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ON BISHOP-PHELPS PARTIAL ORDER, VARIATION MAPPINGS AND CARISTI'S FIXED POINT THEOREM IN QUASI-METRIC SPACES

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Abstract. In this paper we continue the study of those conditions that guarantee the existence of fixed points for variation mapping in the spirit of M.R. Tasković. Concretely, we provide a general fixed point result for variation mappings defined in left-K-sequentially complete T_1 quasi-metric spaces in such a way that only lower semicontinuity from above is required instead of lower semicontinuity. We give examples that elucidate that the assumptions in the statement of our main result cannot be weakened. Moreover, it is shown that the CS-convergence condition by Tasković implies left K-sequentially completeness and, thus, we retrieve the fixed point result for variation mappings in T_1 quasi-metric spaces due to Tasković. Furthermore, some fixed point theorems, among other Caristi type fixed point results, for variation mappings are derived as a particular case of our main result when several different quasi-metric notions of completeness are considered. Finally, we provide a characterization of left K-sequentially completeness for T_1 quasi-metric spaces via variation mappings.
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1. Introduction

In 1976, J. Caristi proved an extension of the Banach fixed point theorem in metric spaces that allows to obtain fixed points of mapping which are not continuous in general. The aforementioned result can be stated as follows (see [2]).

Theorem 1.1. *Let (X,d) be a complete metric space and let $G : X \to \mathbb{R}_+$ be a $\tau(d)$-lower semicontinuous function. If*

$$d(x, f(x)) \leq G(x) - G(f(x)) \tag{1.1}$$

for all $x \in X$, then f has a fixed point.

Nowadays, those mappings f satisfying condition (1.1) are known as Caristi mappings.

More recently, in 2001, W.A. Kirk and L.M. Saliga obtained a more general version of Theorem 1.1 ([9]). To this end, they used, on the one hand, the so-called Bishop-Phelps partial order and, on the other hand, a kind of metric completeness for topological spaces and a weker notion of semicontinuity. Let us recall that given a metric space (X, d) and a function $G : X \to \mathbb{R}_+$ then a Bishop-Phelps partial order $\leq_{d,G}$ can be defined as follows ([1]):

$$x \leq_{d,G} y \Leftrightarrow d(x, y) \leq G(x) - G(y).$$

Moreover, following [9], a Hausdorff topological space (X, τ) endowed with a metric d is τ-d-complete if every Cauchy sequence in (X, d) is convergent with respect to τ. Furthermore, according to Definition 1.2 in [3] (see also Definition 2.1 in [10]), a function $f : X \to \mathbb{R}_+$ is τ-lower semincontinuous from above if and only if the following assertion holds: whenever a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ with respecto to τ and, in addition, the sequence $(f(x_n))_{n\in\mathbb{N}}$ is decreasing then $f(x) \leq \lim_{n\to+\infty} f(x_n)$. Notice that every lower semincontinuous function is lower semicontinous from above. Nevertheless, one can found examples that show that the contrary is not true (see, for instance, Example 1.3 in page 536 of in [3]).

In the light of the preceding notions the above announced result states the following.

Theorem 1.2. *Let (X, τ) be a Hausdorff topological space (X, τ) endowed with a metric d which is τ-d-complete and the function $d(x, \cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous. If $G : X \to \mathbb{R}_+$ is a τ-lower semicontinuous from above function such that*

$$d(x, f(x)) \leq G(x) - G(f(x))$$

for all $x \in X$, then f has a fixed point.

In 1991, M.R. Tasković proved an extension of Theorem 1.1 which also relies on the Bishop-Phelps partial order ([19]) and has some bearing on Theorem 1.2. However, on this occasion, Tasković introduced a general version of Caristi mappings and a new notion of completeness for topological spaces. The new kind of Caristi mappings were called variation mappings.

In order to state the Tasković fixed point result we need to recall a few pertinent notions about quasi-metric spaces.

In our context, by a quasi-metric space we mean a pair (X, d) such that X is a nonempty set and d is a function $d : X \to \mathbb{R}^+$ satisfying the following conditions for all $x, y, z \in X$ (where $\mathbb{R}^+$ denotes the set of nonnegative real numbers):

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$.

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

Moreover, given a quasi-metric space (X, d), the pair (X, d^{-1}) is also a quasi-metric space where the quasi-metric d^{-1} on X is defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$. The quasi-metric space (X, d^{-1}) is called the conjugate quasi-metric space of (X, d) and the d^{-1} is said to be the conjugate quasi-metric of d. Furthermore, every quasi-metric d induces a metric d^s on X defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ for all $x, y \in X$.

It is well known that, given a quasi-metric space (X, d), a topology $\mathcal{T}(d)$ can be induced on X which has as a base the family of open d-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

Am illustrative example of quasi-metric space is given by the pair $(\mathbb{R}_+, d_l)$, where the quasi-metric d_l is defined on $\mathbb{R}_+$ by

$$d_l(x, y) = \max\{x - y, 0\}$$

for all $x, y \in \mathbb{R}_+$.

For a deeper treatment of quasi-metric spaces we refer the reader to [4] and [13].

