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# FIXED POINTS AND SETS OF MULTIVALUED CONTRACTIONS: AN ADVANCED SURVEY WITH SOME NEW RESULTS

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Dedicated to the memory of Professor Andrzej Granas (1929–2019)

**Abstract.** The existence of fixed points and, in particular, coupled fixed points is investigated for multivalued contractions in complete metric spaces. Multivalued coupled fractals are furthermore explored as coupled fixed points of certain induced operators in hyperspaces, i.e. as coupled compact subsets of the original spaces. The structure of fixed point sets is considered in terms of absolute retracts. We also formulate a continuation principle for multivalued contractions as a nonlinear alternative based on the topological essentiality. Two illustrative examples about coupled multivalued fractals are supplied.
**Key Words and Phrases**: Multivalued contractions, fixed points, coupled fixed points, coupled fractals, absolute retracts, continuation principle, essentiality.
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## 1. Introduction

As the title suggests, the present paper can be regarded as an advanced version of our paper [2] from 2001, where the theoretical results were also applied to the existence of multivalued metric fractals and to the existence of almost periodic solutions to differential inclusions. Here some statements are recalled in a new context, or they are new in the sense that they have not yet been published (i.e. not only to be contained in [2]). The latter concerns the existence of coupled fixed points of multivalued maps and, in particular, the existence of multivalued coupled fractals, which will be supplied by two illustrative examples. Furthermore, a continuation method for contractions,

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initiated by Granas in [19] and followed by some further authors, is partly extended for multivalued contractions.

Although the theorems for (multivalued) contractions belong exclusively to the metric fixed point theory (see e.g. [8, 24, 26, 33, 34, 40]), we also decided to make a short excursion into the topological fixed point theory (see e.g. [3, 14, 20], in order to compare the power of the related results about (coupled) fixed points in the both fields. Let us note that the topological essentiality, treated here too, lies between these two theories.

Hence, our paper is organized as follows. Preliminaries concern, as usually, technicalities and basic definitions. Then the existence results for fixed points, including coupled fixed points, are presented. The results for coupled multivalued fractals, which are treated separately, are illustrated by two examples. The structure of fixed point sets is discussed in terms of absolute retracts. A continuation method for multivalued contractions is formulated in the form of a nonlinear alternative. Finally, some concluding remarks are supplied.

## 2. Preliminaries

In this paper, all topological spaces are assumed to be metric.

Let $(X, d)$ be a metric space. For a standard distance between the set $A \subset X$ and the point $x \in X$, we have

$$\operatorname{dist}(x, A) := \inf \{d(x, y); y \in A\}.$$

Evidently, $\operatorname{dist}(x, A) = 0$ if and only if $x \in \operatorname{cl} A$, where $\operatorname{cl} A$ denotes the closure of $A$ in $X$.

We denote

$$\mathcal{CB}(X) := \{A \subset X; A \text{ is nonempty, closed and bounded}\},$$
$$\mathcal{K}(X) := \{A \subset X; A \text{ is nonempty and compact}\}.$$

For a given $\varepsilon > 0$ and $A \in \mathcal{CB}(X)$, we define the neighbourhood of $A$ as

$$O_\varepsilon(A) := \{x \in X; \operatorname{dist}(x, A) < \varepsilon\}.$$

We shall consider the Hausdorff distance $d_H$ as a function

$$d_H : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, +\infty),$$

defined as follows:

$$d_H(A, B) := \inf \{\varepsilon > 0; A \subset O_\varepsilon(B) \wedge B \subset O_\varepsilon(A)\}.$$

It is well known that $(\mathcal{CB}(X), d_H)$ is a metric space which is complete, provided $X$ is complete.

Let $E$ be a Banach space, $A, B, C, D \in \mathcal{CB}(E)$, and $x, y \in E$. It is easy to see that

(i) $d_H(A + B, C + D) \leq d_H(A, C) + d_H(B, D)$,
(ii) $d_H(x + A, \{y\}) = d_H(\{x\}, y - A)$,
(iii) $d_H(tA, tB) = d_H(A, B)$, for $t \in [0, 1]$,
(iv) $d_H(x + A, x + B) = d_H(A, B)$,

where

$$A + B := \{x + y; x \in A, y \in B\} \quad \text{and} \quad tA := \{tx; x \in A\}.$$

Furthermore,

(v) $d_H(A \times B, C \times D) \leq \max\{d_H(A, B), d_H(C, D)\}$,
where $d_H(A \times B, C \times D)$ is considered in $\mathcal{CB}(X \times X)$.

Let us note that, in the Cartesian product of metric spaces, we consider the max-metric.

Let $X, Y$ be two metric spaces. A map $\varphi : X \to \mathcal{CB}(Y)$ will be called a *multivalued map* and

$$\Gamma_\varphi := \{(x, y) \in X \times Y; y \in \varphi(x)\}$$

will be called the *graph* of $\varphi$.

If $\varphi : X \to \mathcal{CB}(Y)$ is a multivalued map, then

- $\varphi$ is called *upper semicontinuous* (u.s.c.) if, for every open $U \subset Y$, the set
$$\varphi^{-1}(U) := \{x \in X; \varphi(x) \subset U\} \quad \text{is open in } X;$$
- $\varphi$ is called *lower semicontinuous* (l.s.c.) if, for every open $U \subset Y$, the set
$$\varphi_+^{-1}(U) := \{x \in X; \varphi(x) \cap U \neq \emptyset\} \quad \text{is open in } X;$$
- $\varphi$ is called *continuous* if it is both u.s.c. and l.s.c.;
- $\varphi$ is called *Hausdorff continuous* (h.c.) if it is continuous with respect to the metrics $d$ in $X$ and $d_H$ in $\mathcal{CB}(Y)$;
- $\varphi$ is called a *Lipschitz map* if there exists a constant $\alpha \in [0, +\infty)$ such that
$$d_H(\varphi(x), \varphi(y)) \leq \alpha d(x, y), \quad \text{for every } x, y \in X;$$
- $\varphi$ is called a *contraction* if it is a Lipschitz map such that $\alpha \in [0, 1)$.

