ON THE BURTON METHOD OF PROGRESSIVE CONTRACTIONS FOR VOLterra INTEGRAL EQUATIONS

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Abstract. In the paper [4] the author gives a new method to study the existence and uniqueness of a solution on \([0, \infty)\) of a scalar integral equation

\[
x(t) = g(t, x(t)) + \int_0^t A(t - s)f(t, s, x(s))ds,
\]

where \(u, v \in \mathbb{R}, t \in [0, \infty]\) imply that there exists \(0 < l < 1\) with

\[|g(t, u) - g(t, v)| \leq l |u - v|
\]

and for each \(b > 0\) there exists \(L_b > 0\) such that

\[|f(t, u) - f(t, v)| \leq L_b |u - v|, \forall t \in [0, b], \forall u, v \in \mathbb{R}.
\]

In this paper we extend the Burton method to the case where instead of scalar equations we consider an equation in a Banach space.

Key Words and Phrases: Progressive contractions, fixed points, existence, uniqueness, integro-differential equations.

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1. Introduction

The purpose of this paper is to present an existence and uniqueness result for integral equations with sum of two operators. The approach is based on proving the existence of a solution on a short interval, then the equation is translated to a new starting time so that the solution on another short interval is fitted onto the first solution and so on.

Our result is connected to some recent papers of T.A. Burton [1]-[4] where it is introduced the technique named progressive contraction. This technique is suited to integral equations and shows that when the equation is defined by the sum of a
contraction and a Lipschitz operator one can prove first existence on arbitrary interval \([0, b]\) and then one can parlay that into a solution on \([0, \infty)\). In our paper we combine the above technique with the classical method of Banach fixed point theorem (see [10]-[13]).

Regarding the integral equations that contain a sum of two operators, one can see the following papers [6]-[7], [12], [13].

Let \((\mathbb{B}, |\cdot|)\) be a Banach space and \(K \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{B}, \mathbb{B})\). We consider the Volterra integral equation corresponding to \(K\)

\[ x(t) = \int_0^t K(t, s, x(s))ds, \quad t \in \mathbb{R}_+, \quad x \in C(\mathbb{R}_+, \mathbb{B}). \tag{1.1} \]

For \(b > 0\), let us consider the same equation defined on \([0, b]\) as follows

\[ x(t) = \int_0^t K(t, s, x(s))ds, \quad t \in [0, b], \quad x \in C([0, b], \mathbb{B}). \tag{1.2} \]

In what follows we consider the Bielecki norm \(\|\cdot\|_\tau\), defined by

\[ \|x\|_\tau = \max_{t \in [0, b]} |x(t)| e^{-\tau t}, \quad \tau > 0, \]

the Chebyshev norm defined by

\[ \|x\|_\infty = \sup_{t \in [0, b]} \{|x(t)|\}. \]

We consider the equation (1.2) with the following conditions

\((C_1)\) \(K \in (C([0, b] \times [0, b] \times \mathbb{B}, \mathbb{B}))\);

\((C_2)\) for each \(b > 0\) there exists \(L_b > 0\) such that

\[ |K(t, s, u) - K(t, s, v)| \leq L_b |u - v|, \quad \forall t, s \in [0, b], u, v \in \mathbb{B}. \]

The following result is well known.

**Theorem 1.1.** If the conditions \((C_1)\) and \((C_2)\) are satisfied, then the equation (1.2) has a unique solution in \(C([0, b], \mathbb{B})\).

Since we have a Volterra integral equation is sufficient to prove the existence and uniqueness of the solution in \(C([0, b], \mathbb{B})\) for any positive \(b\). To prove this we consider on \(C([0, b], \mathbb{B})\) the Bielecki norm, with respect to which the operator

\[ S : C([0, b], \mathbb{B}) \to C([0, b], \mathbb{B}) \]

defined by

\[ S(x)(t) = \int_0^t K(t, s, x(s))ds \]

is a \(\frac{L_b}{\tau}\)-contraction, with \(\tau\) chosen sufficiently large.