Following the terminology of Tasković, given topological space (X, τ) endowed with a quasi-metric d, a mapping $f : X \to X$ is said to be a d-G-variation (locally variation mapping in [19]) provided the existence of a τ-lower semincotinuous function $G : X \to \mathbb{R}_+$ such that for any $x \in X$ with $x \neq f(x)$ there exists $y \in X$ with $y \neq x$ which holds

$$d(x, y) \leq G(x) - G(y). \tag{1.2}$$

Of course, it is easy to check that every Caristi mapping is a variation one. The aforementioned new notion of completeness was called CS-convergence condition. Besides, according to [19], a topological space (X, τ) endowed with a quasi-metric d satisfies the τ-CS-convergence condition provided that every sequence $(x_n)_{n\in\mathbb{N}}$ admits a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges with respecto to τ provided that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Taking into account the exposed notions, Tasković proved the next fixed point result (Theorem 4 in [19]) which generalizes Theorems 1.1 and 1.2.

Theorem 1.3. *Let (X, τ) be a topological space and let d be a T_1-quasi-metric on X such that (X, τ) satisfies that τ-d-CS-convergence condition and $d(x, \cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous. If $G : X \to \mathbb{R}_+$ is a τ-lower semincontinuous function, then every d-G-variation has a fixed point.*

Since both generalizations of Theorem 1.1 share common characteristics it seems natural to wonder whether there exists the possibility of obtaining general result that generalizes Theorem 1.1 and 1.2. In the present paper we focus our attention on exploiting the properties of quasi-metric spaces (not explored by Taskosvić in [19]) in order to answer the posed question. Concretely, we show that such a question has a positive answer and the framework in which one must work with the aim of providing the desired generalization is exactly the left K-sequentially complete quasi-metric spaces.

The remainder of the paper is organized as follows:

Section 2 is devoted to present a property of Bishop-Phelps partial order which will be crucial for our target. Concretely we provide sufficient conditions under which the Bishop-Phelps partial order is in some sense order-complete, namely, every increasing sequence is bounded above by a maximal element. Besides we show that many known notions of completeness in quasi-metric spaces yield the mentioned appropriate conditions for the order-completeness. In Section 3 we take advantage of the above-mentioned order property for providing the desired general fixed point theorem for variation mappings in quasi-metric spaces in such a way that Theorem 1.1 and 1.2

and new Caristi type fixed point theorems are retrieved as a particular case of our new result. Moreover, a few fixed point results are derived from our main result when the variety of completeness for quasi-metric spaces deemed in Section 2 are retaken. Furthermore, we give examples that elucidate that the assumptions in the statement of our main result cannot be weakened.

2. On the completeness of Bishop-Phelps partial order on quasi-metric spaces

In the following we explore the Bishop-Phelps partial order in quasi-metric spaces and provide sufficient conditions in order to guarantee that every increasing sequence is bounded above by a maximal element. With this aim, let us recall several pertinent notions about partially ordered sets (for a fuller treatment of partially ordered sets see, for instance, [7]).

A partially ordered set is pair $(X, \leq)$ such X is a nonempty set X and $\leq$ is a reflexive, antisymmetric and transitive binary relation on X. Moreover, if $(X, \leq)$ is a partially ordered set and $Y \subseteq X$, then an upper bound for Y in $(X, \leq)$ is an element $x \in X$ such that $y \leq x$ for all $y \in Y$. An element $z \in X$ is called maximal provided that if there exits $x \in X$ such that $z \leq x$, then $x = z$. A sequence $(x_n)_{n\in\mathbb{N}}$ is said to be increasing whenever $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of nonnegative integer numbers.).

If (X, d) is a quasi-metric space and $G : X \to \mathbb{R}_+$ is a function, then the pair $(X, \leq_{d,G})$ can be shown to be a partial ordered set following the same reasoning as in the classical metric case (see, for instance, [1] for a detailed treatment in the metric case), where the partial order $\leq_{d,G}$ is defined by

$$x \leq_{d,G} y \Leftrightarrow d(x, y) \leq G(x) - G(y).$$

It must be pointed out that the Bishop-Phelps partial order on quasi-metric spaces has already been considered in [17].

The notion of left K-sequentially completeness for quasi-metric space will allow us to provide the aforesaid sufficient conditions for the order-completeness and, in addition, will be extremely useful in our subsequent discussion in Section 3. Moreover, the quasi-metric spaces holding such completeness will provide an appropriate framework for unifying the completeness notions in the statements of Theorem 1.1 and 1.2.

Let us recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, d) is said to be left K-Cauchy provided that, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n \geq n_0$ (see [4] and [15]).

Motivated by the completeness notion of Kirk and Saliga and the Tasković CS-convergence we will say, in the sequel, that a topological space (X, τ) endowed with a quasi-metric d is τ-d-left K sequentially complete provided that every left K-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) is convergent with respect to τ.

The next example provides an instance of τ-d-left K sequentially complete quasi-metric space. It is also shown that the selection of the topology τ is crucial in order to guarantee the τ-d-left K sequentially completeness of a quasi-metric space.

Example 2.1. Consider the T_1 quasi-metric space $([0,1], d_S)$, where

$$d_S(x,y) = \begin{cases} x-y, & x \geq y \\ \\ 1, & x < y. \end{cases}$$

It is clear that every left K-Cauchy sequence in decreasing with respect to the usual order $\leq$ on $\mathbb{R}_+$. It follows that every left K-Cauchy sequence converges to 0 with respect to the Euclidean topology τ_E on $\mathbb{R}_+$. Thus $([0,1], d_S)$ is τ_E-d_S-left K-sequentially complete. Finally, consider the discrete topology τ_D. Then it is clear that $([0,1], d_S)$ is not τ_D-d_S-left K-sequentially complete, since the sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, is a left K-Cauchy sequence and, however, it is not convergent with respect to τ_D.

