of functions $f_n : [0,1] \times \mathbb{R} \to \mathbb{R}$ with the following properties.*

(a) $|f_n(s,u) - f_n(t,u)| \leq c_{1,r}|s-t|$ *for all $s,t \in [0,1]$, $u \in [-r,r]$ and $n \in \mathbb{N}$, where $c_{1,r}$ is the constant of Lemma 4.3.*

(b) $\lim_{n\to\infty} \sup\big\{|f_n(s,u) - f(s,u)| : s \in [0,1], u \in [-r,r]\big\} = 0$ *for each $u \in [-r,r]$.*

(c) $|\partial_2 f_n(s,u) - \partial_2 f_n(s,v)| \leq L_{n,r}|u-v|$ *for all $s \in [0,1]$, $u,v \in [-r,r]$ and $n \in \mathbb{N}$ and constants $L_{n,r} > 0$.*

(d) $\lim_{n\to\infty} \sup\big\{|\partial_2 f_n(s,u) - \partial_2 f(s,u)| : s \in [0,1], u \in [-r,r]\big\} = 0.$

Before we turn to the quite cumbersome proof of Lemma 4.6, let us make some comments on the statements for the sequence $(f_n)_n$. Assertion (a) says that each individual $f_n$ is Lipschitz continuous with respect to the first variable, uniformly with respect to the second variable. This is precisely condition (a) of Example 4.2. Assertion (b) says that $(f_n)_n$ converges uniformly to $f$ with respect to both variables. Assertion (c) says that the partial derivative with respect to the second variable of each individual $f_n$ satisfies a uniform Lipschitz condition according to Lemma 4.4 (a). Finally, assertion (d) says that the sequence of partial derivatives with respect to the second variable of $f_n$ converges to that of $f$ uniformly in both variables. In particular, since $\partial_2 f(\cdot,\cdot)$ is supposed to be continuous, and hence bounded on the rectangle $[0,1] \times [-r,r]$, each $\partial_2 f_n(\cdot,\cdot)$ must be bounded there as well. Consequently, each $f_n$ also fulfills part (b) of Lemma 4.4, and hence also all requirements of Theorem 4.5.

*Proof of Lemma 4.6.* For the functions $f_n$ we take the Bernstein Polynomials

$$f_n(s,u) := \frac{1}{(2r)^n}\sum_{k=0}^{n}\binom{n}{k} f\left(s, \frac{(2k-n)r}{n}\right)(u+r)^k(r-u)^{n-k} \quad (s\in[0,1],\, u\in\mathbb{R}).$$

Then (a) is obvious. For (b) fix $\varepsilon > 0$. From the hypothesis it follows easily that $f$ is continuous on $[0,1]\times[-r,r]$; in particular, $f$ is bounded by some $M > 0$, say. Moreover, it is also uniformly continuous, and we can pick $\delta > 0$ such that

$$|f(s,u) - f(s,v)| \leq \varepsilon/2$$

whenever $u, v \in [-r,r]$ satisfy $|u-v| \leq \delta$. Then we choose $N \in \mathbb{N}$ so large that

$$N \geq \frac{4Mr^2}{\varepsilon\delta^2}.$$

Now, using the shortcut $\varphi_k(u) := (u+r)^k(r-u)^{n-k}$ we get

$$|f_n(s,u) - f(s,u)| \leq \frac{1}{(2r)^n}\sum_{k=0}^{n}\binom{n}{k}\left|f\left(s,\frac{(2k-n)r}{n}\right) - f(s,u)\right|\varphi_k(u)$$

$$= \frac{1}{(2r)^n}\left\{\sum_{k\in A} + \sum_{k\in B}\right\}\binom{n}{k}\left|f\left(s,\frac{(2k-n)r}{n}\right) - f(s,u)\right|\varphi_k(u),$$

where

$$A := \{k \in \{0,\ldots,n\} : |2kr - n(r+u)| \leq n\delta\}$$

and

$$B := \{k \in \{0,\ldots,n\} : |2kr - n(r+u)| > n\delta\}.$$

For the sum over $A$ we have $|\xi_k - u| \leq \delta$, since $\xi_k$ lies between $(2k-n)r/n$ and $u$. Therefore,

$$\left|f\left(s,\frac{(2k-n)r}{n}\right) - f(s,u)\right| \leq \frac{\varepsilon}{2},$$

hence

$$\frac{1}{(2r)^n}\sum_{k\in A}\binom{n}{k}\left|f\left(s,\frac{(2k-n)r}{n}\right) - f(s,u)\right|\varphi_k(u) \leq \frac{\varepsilon}{2}.$$

On the other hand, for the sum over $B$ we have

$$\frac{\delta^2 n^2}{(2r)^n}\sum_{k\in B}\binom{n}{k}\left|f\left(s,\frac{(2k-n)r}{n}\right) - f(s,u)\right|\varphi_k(u)$$

$$\leq \frac{2M}{(2r)^n}\sum_{k\in B}\binom{n}{k}\varphi_k(u)\big(2kr - n(u+r)\big)^2 = 2Mn(r^2-u^2) \leq 2Mr^2 n,$$

hence

$$\frac{1}{(2r)^n}\sum_{k\in B}\binom{n}{k}\left|f\left(s,\frac{(2k-n)r}{n}\right) - f(s,u)\right|\varphi_k(u) \leq \frac{2Mr^2}{\delta 2n} \leq \frac{\varepsilon}{2},$$

provided $n \geq N$. Combining these estimates we finally obtain

$$|f_n(s,u) - f(s,u)| \leq \varepsilon \qquad (n \geq N,\ s \in [0,1],\ |u| \leq r),$$

and this proves (b).

