Fixed Point Theory, 21(2020), No. 2, 715-726
DOI: 10.24193/fpt-ro.2020.2.51
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

FOUR TO ONE

LECH PASICKI

AGH University of Science and Technology
Faculty of Applied Mathematics
Al. Mickiewicza 30
30-059 Kraków, Poland
E-mail: pasicki@agh.edu.pl

Abstract. In the present paper general theorems on a common fixed point for four mappings in dislocated metric space are proved. By the way, also results for three, two or one mapping are obtained. The assumptions are unified and compact. Numerous basic and sophisticated theorems can be derived from the facts presented in our paper.
Key Words and Phrases: Dislocated metric, weakly compatible mappings, fixed point.
2010 Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

In this paper the ideas presented in [10] are refined and general results are proved. We get rid of a special function φ used in condition (9) of [10], a unified general comparison mapping h is applied (see (1)), in addition, H presented in (2) can differ from h.

The notion of a *dislocated metric* (briefly *d-metric*) is due to Hitzler and Seda [3], and a d-metric p differs from metric, as $p(x, y) = 0$ implies $x = y$ (no equivalence). The topology of a *d-metric space* (X, p) is generated by balls. In this paper, (X, p) is a d-metric space, and f, g, i, j are self mappings on X. The following two conditions with mappings $h, H \colon iX \times jX \times fX \times gX \to [0, \infty)$ are applied:

$$p(fx, gy) > 0 \textit{ yields } p(fx, gy) < h(ix, jy, fx, gy), \quad x, y \in X, \tag{1}$$

$$\begin{aligned} &\textit{for each } \alpha > 0 \textit{ there is an } \epsilon > 0 \textit{ for which} \\ &H(ix, jy, fx, gy) < \alpha + \epsilon \textit{ yields } p(fx, gy) \le \alpha, \quad x, y \in X. \end{aligned} \tag{2}$$

The subsequent properties are assumed to hold for each $\alpha > 0$, and some of them are used in our theorems.

$$[d = a \textit{ or } c = b] \textit{ yields } h(a,b,c,d) \le \max\{p(a,b), p(a,c), p(b,d)\}, \\ \textit{if } \max\{\dots\} > 0, \tag{3a}$$

$$h(a,b,a,b) \le p(a,b), \textit{ if } p(a,b) > 0, \tag{3b}$$

$$[p(b,d) = \alpha \textit{ (b, d fixed) and } p(a,b), p(a,c), p(b,c) \to 0] \textit{ yields} \\ \limsup h(a,b,c,d) < p(b,d), \tag{3c}$$

$$[p(a,c) = \alpha \textit{ (a, c fixed) and } p(a,b), p(a,d), p(b,d) \to 0] \textit{ yields} \\ \limsup h(a,b,c,d) < p(a,c). \tag{3d}$$

$$[p(a,b), p(b,d) \searrow \alpha] \textit{ yields } \limsup H(a,b,b,d) \le \alpha, \tag{4a}$$

$$[p(a,b), p(a,d), p(b,c), p(c,d) \to \alpha \textit{ and } p(a,c), p(b,d) \to 0] \textit{ yields} \\ \limsup H(a,b,c,d) \le \alpha. \tag{4b}$$

In [10] the subsequent mapping was applied (let us call it F):

$$\max\{p(a,b), p(c,a), p(d,b), h(a,b,c,d)\}.$$

It is easily seen, that if $h = H$ satisfies any of the conditions (3a), (3b), (4a), (4b), then F has the same property. In turn [10] (3c), (3d) yield (3c), (3d) for $\varphi(F)$ (see [10] (9) and [10], Corollary 2.2). Therefore, F is unnecessarily complicated, and only h, H are considered in the present paper. Moreover, (1) and (9) from [10] yield our new conditions (1) and (2) (see also Remark 2.2).

Example 1.1. Let us consider $h_1(a,b,c,d) = p(a,b)$. Then the system of conditions (3) holds for $h = h_1$, and (4) is satisfied for $H = h_1$.

Example 1.2. Let us consider $h_2(a,b,c,d) = \max\{p(a,b), p(a,c), p(b,d)\}$. Then for $h = h_2$ condition (3a) clearly holds, and (3b) is satisfied if we assume that $\max\{p(a,a), p(b,b)\} \le p(a,b)$. The system of conditions (4) obviously holds for $H = h_2$.

Example 1.3. Let us consider

$$h_3(a,b,c,d) = \max\{p(a,b), p(a,c), p(b,d), [p(a,d) + p(b,c)]/2\}.$$

Then for $h = h_3$ condition (3a) holds if the following inequalities are satisfied:

(i) $p(b,c) \le p(b,a) + p(a,c) - p(a,a)$,

(ii) $p(a,d) \le p(a,b) + p(b,d) - p(b,b)$

(*e.g.* if p is a *partial metric* in X {see [7], Definition 3.1, or [10] (4)}). Condition (3b) is satisfied if $\max\{p(a,a), p(b,b)\} \le p(a,b)$ (true also for partial metric). In turn, for $H = h_3$ (4a) holds if (ii) is satisfied; (4b) clearly holds.