We can now formulate the result which provides the announced order-completeness of the Bishop-Phelps partial order. To this end, let us recall that, given a topological space (X, τ), a function $f : X \to \mathbb{R}_+$ is τ-lower semincontinuous if and only if f is continuous from (X, τ) into $(\mathbb{R}_+, \tau(d_l))$.

Theorem 2.2. *Let (X,τ) be a topological space and let d be a T_1-quasi-metric on X such that (X,τ) is τ-d-left K-sequentially complete and $d(x,\cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous for every $x \in X$. If $G : X \to \mathbb{R}_+$ is a τ-lower semincontinuous from above function, then every increasing sequence in $(X, \leq_{d,G})$ has an upper bound which is a maximal element.*

Proof. Consider an increasing sequence $(a_n)_{n\in\mathbb{N}}$ in $(X, \leq_G)$. It is clear that we can construct inductively an increasing sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_0 = a_0$ and $x_{n+1} \in \uparrow_{\leq_G} a_n$ and $G(x_{n+1}) < \frac{1}{n+1} + \inf_{x\in\uparrow_{\leq_G} a_n} G(x)$. Next we will show that $(G(x_n))_{n\in\mathbb{N}}$ is convergent with respect to $\tau(|\cdot|)$ and, in addition, that $(x_n)_{n\in\mathbb{N}}$ is a left K-Cauchy sequence.

It is clear that

$$d(x_n, x_{n+1}) \leq G(x_n) - G(x_{n+1})$$

for all $n \in \mathbb{N}$. Whence we deduce that the sequence $(G(x_n))_{n\in\mathbb{N}}$ is decreasing and bounded below by 0. Then the sequence $(G(x_n))_{n\in\mathbb{N}}$ is convergent with respect to $\tau(|\cdot|)$ and it converges to $\alpha \in \mathbb{R}_+$ with $\alpha = \inf_{n\in\mathbb{N}}\{G(x_n)\}$. It follows that given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|G(x_n) - \alpha| < \varepsilon$ for all $n \geq n_0$. Whence we obtain that $G(x_n) - G(x_m) < \varepsilon$ for all $m \geq n \geq n_0$. Indeed, $G(x_n) - G(x_m) \leq |G(x_n) - \alpha| + |\alpha - G(x_m)| < \varepsilon$ for all $m \geq n \geq n_0$. Hence we obtain that the sequence $(x_n)_{n\in\mathbb{N}}$ is left K-Cauchy in (X,d), since

$$d(x_n, x_m) \leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n_0+i+1}) \leq G(x_n) - G(x_m) < \varepsilon$$

for all $m \geq n \geq n_0$. The fact that (X,τ) is τ-d-left K-sequentially complete guarantees the existence of $z \in X$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to $z \in X$ with respect to τ. Since the sequence $(G(x_n))_{n\in\mathbb{N}}$ is decreasing and $G : X \to \mathbb{R}_+$ is τ-lower semiconinuous from above we have that $G(z) \leq \alpha$. Moreover, given $\varepsilon > 0$, there exists

$k_0 \in \mathbb{N}$ such that for all $n \geq k_0$ we have that

$$d(x,z) - d(x,x_n) < \frac{\varepsilon}{2}$$

for all $n \geq k_0$ and $x \in X$, since $d(x,\cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous. It remains to prove that z is an upper bound of the sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \leq_G)$. To this end fix $m \in \mathbb{N}$. Hence we deduce the existence of $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} d(x_m,z) &< \frac{\varepsilon}{2} + d(x_m,x_n) \leq \frac{\varepsilon}{2} + G(x_m) - G(x_n) \\ &\leq \frac{\varepsilon}{2} + (G(x_m) - G(z)) + (\alpha - G(x_n)) \\ &\leq \varepsilon + G(x_m) - G(z). \end{aligned}$$

for all $n \geq k_0$ and $n \geq m$. Hence

$$d(x_m,z) \leq G(x_m) - G(z)$$

and, therefore, $x_m \leq_G z$ for all $m \in \mathbb{N}$. Therefore, we conclude that z is an upper bound of the sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \leq_G)$. Since $a_n \leq_G x_n \leq_G z$ for all $n \in \mathbb{N}$ we deduce that z is an upper bound of the sequence $(a_n)_{n\in\mathbb{N}}$.

It remains to prove that z is maximal. To this end, assume that there exists $y \in X$ such that $z \leq_G y$. It follows that $d(z,y) \leq G(z) - G(y)$. Thus $G(y) \leq G(z)$. Moreover,

$$G(z) - \frac{1}{n} \leq G(x_n) - \frac{1}{n} \leq \inf_{x \in \uparrow a_{n-1}} G(x) \leq G(y) \leq G(z)$$

for all $n \in \mathbb{N}$ with $n \geq 1$. Consequently, $G(y) = G(z)$ and, hence, $d(z,y) = 0$. So $z = y$, since the quasi-metric space (X,d) is T_1. Therefore z is maximal in $(X, \leq_G)$. □

Notice that, under the assumptions in the statement of Theorem 2.2, we have that every element in X is majorized (modulo $\leq_{d,G}$)) by some maximal element.

Since every lower semincontinuous function is lower semicontinous from above we obtain from Theorem 2.2 the following result.