The problem is if we can obtain this result using Chebyshev norm.
2. Burton method in the case of Volterra integral equation

**Theorem 2.1.** If the condition $(C_1)$ and $(C_2)$ are satisfied and $bL_b < 1$, then the equation (1.2) has a unique solution in $C([0, b], \mathbb{R})$.

**Proof.** Following the idea of T.A. Burton [4], [1], [2] and using

\[ [0, b] = \bigcup_{k=0}^{m-1} \left[ \frac{kb}{m}, \frac{(k+1)b}{m} \right], \quad m \in \mathbb{N}^*. \]

we divide the interval $[0, b]$ into $m$ equal parts, denoting the end points by

\[ 0, \frac{b}{m}, \frac{2b}{m}, \ldots, b. \]

**Step 1.** Let $(M_1, \|\cdot\|_1)$ be the complete metric space of continuous functions $x : [0, \frac{b}{m}] \to \mathbb{R}$ with the Chebyshev metric $\|\cdot\|_1$, where

\[ \|x(t)\|_1 = \max_{t \in [0, \frac{b}{m}]} |x(t)|, \quad i = 1, m - 1. \]

We define the following mapping $A_1 : M_1 \to M_1$ with $x \in M_1$

\[ A_1(x)(t) = \int_0^t K(t, s, x(s))ds, \quad t \in [0, \frac{b}{m}]. \]

Then for $x, y \in M_1$ and $0 \leq t \leq \frac{b}{m}$ we have

\[ |A_1(x)(t) - A_1(y)(t)| \leq \int_0^t |f(t, s, x(s)) - f(t, s, y(s))| ds \]

\[ \leq \frac{Lb_b}{m} \max_{t \in [0, \frac{b}{m}]} |x(t) - y(t)| \]

\[ \leq \frac{Lb_b}{m} \|x - y\|_1. \]

So,

\[ \max_{t \in [0, \frac{b}{m}]} |A_1(x)(t) - A_1(y)(t)| \leq \frac{Lb_b}{m} \|x - y\|_1. \]

Thus

\[ \|A_1(x) - A_1(y)\|_1 \leq \frac{Lb_b}{m} \|x - y\|_1. \]

The mapping $A_1$ is a contraction with a unique fixed point $x_1^*$ on $[0, \frac{b}{m}]$ with

\[ (A_1x_1^*)(t) = x_1^*(t) = \int_0^t K(t, s, x_1^*(s))ds, \quad 0 \leq t \leq \frac{b}{m}, \quad (2.1) \]

**Step 2.** Let $(M_2, \|\cdot\|_2)$ be the complete metric space of continuous functions $x : [0, \frac{2b}{m}] \to \mathbb{R}$ with the Chebyshev metric and

\[ x(t) = x_1^*(t) \quad \text{on} \quad \left[ 0, \frac{b}{m} \right]. \]
We define the mapping $A_2 : M_2 \to M_2$ with $x \in M_2$

$$A_2(x)(t) = \int_0^t K(t, s, x(s))ds.$$ 

Notice that for $0 \leq t \leq \frac{b}{m}$ and $x \in M_2$ then $x = x_1^*$ which is a fixed point and from (2.1) we have

$$\begin{cases} x_1^*(t), & t \in \left[0, \frac{b}{m}\right] \\ \int_0^t K(t, s, x(s))ds, & t \in \left[\frac{b}{m}, \frac{2b}{m}\right] \end{cases}$$

For $x, y \in M_2$ we have

$$|A_2(x)(t) - A_2(y)(t)| \leq \int_0^t |K(t, s, x(s)) - K(t, s, y(s))| ds$$

$$\leq \int_0^t L_{b,2} |x(s) - y(s)| ds$$

(and since $x(t) = y(t) = x_1^*(t)$ on $[0, \frac{b}{m}]$)

$$\leq \int_0^t L_{b,2} \frac{b}{m} \max_{t \in \left[\frac{b}{m}, \frac{2b}{m}\right]} |x(t) - y(t)|$$

$$\leq L_{b,2} \frac{b}{m} \max_{t \in [0, \frac{3b}{m}]} |x(t) - y(t)|.$$ 

So,

$$\max_{t \in [0, \frac{3b}{m}]} |A_2(x)(t) - A_2(y)(t)| \leq L_{b,2} \frac{b}{m} \max_{t \in [0, \frac{3b}{m}]} |x(t) - y(t)|.$$ 