Remark 1.4. Let us consider a mapping $\varphi\colon [0,\infty) \to [0,\infty)$ such that for each $\alpha > 0$ we have $\varphi(\alpha) < \alpha$ and $\varphi(\cdot) \le \alpha$ on some interval $(\alpha, \alpha + \epsilon)$ (*i.e.* $\varphi \in \Psi_P$ [9]). Then for $h = \varphi \circ H$ condition (1) implies (2) (means that (2) can be disregarded); for our h, where $H = h_i$, $i = 1, 2, 3$, conditions (3c), (3d) are satisfied, as $p(a,d) \le p(a,b) + p(b,d)$

and $p(b,c) \le p(b,a) + p(a,c)$, and consequently, if p is a partial metric, then (3), (4) hold.

Example 1.5. Let us consider

$$h_4(a,b,c,d) = p(a,b) + \beta(p(a,c), p(b,d))$$

for a mapping $\beta\colon [0,\infty) \times [0,\infty) \to [0,\infty)$. If $\beta(p(a,a), p(b,b)) = 0$ holds, then for $h = h_4$ (3b) is satisfied. If β is continuous at $(0,0)$ and $\beta(0,0) = 0$, then for $H = h_4$ we have (4b).

Example 1.6. Let us consider

$$h_5(a,b,c,d) = h_k(a,b,c,d) + \beta(p(a,c), p(b,d))$$

for $k = 1,2,3$, and β as in Example 1.5. Then for $h = h_5$ and $k > 1$ (3b) holds if p is such that $\max\{p(a,a), p(b,b)\} \le p(a,b)$ (no problem for $k = 1$); for $H = h_5$ we obtain (4b) (k for h and H can differ).

More sophisticated mappings based on Example 1.3 can be found *e.g.* in [6] (see M_1, M_2 or M_3).

2. Extended results

Let us recall ([9], Definition 2.3) that a d-metric space (X,p) is 0-*complete* if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, there exists an $x \in X$ such that $\lim_{n\to\infty} p(x, x_n) = 0$.

The next lemma is an extension of [10], Lemma 2.6.

Lemma 2.1. *Let f,g,i,j be selfmappings on a d-metric space (X,p), $fX \subset jX$, $gX \subset iX$, and let $h,H\colon iX \times jX \times fX \times gX \to [0,\infty)$ be mappings. If* (1), (3a) *are satisfied then there exist sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ such that*

$$x_{2k} = fy_{2k-1},\ x_{2k-1} = iy_{2k-1},\ x_{2k+1} = gy_{2k},\ x_{2k} = jy_{2k}, \quad k \in \mathbb{N}, \tag{5a}$$

$$p(x_{n+2}, x_{n+1}) > 0 \text{ yields } p(x_{n+2}, x_{n+1}) < p(x_{n+1}, x_n), \quad n \in \mathbb{N}. \tag{5b}$$

If (5), (1), (2) *and* (4a) *hold, then* $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ *(so the same conclusion for* (1), (2), (3a), (4a)*). If* $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ *and* (5a), (1), (2), (4b) *hold, then* $\lim_{m,n\to\infty} p(x_n, x_m) = 0$ *(so the same conclusion for* (1), (2), (3a), (4)*); if, in addition, at least one of the sets fX, gX, iX, jX is* 0*-complete, then there exists an $x \in X$ such that*

$$\lim_{n\to\infty} p(x, x_n) = p(x,x) = 0.$$

Proof. From (1) and (3a) we obtain

for each $x \in X$ there is a $y \in X$ for which $gx = iy$, and $p(fy, gx) > 0$ yields $p(fy, gx) < h(gx, jx, fy, gx) \le \max\{p(gx, jx), p(fy, gx)\}$,

and

for each $y \in X$ there is an $x \in X$ for which $fy = jx$, and $p(fy, gx) > 0$ yields $p(fy, gx) < h(iy, fy, fy, gx) \le \max\{p(iy, fy), p(fy, gx)\}$.

If *e.g.* $p(gx, jx) < p(fy, gx)$ holds, then we get a contradiction $p(fy, gx) < p(fy, gx)$. Therefore, $p(fy, gx) > 0$ yields $p(fy, gx) < p(gx, jx)$. Now, it is clear that the following conditions are satisfied:

$$\begin{aligned}&\textit{for each } x \in X \textit{ there is a } y \in X \textit{ for which } gx = iy, \textit{ and}\\ &p(fy, gx) > 0 \textit{ yields } p(fy, gx) < h(gx, jx, fy, gx) \le p(gx, jx),\end{aligned} \tag{6}$$

$$\begin{aligned}&\textit{for each } y \in X \textit{ there is an } x \in X \textit{ for which } fy = jx, \textit{ and}\\ &p(gx, fy) > 0 \textit{ yields } p(gx, fy) < h(iy, fy, fy, gx) \le p(fy, iy).\end{aligned} \tag{7}$$

For an $x_0 \in X$ let us take $x_1 = gx_0 = iy_1$, $x_2 = fy_1$, for y_1 such that $p(x_2, x_1) \le p(x_1, jx_0)$ (see (6)). Now, we take $x_3 = gy_2$ for y_2 such that $x_2 = fy_1 = jy_2$ and $p(x_3, x_2) \le p(x_2, x_1)$ (see (7)). In turn $x_4 = fy_3$, where y_3 is such that $x_3 = gy_2 = iy_3$ and $p(x_4, x_3) \le p(x_3, x_2)$ (see (6)). By induction we obtain sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ satisfying (5).