Let us recall some basic properties of multivalued mappings in the form of the following proposition.

**Proposition 2.1** *Let $X, Y$ be metric spaces and $\varphi$ be a multivalued mapping from $X$ to $Y$.*

- If $\varphi : X \to \mathcal{CB}(Y)$ is u.s.c., then $\Gamma_\varphi$ is a closed subset of $X \times Y$.
- A map $\varphi : X \to \mathcal{K}(Y)$ is u.s.c. if and only if $\Gamma_\varphi$ is a closed subset of $X \times Y$.
- A map $\varphi : X \to \mathcal{K}(Y)$ is h.c. if and only if $\varphi$ is continuous.
- If $\varphi : X \to \mathcal{CB}(Y)$ is h.c., then it is l.s.c.
- Any Lipschitz (and, in particular, contractive) map is h.c.

For the proofs and more details, see e.g. [3, 14, 22].

It will be also convenient to recall some further properties of multivalued mappings.

**Lemma 2.2** (cf. e.g. [3, 14]) *If $\varphi : X \to \mathcal{K}(Y)$ is a u.s.c. map and $A \subset X$ is a compact set, then the set $\varphi(A) = \bigcup_{x \in A} \varphi(x)$ is compact, too.*

**Lemma 2.3** (cf. e.g. [2, Proposition 1.7]) *If $\varphi : X \to \mathcal{CB}(Y)$ is u.s.c. with connected values, then $\varphi(X) = \bigcup_{x \in X} \varphi(x)$ is connected, provided $X$ is connected.*

We shall also use the following notions.

**Definition 2.4** Let $A \subset X$ be a subset of the space $X$. We say that $A$ is a *retract* of $X$ if there exists a continuous map $r : X \to A$ (called a retraction) such that $r(x) = x$, for every $x \in A$.

**Definition 2.5** A space $X$ is called an *absolute retract* (written:$X \in AR$) if, for every space $Y$ and for every embedding $h : Y \to X$ such that $h(Y)$ is a closed subset of $X$, the set $h(Y)$ is a retract of $X$.

Let us note that $X \in AR$ if and only if $X$ has an extension property ($X \in ES$). For more details, see e.g. [6].

**Definition 2.6** A space $X$ is called *n-connected* (written:$X \in C^n$) if any continuous map $f : S^k \to X$, $k \le n$, can be extended over $B^{k+1}$, where $B^{k+1}$ is the unit closed ball in the Euclidean space $\mathbb{R}^{n+1}$ and $S^k$ denotes a unit sphere in $\mathbb{R}^{n+1}$.

**Definition 2.7** A space $X$ is called to be *infinitely connected* (written:$X \in C^\infty$) if it is $n$-connected, for every $n = 1, 2, \ldots$

Evidently, if $X \in AR$, then $X \in C^\infty$.
For more details, see e.g. [6, 14, 22, 25].

## 3. Existence of fixed points

Let $\varphi : X \to \mathcal{CB}(X)$ be a multivalued map and $x_0 \in X$ be a point.

**Definition 3.1** The sequence $\{x_n\}_{n\ge 0}$, where $x_1 \in \varphi(x_0)$, $x_2 \in \varphi(x_1)$, $\ldots$, $x_{n+1} \in \varphi(x_n)$, $\ldots$, is called an *orbit* of $\varphi$, starting from the point $x_0$. If it is convergent, then it is called a *strong orbit*.

**Lemma 3.2** *If $\varphi$ has a closed graph $\Gamma_\varphi$ or if it is a Lipschitz map, then the existence of a strong orbit implies that there exists a point $\hat{x}$ such that $\hat{x} \in \varphi(\hat{x})$.*
*Proof.* Assume that $\{x_n\}_{n\ge 0}$ is a strong orbit and $\Gamma_\varphi$ is a closed subset of $X \times X$. Then we have

$$\{x_n\} \to \hat{x} \qquad \text{and} \qquad x_n \in \varphi(x_{n-1}), \quad \text{for every } n \ge 1,$$

i.e., $(x_{n-1}, x_n) \in \Gamma_\varphi$ for every $n \ge 1$. Consequently, $(\hat{x}, \hat{x}) \in \Gamma_\varphi$, i.e. $\hat{x} \in \varphi(\hat{x})$.
Now, assume that $\varphi$ is a Lipschitz map. Then we have

$$\operatorname{dist}(x_n, \varphi(\hat{x})) \le d_H(\varphi(x_n), \varphi(\hat{x})) \le \alpha \cdot d(x_n, \hat{x}),$$

for every $n \ge 0$. Thus,

$$\operatorname{dist}(\hat{x}, \varphi(\hat{x})) = \lim_{n\to\infty} d(x_n, \varphi(\hat{x})) = 0, \quad \text{i.e. } \hat{x} \in \varphi(\hat{x}).$$

A point $x \in X$ such that $x \in \varphi(x)$ is called a *fixed point* of $\varphi$.
We let

$$\operatorname{Fix}(\varphi) := \{x \in X; x \in \varphi(x)\}.$$

If $\operatorname{Fix}(\varphi) \ne \emptyset$, then there obviously exists a strong orbit of $\varphi$.