Corollary 2.3. *Let (X,τ) be a topological space and let d be a T_1-quasi-metric on X such that (X,τ) is τ-d-left K-sequentially complete and $d(x,\cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous for every $x \in X$. If $G : X \to \mathbb{R}_+$ is a τ-lower semincontinuous function, then every increasing sequence in $(X, \leq_{d,G})$ has an upper bound which is a maximal element.*

It seems natural to consider in the statement of Theorem 2.2 the topology τ as the topology $\tau(d)$. However, the function $d(x,\cdot) : X \to \mathbb{R}_+$ is not in general $\tau(d)$-lower semicontinuous for every $x \in X$. In fact, according to Theorem 5 in [15], the function $d(x,\cdot) : X \to \mathbb{R}_+$ is only $\tau(d)$-upper semicontinuous for every $x \in X$ (i.e. $d(x,\cdot) : X \to \mathbb{R}_+$ is continuous from $(X,\tau(d))$ into $(\mathbb{R}_+, \tau(d_l^{-1}))$, where d_l^{-1} is the conjugate quasi-metric of d_l, that is, $d_l^{-1}(x,y) = \max\{y - x, 0\}$ for all $x, y \in \mathbb{R}_+$). In contrast, it is easy to check that the function $d(x,\cdot) : X \to \mathbb{R}_+$ is always $\tau(d^{-1})$-lower semicontinuous for every $x \in X$. So it makes sense to consider the topology $\tau(d^{-1})$ as the topology τ in the statement of Theorem 2.2.

Note that the quasi-metric spaces (X, d) that are $\tau(d^{-1})$-left K-sequentially complete (backward complete in [6]) are exactly those whose conjugate quasi-metric space (X, d^{-1}) are right K-sequentially complete. Recall that a quasi-metric space (X, d) is right K-sequentially complete whenever every right K-Cauchy sequence is convergent with respect to $\tau(d)$, where a sequence $(x_n)_{n\in\mathbb{N}}$ is called right K-Cauchy provided that, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n \geq m \geq n_0$ (see [4] and [15]).

Taking into account the preceding remark we derive from Theorem 2.2 the next result.

Corollary 2.4. *Let (X, d) be a T_1 quasi-metric space such that the quasi-metric space (X, d^{-1}) is right K-sequentially complete and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. Then every increasing sequence in $(X, \leq_{d,G})$ has an upper bound which is a maximal element.*

On account of [15] (see also [4]), a sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, d) is called weakly right K-Cauchy provided that, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n_0}) < \varepsilon$ for all $n \geq n_0$. In addition, a quasi-metric space (X, d) is said to be weakly right K-sequentially complete provided that every right K-Cauchy sequence is convergent with respecto to $\tau(d)$.

Since every weakly right K-sequentially complete quasi-metric space is right K-sequentially complete Corollary 2.4 yields the following result.

Corollary 2.5. *Let (X, d) be a T_1 quasi-metric space such that the quasi-metric space (X, d^{-1}) is weakly right K-sequentially complete and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. Then every increasing sequence in $(X, \leq_{d,G})$ has an upper bound which is a maximal element.*

The result below presents the relationship between the right K sequentially complete quasi-metric spaces and those that satisfy the CS-convergence condition.

Proposition 2.6. *Let (X, d) be a quasi-metric space. If (X, d) satisfies the $\tau(d^{-1})$-CS-convergence condition, then (X, d^{-1}) is right K-sequentially complete.*

Proof. We will show that (X, d) is $\tau(d^{-1})$-left K-sequentially complete. Indeed, consider a left K-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d). Then, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

for all $m \geq n \geq n_0$. Whence we obtain that

$$d(x_n, x_{n+1}) < \varepsilon$$

for all $n \geq n_0$. Thus $(x_n)_{n\in\mathbb{N}}$ admits a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges with respecto to $\tau(d^{-1})$. It follows that $(x_n)_{n\in\mathbb{N}}$ is convergent with respect tot $\tau(d^{-1})$. So (X, d^{-1}) is right K-sequentially complete. □

It seems natural to question if the converse of Proposition 2.6 holds. Nevertheless, the next example gives a negative answer to the posed question.

Example 2.7. Consider the T_1 quasi-metric space $(\mathbb{R}_+, d_2)$, where the quasi-metric d_2 is given by

$$d_2(x,y) = \begin{cases} y - x, & x \leq y \\ 2(x-y), & x > y. \end{cases}$$

It is not hard to check that $(\mathbb{R}_+, d_2^{-1})$ is right K-sequentially complete. Now, consider the sequence $(x_n)_{n\in\mathbb{N}}$ with

$$x_n = \begin{cases} 1, & n = 1 \\ x_{n-1} + \frac{1}{n}, & n > 1. \end{cases}$$

It is clear that $\lim_{n\to+\infty} d_2^{-1}(x_n, x_{n+1}) = \lim_{n\to+\infty} \frac{2}{n+1} = 0$ but $(x_n)_{n\in\mathbb{N}}$ is not convergent with respec to $\tau(d_2^{-1})$.

From Proposition 2.6 and Corollary 2.4 we deduce the next result.

Corollary 2.8. *Let (X,d) be a T_1 quasi-metric space which satisfies the $\tau(d^{-1})$-CS-convergence and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. Then every increasing sequence in $(X, \leq_{d,G})$ has an upper bound which is a maximal element.*

Following [18], a quasi-metric space (X,d) is called Smyth complete (left Smyth sequentially complete in [4]) whenever every left K-Cauchy sequence is convergent with respect to $\tau(d^s)$.

Corollary 2.4 and the fact that every Smyth complete quasi-metric space (X,d) satisfies that its conjugate (X, d^{-1}) is right K-sequentially complete.