Assume (5). If for some n we have $p(x_{n+1}, x_n) = 0$, then by (5b) $p(x_{n+2}, x_{n+1}) = 0$, and $x_{n+k} = x_n$, $k \in \mathbb{N}$, *i.e.* $x = x_n$ and our lemma is proved. Therefore, we may assume $p(x_{n+1}, x_n) > 0$, $n \in \mathbb{N}$, and then (see (5b))

$$0 < p(x_{n+2}, x_{n+1}) < p(x_{n+1}, x_n), \quad n \in \mathbb{N}$$

holds. Clearly, sequence $(p(x_{n+1}, x_n))_{n\in\mathbb{N}}$ decreases to some $\alpha \ge 0$. Let us assume (1), (2), (4a) and suppose $\alpha > 0$. Then (5a), (1) yield

$$\begin{aligned}&\alpha < p(x_{2k+1}, x_{2k}) = p(gy_{2k}, fy_{2k-1}) = p(fy_{2k-1}, gy_{2k}) <\\ &h(iy_{2k-1}, jy_{2k}, fy_{2k-1}, gy_{2k}) = h(x_{2k-1}, x_{2k}, x_{2k}, x_{2k+1}).\end{aligned} \tag{8}$$

Now, from (4a) we obtain

$$H(iy_{2k-1}, jy_{2k}, fy_{2k-1}, gy_{2k}) = H(x_{2k-1}, x_{2k}, x_{2k}, x_{2k+1}) < \alpha + \epsilon \tag{9}$$

for large k, and (2) yields

$$\alpha < p(x_{2k+1}, x_{2k}) = p(fy_{2k-1}, gy_{2k}) \le \alpha,$$

a contradiction. Thus, $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ is proved.

Now, for $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ conditions (5a), (1), (2) and (4b) are applied to prove that $\lim_{m,n\to\infty} p(x_n, x_m) = 0$. Suppose, $0 < \alpha < p(x_{2k}, x_{2n+1+2k})$ holds for all $k \in \mathbb{K} \subset \mathbb{N}$ ($\mathbb{K}$ infinite) and the respective $n \in \mathbb{N}$. Let $n = n(k)$ be the first number as to satisfy this inequality. From (5a), (1) we obtain

$$\begin{aligned}\alpha &< p(x_{2k}, x_{2n+1+2k}) = p(fy_{2k-1}, gy_{2n+2k})\\ &< h(iy_{2k-1}, jy_{2n+2k}, fy_{2k-1}, gy_{2n+2k}) = h(x_{2k-1}, x_{2n+2k}, x_{2k}, x_{2n+1+2k}).\end{aligned} \tag{10}$$

Let us show that the assumptions of (4b) are satisfied. We have

$$\begin{aligned}&\alpha - p(x_{2k}, x_{2k-1}) - p(x_{2n+2k}, x_{2n+1+2k})\\ &< p(x_{2k}, x_{2n+1+2k}) - p(x_{2k}, x_{2k-1}) - p(x_{2n+2k}, x_{2n+1+2k})\\ &\le p(x_{2k-1}, x_{2n+2k})\\ &\le p(x_{2k-1}, x_{2k}) + p(x_{2k}, x_{2n-1+2k}) + p(x_{2n-1+2k}, x_{2n+2k})\\ &\le p(x_{2k-1}, x_{2k}) + \alpha + p(x_{2n-1+2k}, x_{2n+2k})\end{aligned}$$

which yields

$$\lim_{k\in\mathbb{K}} p(x_{2k-1}, x_{2n+2k}) = \alpha.$$

In addition, from

$$p(x_{2k-1}, x_{2n+2k}) - p(x_{2n+1+2k}, x_{2n+2k}) \le p(x_{2k-1}, x_{2n+1+2k}) \\ \le p(x_{2k-1}, x_{2n+2k}) + p(x_{2n+2k}, x_{2n+1+2k})$$

it follows that

$$\lim_{k\in\mathbb{K}} p(x_{2k-1}, x_{2n+1+2k}) = \alpha.$$

In a similar way we get

$$\lim_{k\in\mathbb{K}} p(x_{2n+2k}, x_{2k}) = \lim_{k\in\mathbb{K}} p(x_{2k}, x_{2n+1+2k}) = \alpha.$$

Now (4b) applies, we get

$$\begin{aligned} &H(iy_{2k-1}, jy_{2n+2k}, fy_{2k-1}, gy_{2n+2k}) \\ &= H(x_{2k-1}, x_{2n+2k}, x_{2k}, x_{2n+1+2k}) < \alpha + \epsilon \end{aligned} \tag{11}$$

for large k, and then (10), (2) yield

$$\alpha < p(x_{2k}, x_{2n+1+2k}) = p(fy_{2k-1}, gy_{2n+2k}) \le \alpha,$$

a contradiction. Thus we have proved that

$$\lim_{k,n\to\infty} p(x_{2k}, x_{2n+1+2k}) = 0.$$

Now, $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ and the triangle inequality yield