**Proposition 3.3** *If $X$ is a complete space and $\varphi$ is a contraction, then there exists a strong orbit of $\varphi$.*

*Proof.* Assume that $\alpha \in [0,1)$ is a Lipschitz constant for $\varphi$. Let $x_0 \in X$ be an arbitrary point. Moreover, let $\alpha_1 \in (\alpha, 1)$. We pick $x_1 \in \varphi(x_0)$ with $d(x_0, x_1) > 0$. If no such $x_1$ exists, then $x_0 \in \mathrm{Fix}(\varphi)$, and $x_0, x_0, \ldots, x_0, \ldots$ is a strong orbit.

Otherwise,

$$\mathrm{dist}(x_1, \varphi(x_1)) \leq d_H(\varphi(x_0), \varphi(x_1)) \leq \alpha_1 d(x_0, x_1),$$

and we can find $x_2 \in \varphi(x_1)$ such that

$$d(x_1, x_2) < \alpha_1 d(x_0, x_1).$$

Inductively, we produce a sequence $\{x_n\}$ such that $x_{n+1} \in \varphi(x_n)$, for every $n \geq 0$, and $d(x_n, x_{n+1}) < \alpha_1^n d(x_0, x_1)$.

It follows, that $\{x_n\}$ is a Cauchy sequence. Since $X$ is a complete space, we get that $\{x_n\}$ is a strong orbit, as claimed.

**Corollary 3.4** (cf. [8]) *If $\varphi$ is a contraction and $X$ is a complete space, then*

$$\mathrm{Fix}(\varphi) \neq \emptyset.$$

Concerning Corollary 3.4, which is due to Covitz and Nadler [8] and its extensions, see also e.g. [3, 14, 16, 20, 24, 26, 34, 40].

Let us note, that Corollary 3.4 is perhaps a mostly known multivalued generalization of the celebrated Banach Contraction Principle. It is not true that, for multivalued mappings, $\mathrm{Fix}(\varphi)$ is a singleton. For example, let $\varphi(x) = A$ be a constant multivalued map, where $A \subset X$ is a set, then $\mathrm{Fix}(\varphi) = A$.

Now, let us analyze the situation locally. Let $A \subset X$ be an arbitrary set and let $\varphi : A \to \mathcal{CB}(X)$ be a contraction. It can be easily seen that the set $\mathrm{Fix}(\varphi)$ can be empty.

Let $B(x_0, r) := \{x \in X; d(x, x_0) < r\}$ be an open ball in $X$ with center $x_0$ and radius $r$. The following proposition was already proved in [2, Proposition 2.2].

**Proposition 3.5** (cf. [2, Proposition 2.2]) *Let $\varphi : B(x_0, r) \to \mathcal{CB}(X)$ be a contraction such that*

$$d_H(x_0, \varphi(x_0)) < (1 - \alpha) \cdot r,$$

*where $\alpha \in [0,1)$ is a Lipschitz constant for $\varphi$. Then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

Now, assume that $X = E$ is a Banach space and let $U$ be an open subset of $E$. With a contraction $\varphi : U \to \mathcal{CB}(E)$, we associate a contractive field

$$\Phi : U \to \mathcal{CB}(E),$$

defined by the formula

$$\Phi(x) := x - \varphi(x), \qquad \text{for every } x \in U,$$

where $x - \varphi(x) := \{x - y; y \in \varphi(x)\}$.

The following Proposition 3.6, whose proof relies on the application of Proposition 3.5, was also already presented as a theorem, under the name of *Invariance of a domain for contractive fields*, in [2, Theorem 2.3].

**Proposition 3.6** (cf. [2, Proposition 2.3]) *Let $U$ be an open subset of a Banach space and let $\varphi : U \to \mathcal{CB}(E)$ be a contraction. Then $\Phi : U \to \mathcal{CB}(E)$ is an open mapping, i.e., for every open $V \subset U$, the set $\Phi(V) := \bigcup_{u \in V} \Phi(u)$ is an open subset of $E$.*

In what follows, an open and connected subset of a Banach space will be called a *domain.*

**Corollary 3.7** *If $U \subset E$ is a domain and $\varphi : U \to \mathcal{CB}(E)$ is a contraction with connected values, then $\Phi(U)$ is a domain, too.*

Corollary 3.7 is a direct consequence of Proposition 3.6 and Lemma 2.3.

**Proposition 3.8** *If $\varphi : E \to \mathcal{CB}(E)$ is a contraction, then $\Phi(E) = E$, where $E$ is a Banach space.*

*Proof.* Let $y \in E$. For the proof, it is sufficient to show that there exists $x \in E$ such that $y \in \Phi(x)$.

For this goal, we define the map $\Psi : E \to E$ by letting

$$\Psi(x) := y + \varphi(x), \quad \text{for every } x \in E.$$

It is obvious that $\Psi$ is a contraction. Using Corollary 3.4, we get a point $x \in \Psi(x)$. It implies that there exists $y_1 \in \varphi(x)$ such that $x = y + y_1$. Hence, $y = (x - y_1) \in \Phi(x)$, and the proof is completed.

Now, we would like to deal briefly with the coupled fixed point theory (see e.g. [21, 27, 28, 29, 30, 31, 37]).

Let $\varphi : X \times X \to \mathcal{CB}(X)$ be a multivalued map.

**Definition 3.9** (cf. e.g. [21]) A pair $(x, y) \in X \times X$ is called a *coupled fixed point* for $\varphi$ if $x \in \varphi(x, y)$ and $y \in \varphi(y, x)$.

For a given $\varphi : X \times X \to \mathcal{CB}(X)$, we define $\psi : X \times X \to \mathcal{CB}(X \times X)$ by putting

$$\psi(x, y) := \varphi(x, y) \times \varphi(y, x).$$

The following property is self-evident.