To prove (c), we still introduce the shortcut

$$\varphi_{k,n}(u) := (r+u)^{k-1}(r-u)^{n-k-1}\big(2kr - n(r+u)\big)^2$$

and get after some straightforward calculations

$$\partial_2 f_n(s,u) = \frac{1}{(2r)^n n} \sum_{k=0}^{n} \binom{n}{k} f\left(s, \frac{(2k-n)r}{n}\right) \varphi_{k,n}(u).$$

In particular,

$$|\partial_2 f_n(s,u) - \partial_2 f_n(s,v)| \leq \frac{1}{(2r)^n n} \sum_{k=0}^{n} \binom{n}{k} \left| f\left(s, \frac{(2k-n)r}{n}\right)\right| |\varphi_{k,n}(u) - \varphi_{k,n}(v)|.$$

Since each $\varphi_{k,n}$ is (being a polynomial) Lipschitz continuous on $[-r,r]$, we find constants $L_{k,n,r} > 0$ such that

$$|\varphi_{k,n}(u) - \varphi_{k,n}(v)| \leq L_{k,n,r}|u-v| \qquad (u, v \in [-r,r]).$$

Again, $f$ is bounded on $[0,1] \times [-r,r]$ by some $M > 0$, say. Letting

$$L'_{n,r} := \max\{L_{k,n,r} : 0 \leq k \leq n\}$$

we obtain

$$|\partial_2 f_n(s,u) - \partial_2 f_n(s,v)| \leq \frac{M}{(2r)^n n}|u-v| \sum_{k=0}^{n} \binom{n}{k} L_{k,n,r} \leq \frac{ML'_{n,r}}{r^n n}|u-v|,$$

and so (c) follows with $L_{n,r} := ML'_{n,r}/nr^n$.

The proof for (d) is almost the same as the one for (b). Again, fix $\varepsilon > 0$. Since $\partial_2 f(\cdot,\cdot)$ is continuous on the compact set $[0,1] \times [-r,r]$, it is bounded by some $M > 0$, say. Moreover, it is also uniformly continuous, and we can pick $\delta > 0$ such that

$$|\partial_2 f(s,u) - \partial_2 f(s,v)| \leq \frac{\varepsilon}{2} \qquad (u, v \in [-r,r],\ |u-v| \leq \delta).$$

Then choose $N \in \mathbb{N}$ so large that

$$N \geq \frac{12Mr^2}{\varepsilon\delta^2}.$$

Writing $\varphi_k(u) := (u+r)^{k-1}(r-u)^{n-k-1}$, we have

$$\partial_2 f_n(s,u) = \frac{1}{(2r)^n} \sum_{k=0}^{n} \binom{n}{k} \left[ f\left(s, \frac{(2k-n)r}{n}\right) - f(s,u)\right] \varphi_k(u)\big(2kr - n(u+r)\big).$$

Applying the Mean Value Theorem we find some $\xi_k$ between $(2k-n)r/n$ and $u$ such that

$$\begin{aligned} f\left(s, \frac{(2k-n)r}{n}\right) - f(s,u) &= \partial_2 f(s,\xi_k)\left(\frac{(2k-n)r}{n} - u\right) \\ &= \frac{1}{n}\partial_2 f(s,\xi_k)\big(2kr - n(r+u)\big). \end{aligned}$$

Consequently,

$$\partial_2 f_n(s,u) = \frac{1}{(2r)^n n}\sum_{k=0}^{n}\binom{n}{k}\partial_2 f\,(s,\xi_k)\,\big(2kr - n(r+u)\big)^2\varphi_k(u).$$

So we obtain

$$|\partial_2 f_n(s,u) - \partial_2 f(s,u)|$$

$$\leq \frac{1}{(2r)^n n}\sum_{k=0}^{n}\binom{n}{k}\big|\partial_2 f(s,\xi_k) - \partial_2 f(s,u)\big|\varphi_k(u)\big(2kr - n(u+r)\big)^2$$

$$= \frac{1}{(2r)^n n}\left\{\sum_{k\in A} + \sum_{k\in B}\right\}\binom{n}{k}\big|\partial_2 f(s,\xi_k) - \partial_2 f(s,u)\big|\varphi_k(u)\big(2kr - n(u+r)\big)^2,$$

where $A$ and $B$ are the same index sets as in the proof of (b). For the sum over $A$ we have $|\xi_k - u| \leq \delta$, since $\xi_k$ lies between $(2k-n)r/n$ and $u$. Therefore,

$$|\partial_2 f(s,\xi_k) - \partial_2 f(s,u)| \leq \frac{\varepsilon}{2},$$

hence

$$\frac{1}{(2r)^n n}\sum_{k\in A}\binom{n}{k}\big|\partial_2 f(s,\xi_k) - \partial_2 f(s,u)\big|\varphi_k(u)\big(2kr - n(u+r)\big)^2 \leq \frac{\varepsilon}{2}.$$

Similarly, for the sum over $B$ we have

$$\frac{\delta^2 n^2}{(2r)^n n}\sum_{k\in B}\binom{n}{k}\big|\partial_2 f(s,\xi_k) - \partial_2 f(s,u)\big|\varphi_k(u)\big(2kr - n(u+r)\big)^2$$

$$\leq \frac{2M}{(2r)^n n}\sum_{k\in B}\binom{n}{k}\varphi_k(u)\big(2kr - n(u+r)\big)^4 = 2M\big((3n-2)r^2 - 3(n-2)u^2\big)$$

$$\leq 6Mr^2 n,$$

hence

$$\frac{1}{(2r)^n n}\sum_{k\in B}\binom{n}{k}\big|\partial_2 f(s,\xi_k) - \partial_2 f(s,u)\big|\varphi_k(u)\big(2kr - n(u+r)\big)^2 \leq \frac{6Mr^2}{\delta^2 n} \leq \frac{\varepsilon}{2},$$

provided $n \geq N$. So we obtain in rather the same way as in (b)

$$|\partial_2 f_n(s,u) - \partial_2 f(s,u)| \leq \varepsilon \qquad (n \geq N,\ s \in [0,1],\ |u| \leq r),$$

and this finally finishes the proof. □

Building on Lemma 4.6, we are now in a position to give another continuity condition for $F$ which is an improved version of Theorem 4.5.