$$\lim_{m,n\to\infty} p(x_n, x_m) = 0.$$

Consequently, if the respective set is 0-complete, then there exists an x such that

$$\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0. \qquad \square$$

Remark 2.2. If $h \le H$ (or $\alpha < H$) holds for $(a, b, b, d) = (x_{2k-1}, x_{2k}, x_{2k}, x_{2k+1})$ as in (4a), or for $(a, b, c, d) = (x_{2k-1}, x_{2n+2k}, x_{2k}, x_{2n+1+2k})$ as in (4b), then the respective part of the proof of Lemma 2.1 works for (2) with "$H(\dots) < \alpha + \epsilon$" replaced by "$\alpha < H(\dots) < \alpha + \epsilon$" (see inequalities (8) and (9), (10) and (11)). This modified condition (2) (or (13)) can be then applied in our Theorems (in Theorems 2.18, 2.20 condition (4a) is not used).

If selfmappings f, i on X commute at their coincidence points, *i.e.*

$$ix = fx \text{ yields } ifx = fix, \quad x \in X,$$

then the pair (f, i) is called *weakly compatible* (see [5]).

Now, [10], Theorem 2.7 can be extended as follows (conditions (3c), (3d) enable us to disregard mapping φ used in [10]).

Theorem 2.3. *Assume that (X, p) is a d-metric space and f, g, i, j are selfmappings on X, (f, i), (g, j) are weakly compatible, $fX \subset jX$, $gX \subset iX$, and at least one of sets fX, gX, iX, jX is* 0*-complete. Let* (1), (2) *hold for mappings $h, H\colon iX \times jX \times fX \times gX \to [0, \infty)$ satisfying* (3), (4)*, respectively. Then f, g, i, j have a single common fixed point x, $p(x, x) = 0$, and neither f, i nor g, j have another common fixed point.*

Proof. Assume *e.g.* that jX or fX is 0-complete, and let us consider $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $x \in jX$ as in Lemma 2.1.
For a v such that $x = jv$ suppose that $p(x, gv) = \alpha > 0$. Then (see (5a), (1)) from

$$p(x_{2k}, gv) = p(fy_{2k-1}, gv) < h(iy_{2k-1}, jv, fy_{2k-1}, gv) = h(x_{2k-1}, x, x_{2k}, gv),$$

we get (see (3c))

$$0 < p(x, gv) \le \liminf_{k\to\infty}[p(x, x_{2k}) + h(x_{2k-1}, x, x_{2k}, gv)] < p(x, gv),$$

a contradiction. Therefore, $p(x, gv) = 0$ and $\underline{x = gv = jv}$.
Now, for w such that $x = iw$ (as $gX \subset iX$) suppose that $p(fw, x) = \alpha > 0$. We have

$$p(fw, x_{2k+1}) = p(fw, gy_{2k}) < h(iw, jy_{2k}, fw, gy_{2k}) = h(x, x_{2k}, fw, x_{2k+1}),$$

and (3d) yields

$$0 < p(fw, x) \le \liminf_{k\to\infty}[h(x, x_{2k}, fw, x_{2k+1}) + p(x_{2k+1}, x)] < p(fw, x),$$

a contradiction. Therefore, $p(fw, x) = 0$ and $\underline{x = fw = iw}$.
The underlined equalities and the weak compatibility yield

$$ix = ifw = fiw = fx \quad \textit{and} \quad jx = jgv = gjv = gx.$$

In the remaining part of our proof conditions (3c), (3d) are not applied.
Suppose $p(fx, x) > 0$. Then we obtain (see (1), (3b))

$$0 < p(fx, x) = p(fx, gv) < h(ix, jv, fx, gv) = h(fx, x, fx, x) \le p(fx, x),$$

a contradiction. Now, it is clear that $ix = fx = x$.
Similarly, for $p(x, gx) > 0$ we get

$$0 < p(x, gx) = p(fw, gx) < h(iw, jx, fw, gx) = h(x, gx, x, gx) \le p(x, gx),$$

a contradiction. We have proved that $jx = gx = x = fx = ix$.
Suppose *e.g.* that y is another common fixed point of i and f. Then we obtain (see (1), (3b))

$$0 < p(y, x) = p(fy, gx) < h(iy, jx, fy, gx) = h(y, x, y, x) \le p(y, x),$$

a contradiction. Consequently, $p(x, y) = 0$ and $x = y$ hold. □

The above theorem further extends [2], Theorem 2.8 (in part concerning a common fixed point). More sophisticated theorems are also included in Theorem 2.3 (see *e.g.* [6]). What is more, from Remark 1.4 it follows that Theorem 2.3 extends our Theorems 2.7, 2.8 from [10].

For $g = f$ Theorem 2.3 concerns three mappings and it can be reformulated as follows:

Theorem 2.4. *Assume that (X, p) is a d-metric space and f, i, j are selfmappings on X, (f, i), (f, j) are weakly compatible, $fX \subset iX \cap jX$, and at least one of sets fX, iX, jX is 0-complete. For $g = f$ let (1), (2) hold, and for mappings $h, H \colon iX \times jX \times (fX)^2 \to [0, \infty)$ let (3), (4) be satisfied, respectively. Then f, i, j have a single common fixed point x, $p(x, x) = 0$, and neither f, i nor f, j have another common fixed point.*

Another three mappings consequence of Theorem 2.3 is the following one.