Corollary 2.9. *Let (X,d) be a Smyth complete T_1 quasi-metric space and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. Then every increasing sequence in $(X, \leq_{d,G})$ has an upper bound which is a maximal element.*

On account of [14], a quasi-metric space (X,d) is said to be weightable provided the existence of a function $w : X \to \mathbb{R}^+$ such that

$$d(x,y) + w(x) = d(y,x) + w(y)$$

for all $x, y \in X$. Moreover, a quasi-metric space (X,d) is called bicomplete whenever the metric space (X, d^s) is complete. Since every weightable bicomplete quasi-metric space is Smyth complete (see [12]), the following result can deduced from Corollary 2.9.

Corollary 2.10. *Let (X,d) be a weightable bicomplete T_1 quasi-metric space and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. Then every increasing sequence in $(X, \leq_{d,G})$ has an upper bound which is a maximal element.*

Let us recall that a quasi-metric space (X,d) is sequentially compact (forward sequentially compact in [6]) provided that every sequence admits a subsequence that converges with respect to $\tau(d)$. It is clear that every quasi-metric space (X,d) such that (X, d^{-1}) is sequentially compact (such quasi-metric spaces are called backward sequentially compact in [6]) is $\tau(d^{-1})$-left K-sequentially complete. Furthermore, according to [11] (see also Proposition 1.2.28 and Corollary 1.2.29 in [4]), sequentially compactness and compactness are equivalent in T_1 quasi-metric spaces. Taking into account the aforesaid information we retrieve as a particular case of Theorem 2.2 the result below.

Corollary 2.11. *Let (X, d) be a T_1 quasi-metric space such that (X, d^{-1}) is sequentially compact and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. Then every increasing sequence in $(X, \leq_{d,G})$ has an upper bound which is a maximal element.*

3. The fixed point theorems

In this section we prove a general fixed point theorem for variation mappings in quasi-metric spaces which will retrieve as particular cases Theorem 1.2 and 1.3 and, therefore, Theorem 1.1. Besides, a few new Caristi type fixed point theorems will be provided from our new result.

3.1. **A general fixed point theorem for variation mappings.** In order to achieve our main goal we will extend the notion of varaition mapping due to Tasković (see Section 1).

Let (X, τ) be a topological space endowed with a quasi-metric d. A mapping $f : X \to X$ will say to be a d-G-variation mapping provided the existence of a τ-lower semincotinuous function from above $G : X \to \mathbb{R}_+$ such that for any $x \in X$ with $x \neq f(x)$ there exists $y \in X$ with $y \neq x$ which holds

$$d(x, y) \leq G(x) - G(y) \tag{3.1}$$

Observe that the new notion of variation mapping retrieves the Tasković one. However, the converse is not true such as the below example shows.

Example 3.1. Consider the pair $(\mathbb{R}, d_{\frac{1}{2}})$, where $\mathbb{R}$ denotes the set of real numbers and $d_{\frac{1}{2}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ is defined by

$$d_{\frac{1}{2}}(x, y) = \begin{cases} \min\{y - x, \frac{1}{2}\}, & x \leq y \\ \frac{1}{2}, & x > y. \end{cases}$$

It is not hard to check that $(\mathbb{R}, d_{\frac{1}{2}})$ is a T_1 quasi-metric space (see [16]). Define the mapping $G : \mathbb{R} \to \mathbb{R}_+$ by

$$G(x) = \begin{cases} x + \frac{1}{2}, & x < 0 \\ x^2 + 1, & x \geq 0. \end{cases}$$

A straightforward computation shows that G is a $\tau(d_{\frac{1}{2}})$-lower semicontinuous function from above which is not $\tau(d_{\frac{1}{2}})$-lower semicontinuous. Define the mapping $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$ for all $x \in \mathbb{R}$. Then it is clear that given $x \in \mathbb{R}$ such that $f(x) \neq x$ we have that

$$d_{\frac{1}{2}}\left(x, \frac{-1}{2}\right) \leq G(x) - G\left(\frac{-1}{2}\right) = 1 + x^2$$

for all $x \geq 0$. In addition we have that

$$d_{\frac{1}{2}}(x, x-1) \leq G(x) - G(x-1) = \frac{1}{2}$$

for all $x < 0$. Thus, f is a $d_{\frac{1}{2}}$-G-variation.

Here we present our new fixed point result which is inspired in Theorem 4 in [19].

Theorem 3.2. *Let (X, τ) be a topological space and let d be a T_1-quasi-metric on X such that (X, τ) is τ-d-left K-sequentially complete and $d(x, \cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous. Let $G : X \to \mathbb{R}_+$ be a τ-lower semincontinuous function from above. If $f : X \to X$ is a d-G-variation mapping, then for every $a_0 \in X$ f has a fixed point in $\uparrow_{\leq_G} a_0 = \{x \in X : a_0 \leq x\}$.*

Proof. Let $a_0 \in X$ such that $a_0 \neq f(a_0)$. Of course if $a_0 = f(a_0)$ we have the desired conclusion. By the d-G-variation condition there exists $y \in X$ such that $d(a_0, y) \leq G(a_0) - G(y)$. Denote y by a_1. Then we have that $d(a_0, a_1) \leq G(a_0) - G(a_1)$. It is clear that we can construct in this way a sequence $(a_n)_{n \in \mathbb{N}}$ in X such that $a_n \neq f(a_n)$ (note that if there exists $n \in \mathbb{N}$ such that $a_n = f(a_n)$ then we obtain the desired conclusion) and

$$d(a_n, a_{n+1}) \leq G(a_n) - G(a_{n+1})$$

for all $n \in \mathbb{N}$. Whence we deduce that the sequence $(a_n)_{n \in \mathbb{N}}$ is increasing. Theorem 2.2 guarantees the existence of an upper bound z of the sequence $(a_n)_{n \in \mathbb{N}}$ which is a maximal element. Assume that $z \neq f(z)$. Since $f : X \to X$ is a d-G-variation condition there exists $y \in X$ such that $d(z, y) \leq G(z) - G(y)$ and, thus, that $z \leq_G y$. The fact that z is maximal provides that $z = y$. So z is a fixed point of f nd $z \in \uparrow_{\leq_G} a_0$. □

The next example shows that the T_1 separation condition of the quasi-metric can not be relaxed in the statement of Theorem 3.2 in order to assure that the mapping has a fixed point.