Thus

$$\|A_2(x) - A_2(y)\|_2 \leq L_{b,2} \frac{b}{m} \|x - y\|_2.$$ 

The mapping $A_2$ is a contraction with a unique fixed point $x_2^*$ on $[0, \frac{3b}{m}]$. Clearly $x_2^*$ is a unique continuous solution of (2.1) with $x_2^*(t) = x_1^*(t)$ on $[0, \frac{b}{m}]$.

**Step 3.** We define the complete metric space $(M_3, \|\cdot\|_3)$ of continuous functions $x : [0, \frac{3b}{m}] \to \mathbb{R}$ with $x(t) = x_2^*(t)$ on $[0, \frac{b}{m}]$. But $x_2^*$ is a fixed point and so $A_3$ is well defined. Analogously we obtain a continuous solution $x_3^*$ on $[0, \frac{3b}{m}]$. 

As follows we get that $A_m$ is a contraction and thus we obtain a unique continuous solution on $[0, b]$, using the induction method.
For $2 < i < m - 1$ let $x^*_{i-1}$ be the unique solution of (2.1) on $[0, \frac{(i-1)b}{m}]$. Let $(M_i, \| \cdot \|_i)$ be the complete metric space of continuous functions $x : [0, \frac{ib}{m}] \to \mathbb{R}$ with the supremum metric and $x(t) = x^*_{i-1}(t)$ on $[0, \frac{(i-1)b}{m}]$. We define $A_i : M_i \to M_i$ by $x \in M_i$ imply

$$ (A_i x)(t) = \int_0^t K(t, s, x(s)) ds $$

$$ = \begin{cases} x^*_{i-1}(t), & t \in \left[0, \frac{(i-1)b}{m}\right] \\ \int_0^t K(t, s, x(s)) ds, & t \in \left[\frac{(i-1)b}{m}, \frac{ib}{m}\right] \end{cases} $$

To prove that $A_i$ is a contraction, let $x, y \in M_i$ and $0 \leq t \leq \frac{ib}{m}$ so that

$$ |A_i(x)(t) - A_i(y)(t)| \leq \int_{\frac{(i-1)b}{m}}^{t} |K(t, s, x(s)) - K(t, s, y(s))| ds $$

$$ \leq \int_{\frac{(i-1)b}{m}}^{t} L_{b,i} |x(s) - y(s)| ds $$

(and since $x(t) = y(t) = x^*_{i-1}(t)$ on $[0, \frac{(i-1)b}{m}]$)

$$ \leq \int_{\frac{(i-1)b}{m}}^{t} L_{b,i} |x(s) - y(s)| ds $$

$$ \leq L_{b,i} \frac{b}{m} \max_{t \in \left[\frac{(i-1)b}{m}, \frac{ib}{m}\right]} |x(t) - y(t)| $$

$$ \leq L_{b,i} \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)| $$

So,

$$ \max_{t \in [0, \frac{ib}{m}]} |A_i(x)(t) - A_i(y)(t)| \leq L_{b,i} \frac{b}{m} \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)|. $$

Thus

$$ \|A_i(x) - A_i(y)\|_i \leq L_{b,i} \frac{b}{m} \|x - y\|_i. $$

We obtain that $A_i$ is a contraction with the unique fixed point $x^*_{i}$ on $[0, \frac{ib}{m}]$. □
3. Applications of Burton method to functional integral equations

We consider the following integral equation

\[ x(t) = g(t, x(t)) + \int_0^t f(t, s, x(s))ds, \quad t \in [0, b) \]  \hspace{1cm} (3.1)

where the functions \( g \in C([0, b] \times \mathbb{R}, \mathbb{R}) \), \( f \in C([0, b] \times [0, b] \times \mathbb{R}, \mathbb{R}) \) are given. We search the solution in the set \( C([0, b], \mathbb{R}) \) for which we consider the Chebyshev norm.