Theorem 2.5. *Assume that (X,p) is a d-metric space and f,g,i are selfmappings on X, (f,i), (g,i) are weakly compatible, $fX \cup gX \subset iX$, and at least one of sets fX, gX, iX is* 0*-complete. For $j=i$ let* (1), (2) *hold, and for mappings $h,H\colon (iX)^2 \times fX \times gX \to [0,\infty)$ let* (3), (4) *be satisfied, respectively. Then f,g,i have a single common fixed point x, $p(x,x)=0$, and neither f,i nor g,i have another common fixed point.*

Let us consider the case $i=j=id$. Then conditions (1), (2) have the following form, respectively

$$p(fx,gy) > 0 \text{ yields } p(fx,gy) < h(x,y,fx,gy), \quad x,y \in X, \tag{12}$$

$$\begin{aligned} &\text{for each } \alpha > 0 \text{ there is an } \epsilon > 0 \text{ for which} \\ &H(x,y,fx,gy) < \alpha + \epsilon \text{ yields } p(fx,gy) \le \alpha, \quad x,y \in X. \end{aligned} \tag{13}$$

The next theorem is a consequence of Theorem 2.3 for two mappings

Theorem 2.6. *Assume that (X,p) is a d-metric space, f,g are selfmappings on X, and at least one of sets X, fX, gX is* 0*-complete. Let* (12), (13) *hold, and for mappings $h,H\colon X^2 \times fX \times gX \to [0,\infty)$ let* (3), (4) *be satisfied, respectively. Then there exists an $x \in X$ such that for any $x_0 \in X$ and $x_{2k+1} = gx_{2k}$, $x_{2k} = fx_{2k-1}$, $k \in \mathbb{N}$, ((5b) holds for $(x_n)_{n\in\mathbb{N}}$) we have $\lim_{n\to\infty} p(x,x_n) = p(x,x) = 0$ and $x = fx = gx$; in addition, neither f nor g has another fixed point.*

Proof. Our sequence $(x_n)_{n\in\mathbb{N}}$ is as in condition (5), because for $i=j=id$ we have $x_n = y_n$, $n \in \mathbb{N}$ (see Lemma 2.1 for (1), (3a)). □

For $g=f$ Theorem 2.6 becomes a pretty general "usual" fixed point theorem.

Theorem 2.7. *Assume that (X,p) is a* 0*-complete d-metric space and f is a selfmapping on X. For $g=f$ let* (12), (13) *hold, and for mappings $h,H\colon X^2 \times (fX)^2 \to [0,\infty)$ let* (3), (4) *be satisfied, respectively. Then there exists an x such that for any $x_0 \in X$ and $x_n = f^n x_0$, $n \in \mathbb{N}$, ((5b) holds for $(x_n)_{n\in\mathbb{N}}$) we have*

$$\lim_{n\to\infty} p(x,x_n) = p(x,x) = 0 \text{ and } x = fx.$$

Proof. For $g=f$ and $i=j=id$ our sequence $(x_n)_{n\in\mathbb{N}}$ is as in condition (5). □

Example 2.8. Let us consider

$$\begin{aligned} &H(a,b,c,d) = \kappa p(a,b) + \lambda p(a,c) + \mu p(b,d) + \nu[p(a,d)+p(b,c)] \\ &\text{for some } \kappa, \lambda\, \mu, \nu \ge 0 \text{ such that } \kappa + \lambda \ + \mu + 2\nu = 1, \\ &\text{and } h = \varphi \circ H \text{ for a nondecreasing mapping } \varphi\colon [0,\infty) \to [0,\infty) \\ &\text{such that for each } \alpha > 0 \text{ we have } \varphi(\alpha) < \alpha \text{ and} \\ &\varphi(\cdot) \le \alpha \text{ on some interval } (\alpha, \alpha+\epsilon). \end{aligned} \tag{14}$$

If condition (1) or (12) is satisfied, then (2) or (13) holds, respectively (see Remark 1.4).

Now, let us check conditions (3) and (4). Assume p satisfies (i) from Example 1.3. Then we have

$$\begin{aligned}
&h(a,b,c,a)\\
&\le \varphi(\kappa p(a,b)+\lambda p(a,c)+\mu p(b,a)+\nu[p(a,a)+p(b,a)+p(a,c)-p(a,a)])\\
&=\varphi((\kappa+\mu+\nu)p(a,b)+(\lambda+\nu)p(a,c))\le\varphi((\kappa+\lambda+\mu+2\nu)\max\{p(a,b),p(a,c)\})\\
&=\varphi(\max\{p(a,b),p(a,c)\})\le\max\{p(a,b),p(a,c)\}.
\end{aligned}$$

In a similar way, if (ii) from Example 1.3 is satisfied, then we obtain

$$\begin{aligned}
&h(a,b,b,d)\\
&\le \varphi(\kappa p(a,b)+\lambda p(a,b)+\mu p(b,d)+\nu[p(a,b)+p(b,d)-p(b,b)+p(b,b)])\\
&=\varphi((\kappa+\lambda+\nu)p(a,b)+(\mu+\nu)p(b,d))\le\varphi((\kappa+\lambda+\mu+2\nu)\max\{p(a,b),p(b,d)\})\\
&=\varphi(\max\{p(a,b),p(b,d)\})\le\max\{p(a,b),p(b,d)\}.
\end{aligned}$$