Example 3.3. Consider the quasi-metric space $(\mathbb{R}_+, d_l^{-1})$. It is clear that $(\mathbb{R}_+, d_l^{-1})$ Smyth complete, since it is weightable and bicomplete and, hence, $\tau(d_l)$-left K-sequentially complete. Moreover, it is not T_1. Of course, $d_l^{-1}(x, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is $\tau(d_l)$-semicontinuous for every $x \in \mathbb{R}_+$. Define the function $G : \mathbb{R}_+ \to \mathbb{R}_+$ by $G(x) = 0$ for all $x \in \mathbb{R}_+$. It is clear that G is $\tau(d_l)$-lower semicontinuos from above. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \neq 0 \\ \\ 1, & x = 0. \end{cases}$$

Then we have that

$$0 = d_l(x, x+1) \leq G(x) - G(x+1)$$

for all $x \in]0, 1]$. Moreover, we have that

$$0 = d_l\left(0, \frac{1}{2}\right) \leq G(0) - G\left(\frac{1}{2}\right).$$

So f is a d_l^{-1}-G-variation but f has no fixed points.

The next example shows that the τ-left K-sequentially completeness can not be relaxed in the statement of Theorem 3.2.

Example 3.4. Consider the T_1 quasi-metric space $(\mathbb{N}, d_{\mathbb{N}})$ where $d_{\mathbb{N}}$ is defined by

$$d_{\mathbb{N}}(x,y) = \begin{cases} \frac{1}{x} - \frac{1}{y}, & x \leq y \\ \\ 1, & x > y. \end{cases}$$

Clearly the sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n = n$ for all $n \in \mathbb{N}$, is left K-Cauchy in $(\mathbb{N}, d_{\mathbb{N}}^{-1})$ and, however, it is not convergent. So $(\mathbb{N}, d_{\mathbb{N}}^{-1})$ is not right K-sequentially complete. Define the function $G : \mathbb{N} \to \mathbb{N}$ by $G(x) = \frac{1}{x}$ for all $x \in \mathbb{N}$. Then G is $\tau(d_{\mathbb{N}}^{-1})$-lower semicontinuous from above. Consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x+1$ for all $x \in \mathbb{N}$. Then we have that

$$d_{\mathbb{N}}(x, x+1) = G(x) - G(x+1)$$

for all $x \in \mathbb{N}$. It follows that f is a $d_{\mathbb{N}}$-G-variation mapping. Nevertheless, f has no fixed points.

In the next example we show that the lower semincontinuity of the function $d(x,\cdot)$ cannot be omitted in the statement of Theorem 3.2 in order to assure the existence of fixed point.

Example 3.5. Consider the topological space $(\mathbb{R}, \tau(d_l))$ endowed with the quasi-metric d_S introduced in Example 2.1. It is easy to see that $(\mathbb{R}, \tau(d_l))$ is $\tau(d_l)$-d_S-sequentially complete. Define the function $G : \mathbb{R} \to \mathbb{R}_+$ by $G(x) = x$. Clearly G is $\tau(d_l)$-lower semicontinuous. from above. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x - 1$ for all $x \in \mathbb{R}$. Then f is a d_S-G-variation, since

$$1 = d_S(x, x-1) \leq G(x) - G(x-1) = 1$$

for all $x \in \mathbb{R}$. It is obvious that f has no fixed points and that the function $d_S(4,\cdot) : \mathbb{R} \to \mathbb{R}_+$ is not $\tau(d_l)$-lower semicontinuous. Indeed, the sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n = 3$ for all $n \in \mathbb{N}$ converges to 0 with respect to $\tau(d_l)$ but

$$1 = d_S(4,0) - d_S(4,3).$$

Since every lower semincontinuous function is lower semicontinous from above we obtain the following result that generalizes Theorem 4 in [19] (notice that the CS-convergence condition in the sense of the aforementioned paper implies the left K-sequentially completeness).

Corollary 3.6. *Let (X, τ) be a topological space and let d be a T_1-quasi-metric on X such that (X, τ) is τ-d-left K-sequentially complete and $d(x,\cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous. Let $G : X \to \mathbb{R}_+$ be a τ-lower semincontinuous function. If $f : X \to X$ is a d-G-variation mapping, then for every $a_0 \in X$ f has a fixed point in $\uparrow_{\leq_G} a_0$.*

From Theorem 3.2 and Corollary 2.4 we obtain the following result.

Corollary 3.7. *Let (X,d) be a T_1 quasi-metric space such that the quasi-metric space (X,d^{-1}) is right K-sequentially complete and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. If $f : X \to X$ is a d-G-variation mapping, then for every $a_0 \in X$ f has a fixed point in $\uparrow_{\leq_G} a_0$.*

As a consequence of Theorem 3.2 and Corollary 2.5 we obtain the following result.