We divide the interval \( [0, b] \) in \( m \) equal parts such that

\[ [0, b] = \bigcup_{k=0}^{m-1} \left[ \frac{kb}{m}, \frac{(k+1)b}{m} \right] , \quad m \in \mathbb{N}^*. \]  \hspace{1cm} (3.2)

We consider the following hypothesis:

\( (H_1) \) there exists \( L_g \in (0, 1) \) such that

\[ |g(t, u) - g(t, v)| \leq L_g |u - v|, \quad \forall u, v \in \mathbb{R}, \quad 0 \leq t < b; \]

\( (H_2) \) for each \( b > 0 \) there exists \( L_{f,k} > 0 \) such that

\[ |f(t, s, u) - f(t, s, v)| \leq L_{f,k}(b) |u - v|, \quad \forall u, v \in \mathbb{R}, \quad 0 \leq t \leq b; \]

**Theorem 3.1.** In the conditions \((H_1)\) and \((H_2)\) the equation \( (3.1) \) has a unique solution on \( C([0, b] \times \mathbb{R}, \mathbb{R}) \).

**Proof.** Following the same steps as in the proof of Theorem 2 and using \( (3.2) \) we divide the interval \( [0, b] \) into \( m \) equal parts, denoting the end points by \( 0, \frac{b}{m}, \frac{2b}{m}, \ldots, b \).

**Step 1.** Let \((M_1, \|\cdot\|_1)\) be the complete metric space of continuous functions \( x: [0, \frac{b}{m}] \rightarrow \mathbb{R} \) with the Chebyshev metric \( \|\cdot\|_1 \), where

\[ \|x(t)\|_i = \max_{t \in [0, \frac{b}{m}]} |x(t)|, \quad i = 1, m - 1. \]

We define the following mapping \( A_1: M_1 \rightarrow M_1 \) with \( x \in M_1 \)

\[ A_1(x)(t) = g(t, x(t)) + \int_0^t f(t, s, x(s))ds, \quad t \in [0, \frac{b}{m}]. \]

Then for \( x, y \in M_1 \) and \( 0 \leq t \leq \frac{b}{m} \) we have

\[ |A_1(x)(t) - A_1(y)(t)| \leq L_g |x(t) - y(t)| + \int_0^t |f(t, s, x(s)) - f(t, s, y(s))|ds \leq L_g \|x - y\|_1 + \frac{L_{f,1}(b)b}{m} \|x - y\|_1. \]

Thus

\[ \|A_1(x) - A_1(y)\|_1 \leq \left( L_g + L_{f,1}(b)\frac{b}{m} \right) \|x - y\|_1. \]

The mapping \( A_1 \) is a contraction with a unique fixed point \( x_*^1 \) on \([0, \frac{b}{m}]\) with

\[ (A_1 x_*^1)(t) = x_*^1(t) = g(t, x_*^1(t)) + \int_0^t f(t, s, x_*^1(s))ds, \quad 0 \leq t \leq \frac{b}{m}. \]  \hspace{1cm} (3.3)
Step 2. Let \((M_2, \| \cdot \|_2)\) be the complete metric space of continuous functions 
\(x : [0, \frac{2b}{m}] \to \mathbb{R}\) with the Chebyshev metric and
\[ x(t) = x_1^*(t) \text{ on } \left[0, \frac{b}{m}\right]. \]

We define the mapping \(A_2 : M_2 \to M_2\) with \(x \in M_2\)
\[ A_2(x)(t) = g(t, x(t)) + \int_0^t f(t, s, x(s))ds. \]

Notice that for \(0 \leq t \leq \frac{b}{m}\) and \(x \in M_2\) then \(x = x_1^*\) which is a fixed point and from (3.3) we have
\[ (A_2x)(t) = \begin{cases} 
  x_1^*(t), & t \in \left[0, \frac{b}{m}\right] \\
  g(t, x(t)) + \int_0^t f(t, s, x(s))ds, & t \in \left[\frac{b}{m}, \frac{2b}{m}\right] 
\end{cases} \]
\[ = \begin{cases} 
  x_1^*(t), & t \in \left(0, \frac{b}{m}\right] \\
  g(t, x(t)) + \int_0^t f(t, s, x_1^*(s))ds + \int_t^{\frac{b}{m}} f(t, s, x(s))ds, & t \in \left[\frac{b}{m}, \frac{2b}{m}\right] 
\end{cases} \]