**Theorem 4.7.** *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies condition* (a) *of Example 4.2 and has a continuous partial derivative $\partial_2 f$ everywhere on $[0,1] \times \mathbb{R}$. Then the operator $F$ generated by $f$ maps the space $J_{\alpha,\beta}$ into itself and is continuous.*

Let us again make some comments on our hypotheses. Condition (a) of Example 4.2 says that $f$ is Lipschitz continuous with respect to its first variable, uniformly with respect to its second variable. We do not need differentiability here. The second assumption, however, says that, for each $s \in [0,1]$, the function $f(s,\cdot)$ is differentiable, and that the function $\partial_2 f(\cdot,\cdot)$ is continuous on each rectangle $[0,1] \times [-r,r]$.

Now, the idea of the proof of Theorem 4.7 is simple. We approximate $F$ locally uniformly by other operators $F_n$ generated by the Bernstein polynomials constructed in Lemma 4.6. Since each $F_n$ is then continuous on the space $J_{\alpha,\beta}$, by Theorem 4.5, our $F$ must be continuous as well. However, dropping the assumption that $\partial_2 f$ satisfies a uniform Lipschitz condition according to Lemma 4.4 (a) may make $F$ no longer Hölder continuous. Here are the details.

*Proof of Theorem 4.7.* Fix $r > 0$. We choose a sequence $(f_n)_n$ of functions $f_n : [0,1] \times [-r,r] \to \mathbb{R}$ as in Lemma 4.6 and put

$$S_n := \sup\left\{|f_n(s,u) - f(s,u)| : s \in [0,1],\, u \in [-r,r]\right\}$$

and

$$T_n := \sup\left\{|\partial_2 f_n(s,u) - \partial_2 f(s,u)| : s \in [0,1],\, u \in [-r,r]\right\}.$$

By Lemma 4.6, we know then that

$$|f_n(s,u) - f_n(t,u)| \le c_{1,r}|s-t| \qquad (s,t \in [0,1],\, u \in [-r,r]), \tag{4.11}$$

where $c_{1,r}$ is the constant occurring in Lemma 4.3,

$$\lim_{n\to\infty} S_n = 0, \tag{4.12}$$

$$|\partial_2 f_n(s,u) - \partial_2 f_n(s,v)| \le L_{n,r}|u-v| \qquad (s \in [0,1],\, u,v \in [-r,r]) \tag{4.13}$$

for some constants $L_{n,r} > 0$, and

$$\lim_{n\to\infty} T_n = 0. \tag{4.14}$$

As mentioned right after Lemma 4.6, each $f_n$ satisfies all requirements of Theorem 4.5, and hence generates an operator $F_n$ which maps the ball $B_r(J_{\alpha,\beta})$ into the ball $B_R(J_{\alpha,\beta})$ and is continuous, where $R$ is given by (4.8). Note that, due to the continuity of $\partial_2 f(\cdot,\cdot)$, the function $f$ particularly fulfills all requirements of Example 4.2, and thus the operator $F$ itself maps $B_r(J_{\alpha,\beta})$ into $B_R(J_{\alpha,\beta})$. Since this is true for all $r > 0$, the operator $F$ maps $J_{\alpha,\beta}$ into itself and is bounded. We now show that the operator sequence $(F_n)_n$ converge uniformly on the ball $B_r(J_{\alpha,\beta})$ to $F$, which then will make $F$ also continuous on that ball.

To this end, we set $g_n := f_n - f$ and let $G_n$ be the corresponding Nemytskij operators, that is, $G_n x(t) := g_n(t, x(t))$. We need to show that the operator sequence $(G_n)_n$ converges uniformly to zero on $B_r(J_{\alpha,\beta})$.

For $x \in B_r(J_{\alpha,\beta})$ we get from (4.12)

$$\|G_n x\|_C = \sup_{0\le t\le 1} |f_n(t,x(t)) - f(t,x(t))| \le S_n \to 0 \quad (n \to \infty). \tag{4.15}$$

Now, for $s \in [0,1]$ and $u, v \in [-r,r]$ we have

$$\begin{aligned} |g_n(s,u) - g_n(s,v)| &\le \sup\{|\partial_2 g_n(\sigma,\xi)||u-v| : \sigma \in [0,1],\, \xi \in [-r,r]\} \\ &\le T_n|u-v|. \end{aligned} \tag{4.16}$$

Moreover, similarly as in the proof of Lemma 4.3, we conclude from (4.11) and (4.12) that, for $s, t \in [0,1]$ and $v \in [-r,r]$ we have

$$|g_n(s,v) - g_n(t,v)| \le |f_n(s,v) - f_n(t,v)| + |f(s,v) - f(t,v)| \le 2c_{1,r}|s-t| \tag{4.17}$$

and

$$|g_n(s,v) - g_n(t,v)| \le |g_n(s,v)| + |g_n(t,v)| \le 2S_n. \tag{4.18}$$

Fix $\theta \in (\alpha, 1)$. By raising (4.17) to the power $\theta$ and (4.18) to the power $1-\theta$, and multiplying both inequalities yields

$$|g_n(s,v) - g_n(t,v)| \le 2c_{1,r}^{\theta} S_n^{1-\theta}|s-t|^{\theta}. \tag{4.19}$$