Consequently, if p satisfies (i), (ii) from Example 1.3, then (3a) holds.
Assume $p(a,a),p(b,b)\le p(a,b)$. Then

$$\begin{aligned}
h(a,b,a,b)&=\varphi(\kappa p(a,b)+\lambda p(a,a)+\mu p(b,b)+2\nu p(a,b))\\
&\le\varphi((\kappa+\lambda+\mu+2\nu)p(a,b))=\varphi(p(a,b))\le p(a,b)
\end{aligned}$$

means that (3b) is satisfied.
Now, let us adopt the assumptions of (3c). We have

$$\begin{aligned}
h(a,b,c,d)&=\varphi(\kappa p(a,b)+\lambda p(a,c)+\mu\alpha+\nu[p(a,d)+p(b,c)])\\
&\le\varphi(\kappa p(a,b)+\lambda p(a,c)+\mu\alpha+\nu[p(a,b)+p(b,d)+p(b,c)])\\
&=\varphi((\kappa+\nu)p(a,b)+\lambda p(a,c)+(\mu+\nu)\alpha+\nu p(b,c))\\
&\le\varphi((\kappa+\lambda+\mu+2\nu)\alpha)=\varphi(\alpha)<\alpha=p(b,d).
\end{aligned}$$

In a similar way we check condition (3d):

$$\begin{aligned}
h(a,b,c,d)&=\varphi(\kappa p(a,b)+\lambda\alpha+\mu p(b,d)+\nu[p(a,d)+p(b,c)])\\
&\le\varphi(\kappa p(a,b)+\lambda\alpha+\mu p(b,d)+\nu[p(a,d)+p(b,a)+p(a,c)])\\
&=\varphi((\kappa+\nu)p(a,b)+(\lambda+\nu)\alpha+\mu p(b,d)+\nu p(a,d))\\
&\le\varphi((\kappa+\lambda+\mu+2\nu)\alpha)=\varphi(\alpha)<\alpha=p(a,c).
\end{aligned}$$

Consequently, the system of conditions (3) is satisfied.
Assume that condition (ii) from Example 1.3 holds. Then for $p(a,b),p(b,d)\to\alpha$ we obtain

$$\begin{aligned}
H(a,b,b,d)&=\kappa p(a,b)+\lambda p(a,b)+\mu p(b,d)+\nu[p(a,d)+p(b,b)]\\
&\le(\kappa+\lambda)p(a,b)+\mu p(b,d)+\nu[p(a,b)+p(b,d)-p(b,b)+p(b,b)]\\
&=(\kappa+\lambda+\nu)p(a,b)+(\mu+\nu)p(b,d)\to(\kappa+\lambda+\mu+2\nu)\alpha=\alpha,
\end{aligned}$$

i.e. (4a) holds. If the assumptions of (4b) are satisfied, then we obtain

$$H(a,b,c,d)\to(\kappa+2\nu)\alpha\le\alpha.$$

Corollary 2.9. *If p is a partial metric, then for h,H as in* (14) *conditions* (3), (4) *hold; in addition,* (1) *yields* (2), *and* (12) *implies* (13).

Now, the subsequent five theorems are immediate consequences of Corollary 2.9, Remark 1.4, and of Theorems 2.3, …, 2.7.

Theorem 2.10. *Assume that* (X, p) *is a partial metric space and* f, g, i, j *are selfmappings on* X*,* (f, i)*,* (g, j) *are weakly compatible,* $fX \subset jX$*,* $gX \subset iX$*, and at least one of sets* fX, gX, iX, jX *is* 0*-complete. Let* (1) *hold for a mapping* $h\colon iX \times jX \times fX \times gX \to [0, \infty)$ *as in* (14) *or as in Remark 1.4. Then* f, g, i, j *have a single common fixed point* x*,* $p(x, x) = 0$*, and neither* f, i *nor* g, j *have another common fixed point.*

Theorem 2.11. *Assume that* (X, p) *is a partial metric space and* f, i, j *are selfmappings on* X*,* (f, i)*,* (f, j) *are weakly compatible,* $fX \subset iX \cap jX$*, and at least one of sets* fX, iX, jX *is* 0*-complete. For* $g = f$ *let* (1) *hold for a mapping* $h\colon iX \times jX \times (fX)^2 \to [0, \infty)$ *as in* (14) *or as in Remark 1.4. Then* f, i, j *have a single common fixed point* x*,* $p(x, x) = 0$*, and neither* f, i *nor* f, j *have another common fixed point.*

Theorem 2.12. *Assume that* (X, p) *is a partial metric space and* f, g, i *are selfmappings on* X*,* (f, i)*,* (g, i) *are weakly compatible,* $fX \cup gX \subset iX$*, and at least one of sets* fX, gX, iX *is* 0*-complete. For* $j = i$ *let* (1) *hold for a mapping* $h\colon (iX)^2 \times fX \times gX \to [0, \infty)$ *as in* (14) *or as in Remark 1.4. Then* f, g, i *have a single common fixed point* x*,* $p(x, x) = 0$*, and neither* f, i *nor* g, i *have another common fixed point.*