**Proposition 3.10** *The map $\varphi$ has a coupled fixed point if and only if* $\mathrm{Fix}(\psi) \neq \emptyset$.

Let us make a short excursion into the topological fixed point theory. It will be suitable to recall the following properties for the Cartesian products:

- the Cartesian product of two (and, in fact, any countable collection) of AR–spaces is an absolute retract (see e.g. [6]);
- the Cartesian product of two acyclic (i.e. homologically equivalent to a point) sets is acyclic (see e.g. [14]);
- the Cartesian product of two (compact) u.s.c. maps is (compact) u.s.c. (see e.g. [3, 14]).

The following theorem is then a direct consequence of Proposition 3.10 and the Eilenberg–Montgomery type fixed point theorem (see e.g. [20, Corollary VI.19.7.5(iv)]).

**Theorem 3.11** *Let $X$ be an AR–space and $\varphi : X \times X \to \mathcal{K}(X)$ be a compact acyclic mapping (i.e. a compact u.s.c. mapping with acyclic values). Then $\varphi$ admits a coupled fixed point.*

**Corollary 3.12** *Let $X$ be a compact AR–space and $\varphi : X \times X \to \mathcal{CB}(X)$ be an acyclic mapping (i.e. a u.s.c. mapping with compact acyclic values). Then $\varphi$ admits a coupled fixed point.*

Coming back to the metric fixed point theory, since the Cartesian product of two complete metric spaces is complete and the Cartesian product of two contractions is a contraction (see the property (v) in Section 2), the following theorem is a direct consequence of Proposition 3.10 and Corollary 3.4.

**Theorem 3.13** *If $X$ is a complete metric space and $\varphi : X \times X \to \mathcal{CB}(X)$ is a contraction, then $\varphi$ admits a coupled fixed point.*

**Remark 3.14** If the mapping $\varphi$ in Theorem 3.11 is still a contraction, then the only AR–space $X$ need not be complete, when comparing the result with Theorem 3.13. Under the same additional assumption, Corollary 3.12 does not bring any new information. On the other hand, in the single–valued case, in Theorem 3.13 and, under the additional assumptions about contractions, in Corollary 3.12 (de facto also in Theorem 3.11) the coupled fixed point is obviously unique and attractive.

## 4. Coupled multivalued fractals

In this section, we will prove the existence of coupled fixed points of certain induced maps in hyperspaces. The related pairs of compact subsets in the original spaces will be called coupled multivalued fractals.

**Theorem 4.1** *Let $X$ be an absolute retract (written: $X \in AR$) and $\varphi_i : X \times X \to \mathcal{K}(X)$, $i = 1, \ldots, n$, be a family of compact continuous maps. Then there exists a pair $(A^*, B^*) \in \mathcal{K}(X) \times \mathcal{K}(X)$ such that*

$$\begin{cases} A^* = \bigcup_{i=1}^{n} \varphi_i(A^*, B^*) = \bigcup_{i=1}^{n} \bigcup_{\substack{x \in A^* \\ y \in B^*}} \varphi_i(x, y), \\ B^* = \bigcup_{i=1}^{n} \varphi_i(B^*, A^*) = \bigcup_{i=1}^{n} \bigcup_{\substack{y \in B^* \\ x \in A^*}} \varphi_i(y, x), \end{cases} \tag{4.1}$$

*which we call a topological multivalued coupled fractal.*

*Proof.* Like in Proposition 3.10, the existence of a pair $(A^*, B^*) \in \mathcal{K}(X) \times \mathcal{K}(X)$ satisfying (4.1) is equivalent to the existence of a fixed point in the Cartesian product of hyperspaces $(\mathcal{K}(X) \times \mathcal{K}(X), d_H^*)$, where

$$d_H^*((A_1, B_1), (A_2, B_2)) := \max \{d_H(A_1, A_2), d_H(B_1, B_2)\},$$

of the *coupled Hutchinson–Barnsley operators* $\mathcal{F} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X) \times \mathcal{K}(X)$, where

$$\mathcal{F}(A,B) := \left( \bigcup_{i=1}^{n} \varphi_i(A,B), \bigcup_{i=1}^{n} \varphi_i(B,A) \right). \tag{4.2}$$

Since $X \in AR$, it is well known (see e.g. [4], and the references therein) that $\mathcal{K}(X) \in AR$, and subsequently (cf. [6]) that $\mathcal{K}(X) \times \mathcal{K}(X) \in AR$. Furthermore, since $\varphi_i$, $i = 1, \ldots, n$, are assumed to be compact and continuous, the same must be true (see e.g. [3, 14]) for their union

$$\bigcup_{i=1}^{n} \varphi_i : X \times X \to \mathcal{K}(X)$$

and the Cartesian product (in the second component, the variables $(x, y)$ are reversed into $(y, x)$)

$$\left( \bigcup_{i=1}^{n} \varphi_i, \bigcup_{i=1}^{n} \varphi_i \right) : X \times X \to \mathcal{K}(X) \times \mathcal{K}(X),$$

as well as its induced hypermap (see [1], and cf. property (v) in Section 2) $\mathcal{F} : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X) \times \mathcal{K}(X)$, i.e. for the coupled Hutchinson–Barnsley operator defined in (4.2).

Hence, applying the Granas version of the Lefschetz fixed point theorem (see e.g. [20, Theorem V.15.4.3]), $\mathcal{F}$ admits a fixed point $(A^*, B^*) \in \mathcal{K}(X) \times \mathcal{K}(X)$, i.e. $(A^*, B^*) = \mathcal{F}(A^*, B^*)$, which completes the proof.

In order to guarantee for coupled topological fractals a sort of a weak local stability, called a nonejectivity in the sense of Browder, let us recall the definition of a nonejective fixed point.