Combining now (4.16) and (4.19) gives

$$\begin{aligned} |g_n(s,u) - g_n(t,v)| &\le |g_n(s,u) - g_n(s,v)| + |g_n(s,v) - g_n(t,v)| \\ &\le T_n|u-v| + 2c_{1,r}^{\theta} S_n^{1-\theta}|s-t|^{\theta}. \end{aligned}$$

In particular,

$$\omega(G_n x;\sigma)^{\beta/\alpha} \le 2^{\beta/\alpha - 1}\left(T_n^{\beta/\alpha}\omega(x;\sigma)^{\beta/\alpha} + 2^{\beta/\alpha} c_{1,r}^{\beta\theta/\alpha} S_n^{\beta(1-\theta)/\alpha}\sigma^{\beta\theta/\alpha}\right),$$

and so

$$\begin{aligned} j_{\alpha,\beta}(G_n x;[0,1])^{\alpha/\beta} &\le 2^{1-\alpha/\beta}\left(T_n j_{\alpha,\beta}(x;[0,1])^{\alpha/\beta} + 2c_{1,r}^{\theta} S_n^{1-\theta}\gamma_\theta^{\alpha/\beta}\right) \\ &\le 2^{1-\alpha/\beta}\left(T_n r + 2c_{1,r}^{\theta} S_n^{1-\theta}\gamma_\theta^{\alpha/\beta}\right), \end{aligned}$$

where $\gamma_\theta$ is as in the proof of Theorem 4.5. Finally, adding (4.15) yields

$$|||G_n x|||_{\alpha,\beta} \le S_n + 2^{1-\alpha/\beta}\left(T_n r + 2c_{1,r}^{\theta} S_n^{1-\theta}\gamma_\theta^{\alpha/\beta}\right). \tag{4.20}$$

Since the right-hand side of (4.20) goes to zero as $n \to \infty$ and does no longer depend on $x$, the proof is complete. □

It is illuminating to illustrate our results for the special case of separated variables, i.e., for $f(t,u) = g(t)h(u)$.

**Example 4.8.** Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be given by

$$f(t,u) = g(t)h(u) \qquad (t \in [0,1],\, u \in \mathbb{R}). \tag{4.21}$$

Then, according to Example 4.2, the operator $F$ maps $J_{\alpha,\beta}$ into itself if $g \in Lip[0,1]$ and $h \in Lip_{loc}(\mathbb{R})$. In addition, Theorem 4.5 tells us that $F$ is locally Hölder continuous if $g \in Lip[0,1]$ and $h \in C^1(\mathbb{R})$ with $h' \in Lip_{loc}(\mathbb{R})$. Finally, Theorem 4.7 guarantees the continuity of $F$ if $g \in Lip[0,1]$ and $h \in C^1(\mathbb{R})$, even if $h'$ does not satisfy a Lipschitz condition. □

## 5. Lipschitz continuity

Theorem 4.5 shows that the Nemytskij operator $F$ is, under natural hypotheses on the generating function $f$, locally Hölder continuous, where we had to impose the constraint $1-\theta < 1-\alpha < 1$ on the corresponding Hölder exponent. The question arises whether or not this is only a technical restriction, or we may choose $\theta = 0$, or even impose a global Lipschitz condition in the form

$$\|Fx_1 - Fx_2\|_X \leq k\|x_1 - x_2\|_X \qquad (x_1, x_2 \in X) \tag{5.1}$$

on the operator $F$. Here one has to be extremely careful, since this may lead to degeneracy phenomena. In fact, it was proved in [15] that only Nemytskij operators $F$ which are generated by affine functions

$$f(t,u) = A(t)u + B(t) \qquad (A, B \in X) \tag{5.2}$$

satisfy (5.1) in the space $X = C^\alpha$. We show now that the same degeneracy phenomenon occurs in $C^\alpha$ if we replace the usual (global) Lipschitz condition (5.1) by the more general condition

$$\|Fx_1 - Fx_2\|_X \leq \phi(\|x_1 - x_2\|_X) \qquad (x, y \in X) \tag{5.3}$$

which is (1.5) for $M = X$ and $T = F$.

**Proposition 5.1.** *Suppose that the operator $F$ from* (1.6) *satisfies a generalized contraction condition of type* (5.3) *in the Hölder space $X = C^\alpha$, i.e.,*

$$\|Fx_1 - Fx_2\|_{\alpha,\infty} \leq \phi(\|x_1 - x_2\|_{\alpha,\infty}) \qquad (x, y \in C^\alpha)$$

*for some comparison function $\phi$. Then the function $f$ is affine, i.e., there exist $A, B \in C^\alpha$ such that $f(t,u) = A(t)u + B(t)$.*

*Proof.* For $0 < \sigma < \tau < 1$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$, we define two functions $x_1, x_2 : [0,1] \to \mathbb{R}$ by

$$x_i(s) = \begin{cases} u_i & \text{for} \quad 0 \leq s < \sigma, \\ \dfrac{(u_i - v_i)(s-\tau)^\alpha}{(\tau-\sigma)^\alpha} + v_i & \text{for} \quad \sigma \leq s \leq \tau, \\ v_i & \text{for} \quad \tau < s \leq 1. \end{cases}$$