Theorem 2.13. *Assume that* (X, p) *is a partial metric space,* f, g *are selfmappings on* X*, and at least one of sets* X, fX, gX *is* 0*-complete. Let* (12) *hold for a mapping* $h\colon X^2 \times fX \times gX \to [0, \infty)$ *as in* (14) *or as in Remark 1.4. Then there exists an* $x \in X$ *such that for any* $x_0 \in X$ *and* $x_{2k+1} = gx_{2k}$*,* $x_{2k} = fx_{2k-1}$*,* $k \in \mathbb{N}$*, ((5b) holds for* $(x_n)_{n\in\mathbb{N}}$*) we have* $\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0$ *and* $x = fx = gx$*; in addition, neither* f *nor* g *has another fixed point.*

Theorem 2.14. *Assume that* (X, p) *is a* 0*-complete partial metric space and* f *is a selfmapping on* X*. For* $g = f$ *let* (12) *hold for a mapping* $h\colon X^2 \times (fX)^2 \to [0, \infty)$ *as in* (14) *or as in Remark 1.4. Then there exists an* x *such that for any* $x_0 \in X$ *and* $x_n = f^n x_0$*,* $n \in \mathbb{N}$*, ((5b) holds for* $(x_n)_{n\in\mathbb{N}}$*) we have* $\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0$ *and* $x = fx$*.*

Remark 2.15. Ćirić in [1] considered the following condition for a metric p:

$$p(fx, fy) \le \alpha p(x, y) + \beta p(x, fx) + \gamma p(y, fy) + \delta[p(x, fy) + p(y, fx)]\},$$
$$\textit{for fixed } \alpha, \beta, \gamma, \delta \ge 0 \textit{ such that } \alpha + \beta + \gamma + 2\delta < 1.$$

It is clear, that (12) for $g = f$ and mapping $\varphi(\alpha) = k\alpha$ with some $k \in [0, 1)$ in (14) is equivalent to the Ćirić's condition. Consequently the Ćirić theorem follows from our Theorem 2.14.

In our further theorems conditions (3c), (3d) are disregarded.

Let us recall (see [11], Definition 2.10) that a selfmapping f on a d-metric space (X, p) is 0*-continuous* if for each sequence $(x_n)_{n\in\mathbb{N}}$, $x \in X$ from $\lim_{n\to\infty} p(x, x_n) = 0$ it follows that $\lim_{n\to\infty} p(fx, fx_n) = 0$.

Remark 2.16. If $\lim_{n\to\infty} p(x, x_n) = 0$ holds for a sequence $(x_n)_{n\in\mathbb{N}}$ in a d-metric space (X, p), then the triangle inequality yields $x \in \mathit{Ker}\, p = \{s \in X : p(s, s) = 0\}$. Consequently, if f is a 0-continuous selfmapping on (X, p), then $f(\mathit{Ker}\, p) \subset \mathit{Ker}\, p$ (consider $x_n = x$). In turn, $\mathit{Ker}\, p$ (if nonempty) with the restricted p is a metric subspace of (X, p). So, if we know that $\mathit{Ker}\, p$ is nonempty, complete and f, g are 0-continuous, then the global condition (3b) guarantees that the fixed point is unique; the remaining part of (3) and conditions (1), (2) and (4) can be local (*i.e.* for X replaced by $\mathit{Ker}\, p$).

Theorem 2.17. *Assume that (X, p) is a d-metric space and f, g are 0-continuous selfmappings on X, and at least one of sets X, fX, gX is 0-complete. Let (12), (13) hold, and for mappings $h, H : X^2 \times fX \times gX \to [0, \infty)$ let (3a), (4) be satisfied, respectively. Then for any $x_0 \in X$ and $x_{2k+1} = gx_{2k}$, $x_{2k} = fx_{2k-1}$, $k \in \mathbb{N}$, (5b) holds, there exists an x such that $\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0$ and $x = fx = gx$. If, in addition, (3b) is satisfied, then x is unique and neither f nor g has another fixed point.*

Proof. For $i = j = id$ from conditions (12), (13), (3a), (4) it follows that $(x_n)_{n\in\mathbb{N}}$ satisfies (5b) and $\lim_{n\to\infty} p(x, x_n) = 0$ (see Lemma 2.1 for (1), (2), (3a), (4)). Now (see (5a)),

$$p(x, gx) \le p(x, x_{2k+1}) + p(x_{2k+1}, gx) = p(x, x_{2k+1}) + p(gx_{2k}, gx),$$

and the 0-continuity of g at x imply

$$p(x, gx) \le \lim_{k\to\infty} [p(x, x_{2k+1}) + p(gx_{2k}, gx)] = 0.$$

In a similar way, we obtain $p(fx, x) = 0$, as

$$p(fx, x) \le p(fx, x_{2k}) + p(x_{2k}, x) = p(fx, fx_{2k-1}) + p(x_{2k}, x).$$

The final part of the proof of Theorem 2.3 shows that neither f nor g have another fixed point provided that (3b) holds. □

From Remark 2.2 and Lemma 2.1 (see also Examples 1.5, 1.6) it follows that the next theorem is a far extension of a theorem of Proinov ([12], Theorem 4.2), and also Theorem 3.7 from [11] is included in Theorem 2.18 (see [11] Corollary 2.6, Lemma 2.9). In addition, those two theorems were proved for $g = f$.