**Definition 4.2** Let $(X, d)$ be a metric space and $f : X \to X$ be a continuous mapping. We say that a fixed point $x_0 \in X$ of $f$ is *ejective* if there exists an open neighbourhood $V$ of $x_0$ such that, for every $x \in V \setminus \{x_0\}$, there is an integer $n = n(x_0) \geq 1$ such that

$$f^n(x) = \underbrace{f \circ \cdots \circ f}_{n\text{-times}}(x) \in X \setminus V.$$

Otherwise (i.e. if $x_0$ is not ejective), a fixed point $x_0 \in X$ of $f$ is called *nonejective.*

It is well known (see e.g. [38]) that the Hilbert cube has the *nonejectivity fixed point property*, i.e. that every continuous mapping on it admits a nonejective fixed point. Moreover, every compact $AR$–space is, up to a homeomorphism, the retract image of the Hilbert cube.

**Theorem 4.3** *Let $X$ be a nondegenerated Peano's continuum, i.e. a compact, connected and locally connected metric space, and $\varphi_i : X \times X \to \mathcal{K}(X)$, $i = 1, \ldots, n$, be a family of continuous maps. Then there exists a pair $(A^*, B^*) \in \mathcal{K}(X) \times \mathcal{K}(X)$ satisfying* (4.1)*, which is nonejective with respect to $\mathcal{F}$ defined in* (4.2).

*Proof.* Since $\mathcal{K}(X)$ is the Hilbert cube (see e.g. [39]) and, in particular, it is a compact AR–space, for the existence part, it is enough to apply the arguments from the proof of Theorem 4.1. The nonejective part follows directly from the nonejectivity fixed point property of the Hilbert cube, because the Cartesian product of two Hilbert cubes is obviously the Hilbert cube.

**Example 1** In order to apply Theorem 4.3, let us consider the following family of (compact) continuous maps

$$\varphi_i : [-2,2]^2 \times [-2,2]^2 \to \mathcal{K}\left([-2,2]^2\right), \quad i = 1,2,3,$$

where

$$\begin{aligned}
\varphi_1(x,y) &:= \left([0.9995, 1.0005]\cos\left(x_1^2 - y_1^2\right), [0.9995, 1.0005]\sin\left(x_2^2 + y_2^2\right)\right), \\
\varphi_2(x,y) &:= \left(1 - \cos\left(\frac{x_1^2}{2} + \frac{y_2^2}{2}\right), 1 + \sin\left(\frac{y_1^2}{2} - \frac{x_2^2}{2}\right)\right), \\
\varphi_3(x,y) &:= \left(\cos\left(\frac{x_1^2}{2} + \frac{y_2^2}{2}\right), 1 - \sin\left(\frac{y_1^2}{2} - \frac{x_2^2}{2}\right)\right),
\end{aligned}$$

$$x = (x_1, x_2),\ y = (y_1, y_2).$$

A nonejective, topological, multivalued, coupled fractal, guaranteed by Theorem 4.3, is plotted in Figure 1.

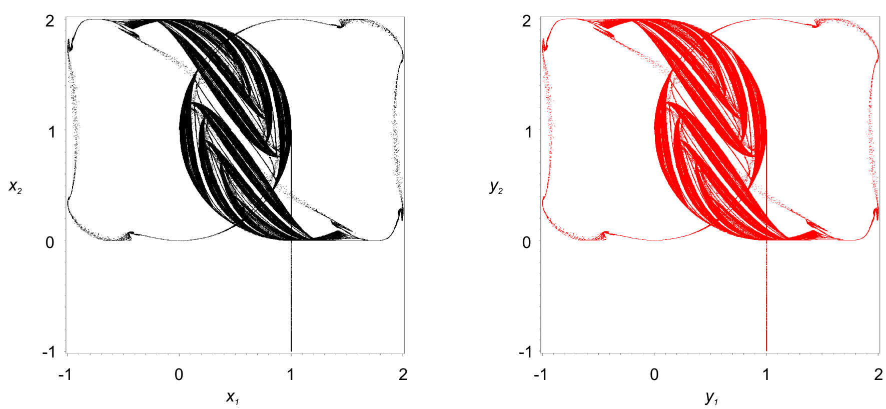


FIGURE 1. Approximation of the coupled fractal from Example 1.

In order to guarantee the attractivity of coupled fractals, let us proceed to metric coupled fractals.

**Theorem 4.4** (metric multivalued coupled fractals) *Let $(X,d)$ be a complete metric space and $\varphi_i : X \times X \to \mathcal{K}(X)$, $i = 1,\ldots,n$, be a family of contractions. Then there exists a unique pair $(A^*, B^*) \in \mathcal{K}(X) \times \mathcal{K}(X)$ satisfying* (4.1)*, which we call a metric multivalued coupled fractal.*

*Proof.* Like in Proposition 3.10, the existence of a pair $(A^*, B^*) \in \mathcal{K}(X) \times \mathcal{K}(X)$ satisfying (4.1) is equivalent to the existence of a fixed point of the coupled Hutchinson–Barnsley operator $\mathcal{F}$, defined in (4.2). The same is obviously true for the uniqueness property.

Since the hyperspace $(\mathcal{K}(X), d_H)$ is well known to be complete (see e.g. [1], and the references therein), so must be $(\mathcal{K}(X)\times\mathcal{K}(X), d_H^*)$, where the metric $d_H^*$ was defined in the proof of Theorem 4.1. Furthermore, since $\varphi_i$, $i = 1,\ldots,n$, are contractions, the same must be true for their union

$$\bigcup_{i=1}^n \varphi_i : X\times X \to \mathcal{K}(X)$$

and the Cartesian product (in the second component, the variables $(x,y)$ are reversed into $(y,x)$)

$$\left(\bigcup_{i=1}^n \varphi_i, \bigcup_{i=1}^n \varphi_i\right) : X\times X \to \mathcal{K}(X)\times\mathcal{K}(X),$$

as well as its induced hypermap (see [1], and cf. property (v) in Section 2) $\mathcal{F}$ : $\mathcal{K}(X)\times\mathcal{K}(X) \to \mathcal{K}(X)\times\mathcal{K}(X)$, defined in (4.2).