Then $x_i \in C^\alpha$ and

$$|(x_1 - x_2)(s) - (x_1 - x_2)(t)| \leq \left(\frac{t-s}{\tau-\sigma}\right)^\alpha (u_1 - v_1 - u_2 + v_2)$$

for $\sigma \leq s < t \leq \tau$, hence

$$\omega(x_1 - x_2; \tau - \sigma) \leq |u_1 - v_1 - u_2 + v_2|.$$

Consequently,

$$\|x_1 - x_2\|_{\alpha,\infty} = |u_1 - u_2| + \frac{|u_1 - v_1 - u_2 + v_2|}{(\tau-\sigma)^\alpha}.$$

Similarly, we have

$$\|Fx_1 - Fx_2\|_{\alpha,\beta} = |f(0,\eta) - f(0,0)| + \frac{|f(\tau,u_1) - f(\tau,u_2) - f(\sigma,v_1) + f(\sigma,v_2)|}{(\tau-\sigma)^\alpha}.$$

Inserting this into the contraction condition (5.3) yields

$$|f(0,\eta) - f(0,0)| + \frac{|f(\tau,u_1) - f(\tau,u_2) - f(\sigma,v_1) + f(\sigma,v_2)|}{(\tau-\sigma)^\alpha} \leq \phi\left(|u_1 - u_2| + \frac{|u_1 - v_1 - u_2 + v_2|}{(\tau-\sigma)^\alpha}\right).$$

Now, multiplying both sides of this estimate by $(\tau-\sigma)^\alpha$ and letting $\tau \to \sigma$ we obtain

$$|f(\sigma,u_1) - f(\sigma,u_2) - f(\sigma,v_1) + f(\sigma,v_2)| \leq \phi\left(|u_1 - v_1 - u_2 + v_2|\right).$$

Finally, substituting $u_1 := \xi + \eta$, $u_2 =: \xi$, $v_1 =: \eta$, and $v_2 =: 0$, we see that the function $t \mapsto f(\sigma,t) - f(\sigma,0)$ is additive and continuous, hence of the form $A(t)\sigma$, where $A(t) = f(t,1)$. So putting $B(t) = f(t,0)$ we get the representation (5.2) as claimed. □

Note that this degeneracy result does not contradict Theorem 4.5, since the function $\phi(u) = u^{1-\theta}$ does not have, for $\theta > 0$, the property $\phi(u) < u$ of a comparison function in the sense of [19]. Proposition 5.1 is of course somewhat disappointing: it shows that we may apply *global* contraction-type conditions like (5.3) in $X = C^\alpha$ only if our problem is actually linear! The next example shows, however, that in the space $J_{\alpha,\beta}$ the situation is different.

**Example 5.2.** Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be of the form (4.21), where $g : [0,1] \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $L_g$, and $h(u) := \sin u$. Taking into account that Corollary 4.6 also holds for globally Lipschitz functions, and $L_{h,r} = L'_{h,r} \equiv 1$ is independent of $r$ for $h(u) = \sin u$, we conclude that

$$\|Fx_1 - Fx_2\|_{\alpha,\beta} \leq k\|x_1 - x_2\|_{\alpha,\beta} \qquad (x_1, x_2 \in J_{\alpha,\beta}),$$

where $k$ depends on $L_g$ and $\|g\|_C$. □

Example 5.2 shows that there is a sufficiently rich variety of nonlinearities $f$ for which the corresponding Nemytskij operator (1.6) even satisfies the global condition (5.1) in the space $X = J_{\alpha,\beta}$. Of course, for some purposes (e.g., for applying the Banach-Caccioppoli fixed point principle), a *local* contraction condition

$$\|Fx_1 - Fx_2\|_X \leq k(r)\|x_1 - x_2\|_X \qquad (x_1, x_2 \in X,\ \|x_1\|_X, \|x_2\|_X \leq r) \tag{5.4}$$

suffices. In many function spaces, the condition (5.4) is equivalent, at least for the autonomous Nemytskij operator $Fx(t) = f(x(t))$ to a local Lipschitz condition for the derivative $f'$ in $\mathbb{R}$, see [1, Theorem 5.51] for the Hölder space $X = C^\alpha$. Interestingly, a local Hölder condition like

$$\|Fx_1 - Fx_2\|_X \leq k(r)\|x_1 - x_2\|_X^\theta \qquad (x_1, x_2 \in X,\ \|x_1\|_X, \|x_2\|_X \leq r) \tag{5.5}$$

for some $\theta \in (0,1)$ is satisfied in $X = J_{\alpha,\beta}$ if $f'$ is merely continuous, but not necessarily Lipschitz continuous, as Theorem 4.5 shows. However, this does *not suffice* for applying the contraction mapping principle.

So we may summarize our discussion in the following synoptic table, where we restrict ourselves to the autonomous case of functions $f : \mathbb{R} \to \mathbb{R}$.

| In the space | (5.1) holds if | (5.3) holds if | (5.4) holds if | (5.5) holds if |
|---|---|---|---|---|
| $X = C^\alpha$ | $f(u) = A(t)u$ | $f(u) = A(t)u$ | $f' \in Lip_{loc}(\mathbb{R})$ | $f' \in C(\mathbb{R})$ |
| $X = J_{\alpha,\beta}$ | $f \in Lip(\mathbb{R})$ | $f \in Lip(\mathbb{R})$ | $f \in Lip_{loc}(\mathbb{R})$ | $f' \in C(\mathbb{R})$ |

Our table shows that contraction conditions are much easier to obtain in the integral-type Hölder space $J_{\alpha,\beta}$ than in the classical Hölder space $C^\alpha$. However, there is another advantage: In contrast to $C^\alpha$, compactness criteria are not hard to obtain in $J_{\alpha,\beta}$. So we may also apply other fixed point principles, like the classical Schauder theorem for compact operators and, more generally, the Darbo theorem for condensing operators. We will explain this in the next section.