Theorem 2.18. *Assume that (X, p) is a d-metric space, f, g are 0-continuous selfmappings on X, and at least one of sets X, fX, gX is 0-complete. Assume that for an $x_0 \in X$, and $x_{2k} = fx_{2k-1}$, $x_{2k+1} = gx_{2k}$, $k \in \mathbb{N}$, we have $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$. Let (12), (13) hold, and for mappings $h, H : X^2 \times fX \times gX \to [0, \infty)$ let H satisfy (4b). Then there exists an x such that $\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0$ and $x = fx = gx$. If, in addition, (3b) is satisfied, then x is unique and neither f nor g has another fixed point.*

Proof. For $i = j = id$ condition (5a) is satisfied. If $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$, then (see Lemma 2.1 for (5a), (1), (2), (4b)) from (5a), (12), (13), (4b) it follows that there exists a point $x \in X$ such that $\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0$. Now, as in the proof of Theorem 2.17 we conclude that $x = fx = gx$ and that f, g have no other fixed points. $\square$

It is worth noting that for $g = f$ and $h = h_1$ (see Example 1.1) condition (1) implies 0-continuity of f. Now, let us present the versions of Theorems 2.17, 2.18 for $g = f$.

Theorem 2.19. *Assume that (X, p) is a 0-complete d-metric space and f is a 0-continuous selfmapping on X. For $g = f$ let (12), (13) hold, and for mappings $h, H\colon X^2 \times (fX)^2 \to [0, \infty)$ let (3a), (4) be satisfied, respectively. Then for any $x_0 \in X$ and $x_n = f^n x_0$, $n \in \mathbb{N}$, (5b) holds, there exists an x such that*

$$\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0 \text{ and } x = fx.$$

If, in addition, (3b) is satisfied, then x is unique.

Theorem 2.20. *Let f be a 0-continuous selfmapping on a 0-complete d-metric space (X, p). Assume that there exists an $x_0 \in X$, such that for $x_n = f^n x_0$, $n \in \mathbb{N}$ we have $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$. For $g = f$ let (12), (13) hold, and for mappings $h, H\colon X^2 \times (fX)^2 \to [0, \infty)$ let H satisfy (4b). Then there exists an x such that*

$$\lim_{n\to\infty} p(x, x_n) = p(x, x) = 0 \text{ and } x = fx.$$

If, in addition, (3b) is satisfied, then x is unique.

From Example 1.3 and Remark 2.2 it follows that for $h = H = h_3$ a theorem of Jachymski ([4], Theorem 2) is included in Theorem 2.20.

Acknowledgements. This work was partially supported by the Faculty of Applied Mathematics AGH UST statutory tasks within subsidy of the Polish Ministry of Science and Higher Education, grant no. 11.11.420.004.

References

[1] L.B. Ćirić, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. (Beograd), **12**(1971), no. 26, 19-26.

[2] R.H. Haghi, Sh. Rezapour, N. Shahzad, *Be careful on partial metric fixed point results*, Topology Appl., **160**(2013), 450-454.

[3] P. Hitzler, A. Seda, *Mathematical Aspects of Logic Programming Semantics*, Studies in Informatic Series, Chapman & Hall, CRC Press, 2011.

[4] J. Jachymski, *Equivalent conditions and the Meir-Keeler type theorems*, J. Math. Anal. Appl., **194**(1995), 293-303.

[5] G. Jungck, B.E. Rhoades, *Fixed points for set valued functions without continuity*, Indian J. Pure Appl. Math., **29**(1998), 227-238.

[6] Z. Liu, X. Zhang, J.S. Ume, S.M. Kang, *Common fixed point theorems for four mappings satisfying ψ-weakly contractive conditions*, Fixed Point Theory Appl., **2015:20**(2015).

[7] S.G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., **728**(1994), 183-197.

[8] D. O'Regan, N. Shahzad, R.P. Agarwal, *Fixed point theory for generalized contractive maps on spaces with vector-valued metrics*, Fixed Point Theory and Applications, (Eds. Y.J. Cho, J.K. Kim, S. M. Kang), Vol. 6, Nova Sci. Publ., New York, 2007, 143-149.

[9] L. Pasicki, *Dislocated metric and fixed point theorems*, Fixed Point Theory Appl., **2015:82**(2015).

[10] L. Pasicki, *Four mappings and generalized contractions*, Fixed Point Theory Appl., **2016:99**(2016).
[11] L. Pasicki, *Some extensions of the Meir-Keeler theorem*, Fixed Point Theory Appl., **2017:1**(2017).
[12] P.D. Proinov, *Fixed point theorems in metric spaces*, Nonlinear Anal., **64**(2006), 546-557.
[13] I.A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory*, Cluj University Press, 2008.
[14] I.A. Rus, A. Petruşel, M.A. Şerban, *Weakly Picard operators: equivalent definitions, applications and open problems*, Fixed Point Theory, **7**(2006), no. 1, 3-22.

Received: June 21, 2018; Accepted: October 20, 2018.