Hence, applying the celebrated Banach Contraction Principle, $\mathcal{F}$ admits a unique fixed point $(A^*, B^*) \in \mathcal{K}(X)\times\mathcal{K}(X)$, i.e. $(A^*, B^*) \in \mathcal{F}(A^*, B^*)$, which completes the proof.

**Remark 4.5** Because of the application of the Banach Contraction Principle, the pair $(A^*, B^*)$ is an attractor of the coupled Hutchinson–Barnsley operator $\mathcal{F}$ in $\mathcal{K}(X)\times\mathcal{K}(X)$. In other words,

$$\lim_{m\to\infty} d_H^*\left((A^*, B^*), \mathcal{F}^m(A_0, B_0)\right) = 0$$

holds, for any pair $(A_0, B_0) \in \mathcal{K}(X)\times\mathcal{K}(X)$.

**Example 2** In order to apply Theorem 4.4, let us consider the following family of contractions

$$\varphi_i : [0,1]^2 \times [0,1]^2 \to \mathcal{K}\left([0,1]^2\right), \quad i = 1,2,3,4,$$

where

$$\begin{aligned}
\varphi_1(x,y) &:= \left([0.9995, 1.0005]\frac{x_1}{2}, [0.9995, 1.0005]\frac{x_2}{2}\right),\\
\varphi_2(x,y) &:= \left(\frac{1}{2} + \frac{x_1 y_1}{3}, \frac{x_2}{3}\right),\\
\varphi_3(x,y) &:= \left(\frac{x_1}{3}, \frac{1}{2} + \frac{x_2 y_2}{3}\right),\\
\varphi_4(x,y) &:= \left(\frac{1}{2} + \frac{x_1}{3}, \frac{1}{2} + \frac{x_2}{3}\right),
\end{aligned}$$

$x = (x_1, x_2)$, $y = (y_1, y_2)$.

A unique, metric, multivalued, coupled fractal, guaranteed by Theorem 4.4, is plotted in Figure 2.

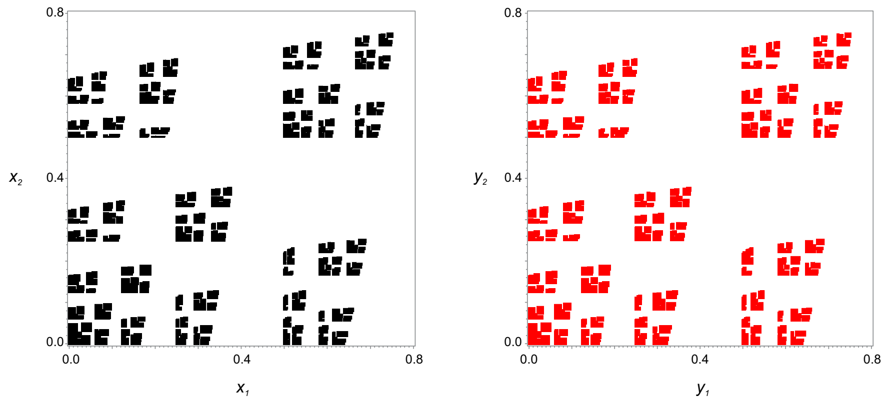


FIGURE 2. Approximation of the coupled fractal from Example 2.

**Remark 4.6** If the maps $\varphi_i$, $i = 1, \ldots, n$, in Theorem 4.1 are still contractions, then the only AR–space $X$ need not be complete, when comparing the result with Theorem 4.4. Under the same additional assumption, Theorem 4.3 does not bring any new information, because the notion of attractivity is stronger than the notion of nonejectivity. For some further results concerning the metric "single–valued" coupled fractals, see e.g. [28, 31].

## 5. Fixed point sets of contractions

In this section, we will briefly characterize the fixed point sets of contractions in terms of absolute retracts.

In 1994, Saint Raymond proved the following theorem.

**Theorem 5.1** (cf. [35, 36]) *If $X$ is a complete space and $\varphi : X \to \mathcal{K}(X)$ is a contraction, then the fixed point set* $\mathrm{Fix}(\varphi)$ *is nonempty and compact.*

In 1987, Ricceri improved Saint Raymond's Theorem 5.1 for Banach spaces as follows.

**Theorem 5.2** (cf. [32]) *If $E$ is a Banach space and $\varphi : E \to \mathcal{CB}(E)$ is a contraction with convex values, then* $\mathrm{Fix}(\varphi)$ *is an absolute retract. If, additionally, $\varphi : E \to \mathcal{K}(E)$ is a contraction with convex values, then* $\mathrm{Fix}(\varphi)$ *is a compact absolute retract.*

In order to generalize Theorem 5.2, we need the notion of Michael's family of sets.

**Definition 5.3** (cf. [3, 16]) Let $M(X)$ be a family of closed subsets of $X$ satisfying the following conditions:

(i) $X \in M(X)$ and $\{x\} \in M(X)$, for every $x \in X$;
(ii) for any subclass $\{A_i\}_{i\in J} \subset M(X)$, we have $\left(\bigcap_{i\in J} A_i\right) \in M(X)$;
(iii) for every $k = 1, 2, 3, \ldots$, and every $x_1, \ldots, x_k \in X$, the set $A(x_1, \ldots, x_k) := \bigcap\{A; A \in M(X) \text{ and } x_1, \ldots, x_k \in A\}$ is infinitely connected;
(iv) for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $A \in M(X)$ and for every $x_1, \ldots, x_k \in \mathcal{O}_\delta(A)$, we have $A(x_1, \ldots, x_k) \subset \mathcal{O}_\varepsilon(A)$;

(v) the closure $\overline{A \cap B(x,r)}$ of $A \cap B(x,r)$ belongs to $M(X)$, for every $x \in X$ and $r > 0$.