Corollary 3.8. *Let (X,d) be a T_1 quasi-metric space such that the quasi-metric space (X,d^{-1}) is weakly right K-sequentially complete and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. If $f : X \to X$ is a d-G-variation mapping, then for every $a_0 \in X$ f has a fixed point in $\uparrow_{\leq_G} a_0$.*

As a consequence of Theorem 3.2 and Corollary 2.8 we obtain the following result.

Corollary 3.9. *Let (X,d) be a T_1 quasi-metric space which satisfies the $\tau(d^{-1})$-CS-convergence and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. If $f : X \to X$ is a d-G-variation mapping, then for every $a_0 \in X$ f has a fixed point in $\uparrow_{\leq_G} a_0$.*

As a consequence of Theorem 3.2 and Corollary 2.9 we obtain the following result.

Corollary 3.10. *Let (X,d) be a Smyth complete T_1 quasi-metric space and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. If $f : X \to X$ is a d-G-variation mapping, then for every $a_0 \in X$ f has a fixed point in $\uparrow_{\leq_G} a_0$.*

As a consequence of Theorem 3.2 and Corollary 2.10 we obtain the following result.

Corollary 3.11. *Let (X,d) be a weightable bicomplete T_1 quasi-metric space and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function. If $f : X \to X$ is a d-G-variation mapping, then for every $a_0 \in X$ f has a fixed point in $\uparrow_{\leq_G} a_0$.*

As a consequence of Theorem 3.2 and Corollary 2.11 we obtain the following result.

Corollary 3.12. *Let (X,d) be a T_1 quasi-metric space such that (X,d^{-1}) is sequentially compact and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semincontinuous function from above. If $f : X \to X$ is a d-G-variation mapping, then for every $a_0 \in X$ f has a fixed point in $\uparrow_{\leq_G} a_0$.*

3.2. **An application: Caristi type fixed point theorems.** In this subsection we show that Caristi type fixed point theorems can be derived form the developed theory.

Theorem 3.13. *Let (X,τ) be a topological space and let d be a T_1-quasi-metric on X such that (X,τ) is τ-left K-sequentially complete and $d(x,\cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous. Let $G : X \to \mathbb{R}_+$ be a τ-lower semincontinuous function from above. If a mapping $f : X \to X$ satisfies the condition*

$$d(x, f(x)) \leq G(x) - G(f(x))$$

for all $x \in X$, then for every $x \in X$ f has a fixed point in $\uparrow_{\leq_G} x$.

Proof. It is clear that f is a d-G-variation mapping, since for every $x \in X$ such that $x \neq f(x)$ we have that

$$d(x, f(x)) \leq G(x) - G(f(x)).$$

So the conclusion follows from Theorem 3.2. □

The preceding result and Corollary 3.7 give as a particular case the Caristi fixed point type result given by S. Cobzaş (see assertion 2) in statement of Theorem 2.3 in [5]). Note that the theorem by Cobzaş is given for lower semincontinuous functions. So our result below generalizes the aforementioned theorem.

Corollary 3.14. *Let (X, d) be a T_1 quasi-metric space such that the quasi-metric space (X, d^{-1}) is right K-sequentially complete and let $G : X \to \mathbb{R}_+$ be a $\tau(d^{-1})$-lower semicontinuous function from above. If a mapping $f : X \to X$ satisfies the condition*

$$d(x, f(x)) \leq G(x) - G(f(x))$$

for all $x \in X$, then for every $x \in X$ f has a fixed point in $\uparrow_{\leq_G} x$

Of course when the quasi-metric in statement of Theorem 3.13 is considered exactly as a metric then the aforesaid theorem provides immediately the result for metric spaces given by W. Kirk and L.M. Saliga in Remark 1 in [9]). In order to introduce the aforesaid results we need recall the next concept.

Following [9], a Hausdorff topological space (X, τ) endowed with a metric d is τ-d-complete if every Cauchy sequence in (X, d) is convergent with respect to τ.

Corollary 3.15. *Let (X, τ) be a Hausdorff topological space (X, τ) endowed with a metric d which is τ-d-complete and the function $d(x, \cdot) : X \to \mathbb{R}_+$ is τ-lower semicontinuous. If $G : X \to \mathbb{R}_+$ is a τ-lower semicontinuous from above function such that*

$$d(x, f(x)) \leq G(x) - G(f(x))$$

for all $x \in X$, then for every $x \in X$ f has a fixed point in $\uparrow_{\leq_G} x$.

When in statement of Corollary 3.14 the quasi-metric is considered exactly as a metric, then the aforesaid corollary provides immediately the next result (given as Theorem 2.1 in [9] and Theorem 2.3 in [10]).

Corollary 3.16. *Let (X, d) be a complete metric space and let $G : X \to \mathbb{R}_+$ be a $\tau(d)$-lower semicontinuous from above function. If*

$$d(x, f(x)) \leq G(x) - G(f(x))$$

for all $x \in X$, then for every $x \in X$ f has a fixed point in $\uparrow_{\leq_G} x$.

3.3. **A characterization of left K-sequentially completeness via variation mappings.** In [8], Kirk proved the following characterization of metric completeness.

Theorem 3.17. *Let (X, d) be a space.Then the following statements are equivalent:*

1) *(X, d) is complete.*
2) *If there exists a $\tau(d)$-lower semincontinuous function $G : X \to \mathbb{R}_+$ such that a mapping $f : X \to X$ holds*

$$d(x, f(x)) \leq G(x) - G(f(x))$$

for all $x \in X$, then f has a fixed point.