For \(x, y \in M_2\) we have
\[ |A_2(x)(t) - A_2(y)(t)| \leq L_g |x(t) - y(t)| + \int_{\frac{b}{m}}^t |f(t, s, x(s)) - f(t, s, y(s))|ds \]
\[ \leq \left(L_g + L_{f,2}(b) \frac{b}{m}\right) \max_{t \in [0, \frac{2b}{m}]} |x(t) - y(t)|. \]

Thus
\[ \|A_1(x) - A_1(y)\|_2 \leq \left(L_g + L_{f,2}(b) \frac{b}{m}\right) \|x - y\|_2. \]

The mapping \(A_2\) is a contraction with a unique fixed point \(x_2^*\) on \([0, \frac{2b}{m}]\). Clearly \(x_2^*\) is a unique continuous solution of (3.1) with \(x_2^*(t) = x_1^*(t)\) on \([0, \frac{b}{m}]\).

Step 3. We define the complete metric space \((M_3, \| \cdot \|_3)\) of continuous functions 
\(x : [0, \frac{3b}{m}] \to \mathbb{R}\) with \(x(t) = x_2^*\) on \([0, \frac{2b}{m}]\). But \(x_2^*\) is a fixed point and so \(A_3\) is well defined. Analogously we obtain a continuous solution \(x_4^*\) on \([0, \frac{3b}{m}]\).

By induction we get a unique continuous solution on \([0, b]\). We give below some induction details. For \(2 < i < m - 1\) let \(x_{i-1}^*\) be the unique solution of (3.1) on \([0, \frac{(i-1)b}{m}]\). Let \((M_i, \| \cdot \|_i)\) be the complete metric space of continuous functions \(x : [0, \frac{ib}{m}] \to \mathbb{R}\) with the supremum metric and \(x(t) = x_{i-1}^*(t)\) on \([0, \frac{(i-1)b}{m}]\). We define \(A_i : M_i \to M_i\).
by \( x \in M_i \) imply
\[
(A_i x)(t) = g(t, x(t)) + \int_0^t f(t, s, x(s)) \, ds
\]
\[
= \begin{cases}
  x^*_{i-1}(t), & t \in \left[0, \frac{(i-1)b}{m}\right] \\
  g(t, x(t)) + \int_0^t f(t, s, x(s)) \, ds, & t \in \left[\frac{(i-1)b}{m}, \frac{ib}{m}\right] \\
  x^*_{i-1}(t), & t \in \left[0, \frac{(i-1)b}{m}\right]
\end{cases}
\]
\[
= g(t, x(t)) + \int_0^{(i-1)b/m} f(t, s, x^*_{i-1}(s)) \, ds + \int_{(i-1)b/m}^t f(t, s, x(s)) \, ds,
\]
\[
t \in \left[\frac{(i-1)b}{m}, \frac{ib}{m}\right].
\]
To prove that \( A_i \) is a contraction, let \( x, y \in M_i \) and \( 0 \leq t \leq \frac{ib}{m} \) so that
\[
|A_i(x)(t) - A_i(y)(t)| \leq L_g |x(t) - y(t)| + \int_0^t |f(t, s, x(s)) - f(t, s, y(s))| \, ds
\]
\[
\leq \left( L_g + L_{f,i} \frac{b}{m} \right) \max_{t \in [0, \frac{ib}{m}]} |x(t) - y(t)|.
\]
Thus
\[
\|A_i(x) - A_i(y)\|_i \leq \left( L_g + L_{f,i} \frac{b}{m} \right) \|x - y\|_i.
\]
We obtain that \( A_i \) is a contraction with the unique fixed point \( x^*_i \) on \([0, \frac{ib}{m}]\). \( \square \)

**Remark.** Since on each subinterval we apply the contraction principle, the unique solution of the problem can be obtained on each subinterval using successive approximation method, see [9, 11].

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**References**


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