Then $M(X)$ is called the *Michael family of subsets* of $X$.

There are several natural examples of Michael's families.

**Example 3** (convex sets) Let $X$ be a convex subset of a normed space $E$, and let

$$M(X) := \{A; A = \emptyset \text{ or } A \subset X \text{ is convex and closed}\}.$$

Then $M(X)$ is the Michael family of sets.

**Example 4** (simplicially convex sets, cf. [5]) Let $(X,d)$ be a metric space and

$$M(X) := \{A; A = \emptyset \text{ or } A \text{ is a closed and simplicially convex subset of } X\}.$$

Then $M(X)$ is the Michael family of subsets of $X$.

**Example 5** ($\alpha$-convex sets, cf. [9]) Let $(X,d)$ be a metric space and let

$$M(X) := \{A; A = \emptyset \text{ or } A \text{ is a closed and } \alpha\text{-convex subset of } X\}.$$

Then $M(X)$ is the Michael family of sets.

Note that the concept of Michael's family is strictly related to the existence of continuous (single–valued) selections of certain suitable multivalued mappings.

Namely, we have the following theorem, which we state here in the form of proposition.

**Proposition 5.4** (cf. [2, 14, 16]) *Let $\varphi : X \to 2^Y$ be a multivalued map of metric spaces, where the symbol $2^Y$ denotes a power set of $Y$, i.e. the set of all subsets of $Y$. Assume that $\varphi$ is l.s.c. with nonempty values such that $\varphi(x) \in M(Y)$, for every $x \in X$. Then there exists a continuous (single–valued) map $f : X \to Y$ such that $f(x) \in \varphi(x)$, for every $x \in X$.*

We shall write that $\varphi \in SP(X)$ if $\varphi$ satisfies the assumptions of Proposition 5.4 for $Y = X$, where $X$ is a complete space.

Following [15, 16], we can formulate the following generalization of the first part of Ricceri's Theorem 5.2.

**Theorem 5.5** *If $X$ is a complete absolute retract and $\varphi : X \to 2^X$ is a contraction such that $\varphi \in SP(X)$, then the set $\mathrm{Fix}(\varphi)$ is a complete absolute retract.*

**Remark 5.6** For some further results in this field, see e.g. [7, 33] The structure of fixed point sets was also considered for contractions in terms of topological (covering) dimensions in [10] and [3, Chapter II.3].

## 6. Continuation method

In this section, we would like to generalize the Granas continuation method for single–valued contractions in [19] into the multivalued setting. For some further papers related to continuation methods, based on the notion of essentiality, see e.g. [2, 9, 11, 12, 13, 17, 18, 23].

Let $U$ be a domain contained in a complete metric space $E$. As usually, by $\overline{U}$ we shall denote the closure of $U$ in $X$ and by $\partial U$ the boundary of $U$ in $X$.

We let

$$\mathscr{K}(U) := \left\{\varphi : \overline{U} \to \mathcal{K}(X); \varphi \text{ is a contraction}\right\}.$$

Let us assume that $\chi : [0,1] \times \overline{U} \to \mathcal{K}(X)$ is a u.s.c. mapping. For every $t \in [0,1]$, we define the map $\chi_t : \overline{U} \to \mathcal{K}(X)$ by the formula

$$\chi_t(x) = \chi(t,x), \quad \text{for every } x \in \overline{U}.$$

We put

$$\mathrm{Fix}(\chi) := \bigcup_{t\in[0,1]} \mathrm{Fix}(\chi_t) = \left\{x \in \overline{U}, \exists t \in [0,1]; x \in \chi(t,x)\right\}.$$

In what follows, we shall assume that our map $\chi : [0,1] \times \overline{U} \to \mathcal{K}(X)$ satisfies additionally the following condition:

$$\exists M > 0 \ \forall t_1, t_2 \in [0,1] \ \forall x \in \overline{U} : d_H(\chi(t_1,x), \chi(t_2,x)) \le M(t_1 - t_2). \tag{6.1}$$

We let

$$\mathscr{K}_0(U) := \left\{\varphi : \overline{U} \to \mathcal{K}(X); \mathrm{Fix}(\varphi) \cap \partial U = \emptyset\right\}.$$

**Definition 6.1** The above map $\chi : [0,1] \times \overline{U} \to \mathcal{K}(X)$ is called a *homotopy* in $\mathscr{K}_0(U)$ if the following conditions are satisfied:

- $\mathrm{Fix}(\chi) \cap \partial U = \emptyset$,
- there exists $t \in [0,1]$ such that $\chi_t$ is an $\alpha$-contraction (i.e. a contraction with the Lipschitz constant $\alpha \in [0,1)$), for every $t \in [0,1]$.

The two maps $\varphi, \psi \in \mathscr{K}_0(U)$ are called *homotopic* ($\varphi \sim \psi$) if there exists a homotopy $\chi$ in $\mathscr{K}_0(U)$ such that $\chi(0,x) = \varphi(x)$ and $\chi(1,x) = \psi(x)$, for every $x \in \overline{U}$.

**Definition 6.2** A map $\varphi \in \mathscr{K}_0(U)$ is called *essential* if $\mathrm{Fix}(\varphi) \neq \emptyset$.

Now, we are ready to formulate the following theorem.