In this subsection, motivated by the preceding result, we provide a converse of Theorem 3.2 and, thus, we characterize the τ-left K-sequentially completeness in terms of variation mapping. It must be stressed that the proof is inspired by a characterization of Smyth completeness in terms of Caristi's mappings due to S. Romaguera and P. Tirado (see [17]).

Theorem 3.18. *Let (X,τ) be a topological space and let d be a T_1-quasi-metric on X such that $d^s(x,\cdot): X\to\mathbb{R}_+$ is τ-lower semicontinuous. Then the following statements are equivalent:*

1) *(X,τ) is τ-d-left K-sequentially complete.*
2) *If there exists a τ-lower semincontinuous from above function $G: X\to\mathbb{R}_+$ such that a mapping $f: X\to X$ is a d-G-variation, then for every $x\in X$ f has a fixed point in $\uparrow_{\leq_G} x$.*
3) *If there exists a τ-lower semincontinuous from above function $G: X\to\mathbb{R}_+$ such that a mapping $f: X\to X$ holds*

$$d(x,f(x))\leq G(x)-G(f(x))$$

for all $x\in X$, then for every $x\in X$ f has a fixed point in $\uparrow_{\leq_G} x$.

Proof. By Theorem 3.2 we have that 1) $\to$ 2). Theorem 3.13 guarantees that 2) $\to$ 3). Next we show that 3) $\to$ 1). To obtain a contradiction assume that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ which is left K-Cauchy and, in addition, it does not converge with respect to τ. Then there exists a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which does admit any convergent subsequence. Since $(x_n)_{n\in\mathbb{N}}$ is left K-Cauchy, then $(y_n)_{n\in\mathbb{N}}$ is left K-Cauchy. It follows that for each $k\in\mathbb{N}$ there exists $n_k\in\mathbb{N}$ such that $d(y_{n_k},y_n)<2^{-(k+1)}$ for all $n\geq n_k\geq k$. Thus $d(y_{n_k},y_{n_{k+1}})<2^{-(k+1)}$ for all $k\in\mathbb{N}$. Next call $z_k=y_{n_k}$ for all $k\in\mathbb{N}$. Then we have that there exists $k_0\in\mathbb{N}$ such that $z_k\neq z_m$ for all $k,m\geq k_0$ because otherwise the subsequence $(z_k)_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ will be convergent with respecto to τ.

Define the function $G: X\to\mathbb{R}_+$ as follows:

$$G(x)=\begin{cases} 2^{-k}, & x=z_k\\ \\ d^s(x,z_1)+\frac{1}{2}, & x\notin\{z_k : k\in\mathbb{N}\}.\end{cases}$$

A straightforward computation shows that G is τ-lower semicontinuous from above (in fact G is τ-lower semicontinuous). Consider the mapping $f: X\to X$ defined by

$$f(x)=\begin{cases} z_{k+1}, & x=z_k\\ \\ z_1, & x\notin\{z_k : k\in\mathbb{N}\}.\end{cases}$$

Hence we have that

$$d(z_k,f(z_k))=d(z_k,z_{k+1})<2^{-(k+1)})=G(z_k)-G(z_{k+1})$$

for all $k\in N$, and

$$d(x,f(x))=d(x,y_1)\leq d^s(x,y_1)=G(x)-G(f(x))$$

for all $x \notin \{z_k : k \in \mathbb{N}\}$. Then, by hypothesis, f must have a fixed point in X. Nevertheless, f has not fixed points. So the left K-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is convergent with respect to τ and, thus, (X,τ) is τ-d-left K sequentially complete. □

Example 3.5 shows that the τ-lower semicontinuity of the function $d^s(x,\cdot) : X \to \mathbb{R}_+$ cannot deleted in the statement of Theorem 3.18.
When the topology τ in the statement of Theorem 3.18 is exactly $\tau(d^s)$ we immediately obtain that the function $d^s(x,\cdot) : x \to \mathbb{R}_+$ is $\tau(d^s)$-lower semicontinuous and, hence, Theorem 3.18 retrieves the following characterization in the spirit of [17].

Corollary 3.19. *Let (X,d) be a T_1-quasi-metric space.Then the following statements are equivalent:*

1) *(X,d) is Smyth complete.*
2) *If there exists a $\tau(d^s)$-lower semincontinuous from above function $G : X \to \mathbb{R}_+$ such that a mapping $f : X \to X$ is a d-G-variation, then for every $x \in X$ f has a fixed point in $\uparrow_{\leq_G} x$.*
3) *If there exists a $\tau(d^s)$-lower semincontinuous from above function $G : X \to \mathbb{R}_+$ such that a mapping $f : X \to X$ holds*

$$d(x, f(x)) \leq G(x) - G(f(x))$$

for all $x \in X$, then for every $x \in X$ f has a fixed point in $\uparrow_{\leq_G} x$.

4. Further work

We end the paper with a proposal to continue this research line. In [17], Romaguera and Tirado have proved a fixed point theorem for Caristi's mapping in quasi-metric spaces (X,d) that, unlike Theorem 2.3 by S. Cobzaş ([5]), the T_1 separation condition for d is not assumed (see Theorem 4 in [17]). This goal is accomplished considering a different notion of completeness in which every left K-Cauchy net is convergent with respect to $\tau(d)$. In addition, in the aforesaid reference, they provided a converse of Theorem 4, in the spirit of Theorem 3.17, when Smyth complete quasi-metric spaces (every left K-Cauchy net is convergent with respecto to $\tau(d^s)$) are considered. So it seems natural to wonder whether our Theorems 3.2, 3.13 and 3.18 remain valid when we remove the T_1 separation condition from their statements and we consider a τ-left K-sequentially completeness for nets.

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