## 6. Condensing operators

To apply our abstract results to the initial value problem

$$\begin{cases} D_c^\tau x(t) = f(t, x(t)) \quad (0 < t < 1), \\ x(0) = \theta_1, \ x'(0) = \theta_2, \end{cases} \tag{6.1}$$

where $\tau$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $\theta_1$, $\theta_2$ and $D_c^\tau$ are the same as in problem (1.3), we still have to impose some topological conditions on the Nemytskij operator $F$ generated by the right-hand side of (6.1). It is not hard to see that a function $x : [0,1] \to \mathbb{R}$ solves (6.1) if and only if $x$ is a fixed point of the operator

$$Tx(t) = \theta_1 + \theta_2 t + (I^\tau \circ F)x(t), \tag{6.2}$$

with $I^\tau$ given by (3.1) and $F$ by (1.6). In fact, this follows by applying the operator $I^\tau$ to both sides of the equation $D_c^\tau x = Fx$ in (6.1) and using the fact that for $n = 2$ the right-hand side of (3.3) simply becomes $x(t) - x(0) - x'(0)t$.

The structure of the fixed point operator $T$ in (6.2) shows again that it is of utmost importance to give conditions under which the operator $I^\tau$ maps some space $J_{\alpha,\beta}$ *into itself*. In fact, if $I^\tau$ would map $J_{\alpha,\beta}$ into some larger space $J_{\gamma,\delta}$ with $\gamma < \alpha$, then the Nemytskij operator $F$ should go back from $J_{\gamma,\beta}$ into the smaller space $J_{\alpha,\beta}$. Unfortunately, this would also lead to a drastic degeneracy of the generating function $f$, as was shown in [2].

So solving the initial value problem (1.3) reduces, as usual, to finding a fixed point of the operator (6.2) in the space $J_{\alpha,\beta}$. To this end, we will use the Darbo-Sadovskij fixed point principle [9,20] which is closely related to the notions of measures of noncompactness and condensing operators. Let us briefly recall the necessary definitions.

**Definition 6.1.** The *Hausdorff measure of noncompactness* of a bounded set $M$ in a Banach space $X$ is defined by

$$\chi(M) := \inf \{\varepsilon > 0 : M \text{ admits a finite } \varepsilon\text{-net in } X\}.$$

A (usually, nonlinear) operator $T : D \to X$ ($D \subseteq X$) is called *k-set contraction* if there exists a $k > 0$ such that

$$\chi(T(M)) \leq k\chi(M) \qquad (M \subseteq D \text{ bounded}). \tag{6.3}$$

The smallest possible constant $k$ in (6.3) is denoted by $\chi(T)$. In case $\chi(T) < 1$ the operator $T$ is called *condensing*.

The following important fixed point theorem has been proved independently by the Italian mathematician G. Darbo [9] and the Russian mathematician B.N. Sadovskij [20].

**Theorem 6.2.** *Let $X$ be a Banach space, $M \subset X$ bounded, closed, and convex, and $T : M \to M$ continuous and condensing. Then $T$ has a fixed point in $M$.*

Since every compact operator $T$ is condensing with $\chi(T) = 0$, and every contraction is condensing, where $\chi(T)$ is not larger than the minimal Lipschitz constant of $T$, Theorem 5.2 bridges the gap between the fixed point theorems of Schauder and Banach-Caccioppoli.

In order to apply Theorem 6.2 to a problem in a specific function space, one needs of course bilateral estimates, or even explicit formulas, for the Hausdorff measure of noncompactness in that space. Usually, this is achieved by introducing a certain "intrinsic" set function $\eta$ in $X$ which is equivalent to $\chi$ in the sense that

$$c\eta(M) \leq \chi(M) \leq C\eta(M) \qquad (M \subseteq X \text{ bounded}) \tag{6.4}$$

with two constants $c, C > 0$ independent of $M$. In particular, a bounded set $M \subset X$ is then precompact if and only if $\eta(M) = 0$. In the space $X = J_{\alpha,\beta}$, an estimate of this type is given in the following

**Proposition 6.3.** *The Hausdorff measure of noncompactness in the space $J_{\alpha,\beta}$ is equivalent to the set function*

$$\eta(M) := \limsup_{s\to 0} \sup_{x\in M} j_{\alpha,\beta}(x; [0, s]), \tag{6.5}$$

*where $j_{\alpha,\beta}(x; [0, s])$ is given by* (2.3). *More precisely, the estimate* (6.4) *holds in $J_{\alpha,\beta}$ with $c = 2^{-\beta/\alpha}$ and $C = 2^{\beta/\alpha}$. In particular, a subset $M \subset J_{\alpha,\beta}$ is compact if and only if it is bounded, closed, and satisfies $\eta(M) = 0$.*

*Proof.* First of all, we remark that there is no need to take into account the first (scalar) term in the norm (2.4), because every bounded set in $\mathbb{R}$ is precompact.