**Theorem 6.3** (transversality property) *If $\chi : [0,1] \times \overline{U} \to \mathcal{K}(X)$ is a homotopy in $\mathscr{K}_0(U)$ and $\chi_0$ is essential, then $\chi_t$ is essential, for every $t \in [0,1]$.*
*Proof.* For the proof, we consider the following set

$$T := \{t \in [0,1]; \mathrm{Fix}(\chi_t) \neq \emptyset\}.$$

By the hypothesis, $T \neq \emptyset$.

(i) Firstly, we prove that $T$ is a closed subset of $[0,1]$. Let $\{t_n\}$ be a sequence in $T$ such that $\lim_{n\to\infty} t_n = t_0$. We have to prove that $t_0 \in T$; i.e. $\mathrm{Fix}(\chi_{t_0}) \neq \emptyset$.

For every $n = 1, 2, \dots$, we choose a point $x_n \in \mathrm{Fix}(\chi_{t_n})$.
We have

$$\begin{aligned} d(x_n, x_m) &\leq d_H(\chi(t_n, x_n), \chi(t_m, x_m)) \\ &\leq d_H(\chi(t_n, x_n), \chi(t_m, x_n)) + d_H(\chi(t_m, x_n), \chi(t_m, x_m)). \end{aligned}$$

In view of condition (6.1) and our assumption that $\chi_{t_m}$ is an $\alpha$-contraction, we get

$$d(x_n, x_m) \leq M|t_n - t_m| + \alpha d(x_n, x_m),$$

and so, we obtain

$$d(x_n, x_m) \leq \frac{M}{1-\alpha}|t_n - t_m|.$$

Thus, $\{x_n\}$ is a Cauchy sequence, and therefore $\lim_{n\to\infty} x_n = x_0$, where $x_0 \in \overline{U}$.
We have $(t_n, x_n, x_n) \in \Gamma_\chi = \{(t, x, y); y \in \chi(t, x)\}$. Since the graph $\Gamma_\chi$ of $\chi$ is a closed subset of $[0, 1] \times \overline{U} \times X$, we get $(t_0, x_0, x_0) \in \Gamma_\chi$, and $x_0 \in \chi(t_0, x_0)$ implies that $T$ is closed.

(ii) Now, we show that $T$ is open. Let $t_0 \in T$. Then $\mathrm{Fix}(\chi_{t_0})$ is a nonempty and compact set such that $\mathrm{Fix}(\chi_{t_0}) \cap \partial U = \emptyset$. We choose a point $x_0 \in \mathrm{Fix}(\chi_{t_0})$ and a real number $r$ such that $0 < r < \mathrm{dist}(x_0, \partial U)$.
Fix $\varepsilon > 0$ such that $\varepsilon < \frac{(1-\alpha)r}{M}$, where $M$ satisfies condition (6.1).
Let $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.
In view of the inequalities

$$\begin{aligned} d_H(\chi(t, \{x_0\}) &\leq d_H(\chi(t, x), \chi(t, x_0)) + d_H(\chi(t, x_0), \chi(t_0, x_0)) \\ &\leq \alpha d(x, x_0) + (1-\alpha)r, \end{aligned}$$

we can conclude that if $d(x, x_0) \leq r$, then $d_H(\chi(t, x), \{x_0\}) \leq r$. It implies that $\chi_t$ maps the closed ball $\mathrm{cl}\, B(x_0, r) \subset U$ into itself. Therefore, our claim follows from Corollary 3.7, which completes the proof.

As a consequence of Theorem 6.3, we can formulate the following nonlinear alternative.

**Theorem 6.4** (nonlinear alternative) *Let $U$ be a bounded domain in a Banach space $E$ such that $0 \in U$.*

*If $\varphi \in \mathscr{K}(U)$, then at least one of the following possibilities occurs:*

(1) $\mathrm{Fix}(\varphi) \neq \emptyset$,
(2) *there exists $x_0 \in \partial U$ and $t \in (0, 1)$ such that $x_0 \in \lambda\varphi(x_0)$.*

*Proof.* For the proof, consider the homotopy $\chi : [0, 1] \times \overline{U} \to \mathcal{K}(E)$ defined by the formula

$$\chi(t, x) = t\varphi(x).$$

Assume that $\chi$ is a homotopy in $\mathscr{K}_0(U)$. Since $\chi_0$ is essential, we infer from Theorem 6.3 that $\chi_1 = \varphi$ is essential, and so $\mathrm{Fix}(\varphi) \neq \emptyset$. If $\chi$ is not a homotopy in $\mathscr{K}_0(U)$, then there exists a fixed point $x \in \mathrm{Fix}(\chi_t) \cap \partial U$, for some $t \in (0, 1)$, and the proof is complete.

As an immediate consequence of Theorem 6.4, we obtain the following corollary.

**Corollary 6.5** *Let $U$ be the same as in Theorem 6.4 and $\varphi \in \mathscr{K}_0(U)$. Then:*

(i) *if $\mu x \notin \varphi(x)$, for all $x \in \partial U$ and $\mu > 1$, then $\varphi$ is essential;*

(ii) *if there is a point $x \in E$, $x \neq 0$, such that $x \notin (\varphi(y) + \mu x)$, for all $y \in \partial U$ and $\mu > 0$, then $\varphi$ is inessential.*

## 7. Concluding remarks

Our main ambition was to actualize the earlier results in [2] rather than to prepare a complete survey in the current metric fixed point theory for multivalued contractions.

Besides other things, we applied in the proof of Theorem 4.1 the Granas version of the Lefschetz fixed point theorem and, especially, we followed in Section 6 his seminal idea to develop a continuation principle for (multivalued) contractions.

In our next paper, we would like to randomize and fuzzify the deterministic results, jointly with some applications.

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