Given a bounded set $M \subset J_{\alpha,\beta}$ and $\lambda > \chi(M)$, let $\{z_1, \dots, z_m\}$ be a $\lambda$-net for $M$ in $J_{\alpha,\beta}$, which means that $M \subseteq B_\lambda(z_1) \cup \dots \cup B_\lambda(z_m)$, with $B_r(z)$ denoting the closed ball of radius $r > 0$ around $z$. Fix $x \in M$, and choose $z_j$ such that $j_{\alpha,\beta}(x - z_j) \leq \|x - z_j\|_{\alpha,\beta} \leq \lambda$. Since $j_{\alpha,\beta}(z_j)$ is finite, we find a $\delta > 0$ such that

$$\int_0^s \sigma^{-(\beta+1)}\omega(z_j; \sigma)^{\beta/\alpha}\, d\sigma \leq \varepsilon \qquad (j = 1, \dots, m)$$

for $0 \leq s \leq \delta$. So from

$$\omega(x; \sigma)^{\beta/\alpha} \leq (\omega(x - z_j; \sigma) + \omega(z_j; \sigma))^{\beta/\alpha} \leq 2^{\beta/\alpha}\omega(x - z_j; \sigma)^{\beta/\alpha} + 2^{\beta/\alpha}\omega(z_j; \sigma)^{\beta/\alpha}$$

it follows that

$$j_{\alpha,\beta}(x;[0,s]) \leq 2^{\beta/\alpha} j_{\alpha,\beta}(x-z_j;[0,s]) + 2^{\beta/\alpha} j_{\alpha,\beta}(z_j;[0,s]) \leq 2^{\beta/\alpha}(\lambda+\varepsilon)$$

for $0 \leq s \leq \delta$, hence $\eta(M) \leq 2^{\beta/\alpha}\chi(M)$ which proves one estimate.

To prove the other estimate, let $\mu > \eta(M)$. Given $\varepsilon > 0$, we may find $\delta > 0$ such that $j_{\alpha,\beta}(x;[0,s]) \leq \mu + \varepsilon$ for $0 \leq s \leq \delta$, uniformly in $x \in M$. Since $M$ is bounded in $J_{\alpha,\beta}$, $M$ is equicontinuous, hence precompact in $C$. So there exists a finite $\varepsilon$-net $\{z_1, \ldots, z_m\}$ for $M$ in $C$, where we may assume without loss of generality that all functions $z_j, \ldots, z_m$ belong to $J_{\alpha,\beta}$.

Fix $x \in M$, and choose $z_j$ such that $\|x - z_j\|_C \leq \varepsilon$. Now we distinguish two cases:

<u>1st case</u>: $\delta < s \leq 1$. Then $\omega(x - z_j; s) \leq 2\|x - z_j\|_C \leq 2\varepsilon$, and therefore

$$\int_\delta^1 s^{-(\beta+1)}\omega(x-z_j;s)^{\beta/\alpha}\,ds \leq (2\varepsilon)^{\beta/\alpha}\frac{\delta^{-\beta}-1}{\beta},$$

and the last expression may be made arbitrarily small.

<u>2nd case</u>: $0 \leq s \leq \delta$. Then

$$\int_0^\delta s^{-(\beta+1)}\omega(x-z_j;s)^{\beta/\alpha}\,ds \leq 2^{\beta/\alpha}\int_0^\delta s^{-(\beta+1)}\omega(x;s)^{\beta/\alpha}\,ds$$
$$+2^{\beta/\alpha}\int_0^\delta s^{-(\beta+1)}\omega(z_j;s)^{\beta/\alpha}\,ds \leq 2^{\beta/\alpha}(\mu+2\varepsilon).$$

We conclude that $\chi(M) \leq 2^{\beta/\alpha}\eta(M)$ which proves the other estimate. □

The two-sided estimate for $\chi(M)$ in terms of the function (6.5) is of course similar to the well-known Arzelà-Ascoli compactness criterion which states that a subset $M \subset C$ is compact if and only if it is bounded, closed, and satisfies

$$\limsup_{s\to 0}\sup_{x\in M}\omega(x;s) = 0.$$

One could expect that a similar result holds in the Hölder space $C^\alpha$. However, the analogous condition

$$\limsup_{s\to 0}\sup_{x\in M} s^{-\alpha}\omega(x;s) = 0$$

is *not* satisfied even for a singleton $\{x_0\}$, as the example $x_0(t) = t^\alpha$ shows. Some authors (e.g., [11,18]) claim to give compactness criteria in the Hölder space $C^\alpha$; unfortunately, all these criteria are false, since they are either sufficient, but not necessary, or necessary, but not sufficient. Our Proposition 5.3 above shows that it is easier to obtain compactness criteria in the integral-type Hölder space $J_{\alpha,\beta}$. More generally, the following is true.

**Theorem 6.4.** *Under the hypothesis* (4.3), *the Nemytskij operator* (1.6) *satisfies the estimate* (6.3) *for* $X = J_{\alpha,\beta}$ *and* $D = B_r(J_{\alpha,\beta})$ *with* $k = 2^{2\beta/\alpha}b_r$, *where* $b_r$ *is the constant appearing in* (4.3).

*Proof.* Replacing in (4.5) the integration over $[0,1]$ by the integration over $[0,s]$ with $0 < s < 1$ we get

$$j_{\alpha,\beta}(Fx;[0,s]) \leq \int_0^s a_r(\sigma)\,d\sigma + b_r j_{\alpha,\beta}(x;[0,s]). \tag{6.6}$$

Since the integral in (6.6) tends to zero for $s \to 0+$, the claim follows from the definition of $\eta$ and Proposition 6.3. □

## 7. An existence result

Using the results obtained so far for the operators $I^\tau$ and $F$ we are now in a position to prove our existence result for solutions of the problem (6.1) in the Hölder space of integral type $J_{\alpha,\beta}$. We achieve this by applying Theorem 6.2 to the operator (6.2), verifying the hypotheses in 4 steps.

1st step: *The operator $T$ maps $B_r(J_{\alpha,\beta})$ into $J_{\alpha,\beta}$ and is bounded on $B_r(J_{\alpha,\beta})$.* By Theorem 3.5 and Theorem 4.1, this is true if $0 < \tau < 1$, $0 < \alpha < \tau$, $\beta > \alpha$, and (4.3) holds for some $a_r \in L_1$ and $b_r \geq 0$.

2nd step: *The operator $T$ is continuous on $B_r(J_{\alpha,\beta})$.* Continuity of $I^\tau$ is equivalent to boundedness, while continuity of $F$ follows from (4.6) and (4.9), by Theorem 4.7.

3rd step: *The operator $T$ is condensing on $B_r(J_{\alpha,\beta})$.* Since the term $\theta_1 + \theta_2 t$ in the operator (6.2) is finite dimensional, we only have to make sure that (6.3) holds for the operator $I^\tau \circ F$ on $M = B_r(J_{\alpha,\beta})$ with $k < 1$. Theorem 3.5 and Theorem 6.4 show that this is true if $2^{2\beta/\alpha} b_r c_3 < 1$, with $b_r$ given in (4.3) and $c_3$ given in (3.6)

4th step: *The operator $T$ maps $B_r(J_{\alpha,\beta})$ into itself.* This step requires some calculation. Since the linear function $u(t) = t$ satisfies $\omega(u;\sigma) = \sigma$ and $j_{\alpha,\beta}(u;[0,1]) = \alpha/\beta(1-\alpha)$, we have

$$|||u|||_{\alpha,\beta} = 1 + \left(\frac{\alpha}{\beta(1-\alpha)}\right)^{\alpha/\beta}.$$

For estimating $|||(I^\tau \circ F)x|||_{\alpha,\beta}$ we may use Theorem 3.5 and Theorem 4.1.
As a result we get

$$|||(I^\tau \circ F)x|||_{\alpha,\beta} \leq c_3 |||Fx|||_{\alpha,\beta} \leq c_r R,$$

where $R$ depends on $r$ and is given by (4.8). So the following existence theorem is true.

**Theorem 7.1.** *Let $0 < \tau < 1$, $0 < \alpha < \tau$, and $\beta > \alpha$. Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the hypotheses of Theorem 4.7. Assume that*

$$|\theta_1| + |\theta_2| + |\theta_2| \left(\frac{\alpha}{\beta(1-\alpha)}\right)^{\alpha/\beta} + c_3 c_{0,r} + c_3 R \leq r \tag{7.1}$$

*for some $r > 0$, where $c_3$ is given by* (3.6), *$c_{0,r}$ by* (4.2), *and $R$ by* (4.4). *Then the problem* (6.1) *has a solution $x \in B_r(J_{\alpha,\beta})$.*

Of course, the invariance condition (7.1) is not very specific, so the question arises how to adjust the constants involved to achieve it. If the left-hand side of condition (7.1) is small enough, this condition will be satisfied for sufficiently large $r > 0$. Now, the only terms in (7.1) which depend on $r$ are $c_{0,r}$ and $R$ (containing $c_{1,r}$ and $c_{2,r}$). If the function $f(t, \cdot)$ has strictly sublinear growth, uniformly in $t$, i.e.,

$$\lim_{|u|\to\infty} \sup_{0\leq t\leq 1} \frac{|f(t,u)|}{|u|} = 0,$$

then $c_{0,r} = o(r)$ as $r \to \infty$, as needed. A similarly condition for $\partial_1 f(t, \cdot)$ and $\partial_1 f(t, \cdot)$ will work for $c_{1,r}$ and $c_{2,r}$. This reasoning, however, restricts very much the applicability of Theorem 7.1. Instead, observe that the terms $c_{0,r}$ and $R$ depend linearly on $r$. So we may multiply the function $f$ with some free parameter $\lambda$ which is so small that the slope of the right-hand side of (7.1), as an affine function of $r > 0$, is less than 1. Here is an example, where the variables $t$ and $u$ of the nonlinearity are not separated as in (4.21).

**Example 7.2.** Consider the problem

$$\begin{cases} D_c^\tau x(t) = \lambda \sin(t + x(t)) \quad (0 < t < 1), \\ x(0) = \theta_1,\ x'(0) = \theta_2, \end{cases} \tag{7.2}$$

where $\lambda > 0$. Here the constants $c_{0,r} = c_{1,r} = c_{2,r} = \lambda$ are independent of $r > 0$, and all condition (4.6), (4.7), (4.9) and (4.10) are satisfied with $A_r := \lambda$ and $B_r := \lambda(r+1)$.

As we have seen in Example 4.2, the crucial growth estimate (4.3) holds here, using the shortcut $\gamma := \beta/\alpha$, with

$$a_r(\sigma) := 2^{\gamma-1}\lambda^\gamma\sigma^{\gamma-\beta-1}, \qquad b_r := 2^{\gamma-1}\lambda^\gamma.$$

Moreover, the radius (4.8) of the ball $B_R(J_{\alpha,\beta})$ becomes

$$R = \lambda + \lambda(1 + r^\gamma)^{1/\gamma},$$

which illustrates the linear dependence on $r$. Finally, the fixed point operator (6.2) is condensing if

$$2^{3\gamma-1}\lambda^{\gamma+1} < 1.$$

All these requirements may be achieved if $\lambda > 0$ is sufficiently small, and so we get existence and uniqueness of a solution $x \in J_{\alpha,\beta}$ for arbitrary initial values $\theta_1$ and $\theta_2$. □

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