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VISCOSITY METHOD WITH A ϕ-CONTRACTION MAPPING FOR HIERARCHICAL VARIATIONAL INEQUALITIES ON HADAMARD MANIFOLDS

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Abstract. In this paper, we propose the viscosity method for solving variational inequality problems defined over a set of fixed points of a nonexpansive mapping and involving a ϕ-contraction mapping and another nonexpansive mapping in the setting of Hadamard manifolds. Several special cases of such a variational inequality problem are also considered. The convergence analysis of the proposed method is studied. We illustrate proposed algorithm and convergence result by a numerical example. The algorithms and convergence results of this paper extend and improve several known algorithms and results from linear structure to Hadamard manifolds.
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1. Introduction

The viscosity method for solving variational inequalities was first introduced by Moudafi [18] in the setting of Hilbert spaces which is further extended by Xu [29] in the frame work of uniformly smooth Banach spaces. In the viscosity method, Moudafi [18] and Xu [29] considered a contraction mapping which is involved in the formulation of the variational inequality. Several researchers replaced contraction mapping by some weak form of contraction mappings, namely, pseudo-contraction mapping, weakly contraction mapping and developed the viscosity method for solving variational inequalities defined over the set of fixed points of a nonexpansive mapping in the setting of Hilbert spaces or Banach spaces, see, e.g., [18, 17, 29, 28, 30, 27] and the references therein.

During the last decades, the study of variational inequalities, variational inclusions and optimization problems over manifolds is emerged as promissing, interesting and hot topic for researchers, see, e.g., [2, 3, 4, 5, 7, 9, 10, 12, 13, 14, 15, 16, 19, 20, 21, 22, 25, 11] and the references therein. Recently, Huang [13] extended viscosity method with a weak contraction mapping for solving variational inequalities in the setting of Hadamard manifolds and discussed its convergence criterion.

The main purpose of this work is to give a viscosity method with a ϕ-contraction mapping for solving variational inequalities defined over the set of fixed points of a nonexpansive mapping in the setting of Hadamard manifolds.

The following paper is organized as follows: In the next section, we collect several basic definitions, terminologies and results from manifolds which are needed in the sequel. In Section 3, we give the formulation of a variational inequality problem defined over the set of fixed points of a nonexpansive mapping and involving a ϕ-contraction mapping and another nonexpansive mapping in the setting of Hadamard manifolds. Several special cases, namely, variational inequality problems defined over the set of zeros of a set-valued monotone vector field, Moreau-Yosida regularization problems of our variational inequality problem are also considered. Section 4 proposes a viscosity method for solving considered variational inequality problem and studies convergence analysis of the proposed method. Several special cases are also discussed. In the last section, we illustrate the proposed viscosity method and convergence result by a numerical example. The algorithms and results of the paper extended and improve several known results from the setting of linear structure to Hadamard manifolds.

2. Preliminaries

In this section, we recall some known notions, definitions and results from manifold, which can be found in any standard book on manifolds, see, e.g., [23, 11, 25].

Let $\mathbb{M}$ be a finite dimensional differentiable manifold. For each $x \in \mathbb{M}$, let $T_x\mathbb{M}$ be the tangent space at x, which is a real vector space of the same dimension as $\mathbb{M}$. The collection of all tangent spaces on $\mathbb{M}$ is called a tangent bundle and it is denoted by $T\mathbb{M}$. A single-valued vector field on a manifold $\mathbb{M}$ is a C^∞ mapping $A : \mathbb{M} \to T\mathbb{M}$ such that for each $x \in \mathbb{M}$, it assigns a tangent vector $A(x) \in T_x\mathbb{M}$. A smooth mapping $\langle \cdot, \cdot \rangle : T\mathbb{M} \times T\mathbb{M} \to \mathbb{R}$ is said to be a Riemannian metric on $\mathbb{M}$ if $\langle \cdot, \cdot \rangle_x : T_x\mathbb{M} \times T_x\mathbb{M} \to \mathbb{R}$ is an inner product for all $x \in \mathbb{M}$. We denote by $\| \cdot \|_x$ the corresponding norm to the inner product $\langle \cdot, \cdot \rangle_x$ on $T_x\mathbb{M}$. We omit the subscript x if no confusion arises. A differentiable manifold $\mathbb{M}$ endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$ is called a Riemannian manifold.

The length of a piecewise smooth curve $\gamma : [a, b] \to \mathbb{M}$ joining $x = \gamma(a)$ to $y = \gamma(b)$ in $\mathbb{M}$ is given by

$$L(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt,$$

where $\dot{\gamma}(t)$ denotes the tangent vector at $\gamma(t)$ in the tangent space $T_{\gamma(t)}\mathbb{M}$. The minimal length of all such curves joining x to y is called a Riemannian distance and it is denoted by $d(x, y)$.

Let ∇ be the Levi-Civita connection associated with the Riemannian manifold $\mathbb{M}$. A smooth vector field X along γ is said to be parallel if $\nabla_{\dot{\gamma}(t)} X = \mathbf{0}$, where $\mathbf{0}$ denotes the zero tangent vector. If $\dot{\gamma}$ is parallel along γ, i.e., $\nabla_{\dot{\gamma}(t)}\dot{\gamma}(t) = \mathbf{0}$, then γ is said to be a geodesic and in this case $\|\dot{\gamma}(t)\|$ is a constant. In addition, if $\|\dot{\gamma}(t)\| = 1$, then γ is called a normalized geodesic. A geodesic joining x to y in a Riemannian manifold $\mathbb{M}$ is said to be a minimal geodesic if its length is equal to $d(x, y)$.

A Riemannian manifold $\mathbb{M}$ is said to be complete if for any $x \in \mathbb{M}$, all geodesics emanating from x are defined for all $t \in \mathbb{R}$. By Hopf-Rinow Theorem [23], any pair of points in a complete Riemannian manifold $\mathbb{M}$ can be joined by a minimal geodesic; moreover, $(\mathbb{M}, d)$ is a complete metric space. If $\mathbb{M}$ is a complete Riemannian manifold, then the exponential mapping $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ at $x \in \mathbb{M}$ is defined by

$$\exp_x v = \gamma_v(1; x), \quad \forall v \in T_x\mathbb{M},$$

where $\gamma_v(\cdot; x)$ is the geodesic starting from x with velocity v, i.e., $\gamma_v(0; x) = x$ and $\dot{\gamma}_v(0; x) = v$. It is known that $\exp_x tv = \gamma_v(t; x)$ for any real number t, and $\exp_x \mathbf{0} = \gamma_v(0; x) = x$. Note that the exponential mapping $\exp_x$ is differentiable on $T_x\mathbb{M}$ for any $x \in \mathbb{M}$. It is well-known that the derivative $\mathrm{D}\exp_x(\mathbf{0})$ of $\exp_x(\mathbf{0})$ is equal to the identity vector of $T_x\mathbb{M}$. Therefore, by the inverse mapping theorem, there exists an inverse exponential mapping $\exp_x^{-1} : \mathbb{M} \to T_x\mathbb{M}$. Moreover, for any $x, y \in \mathbb{M}$, we have $d(x, y) = \|\exp_x^{-1} y\|$. For further details, we refer to [23].

A complete simply connected Riemannian manifold with nonpositive sectional curvature is called a Hadamard manifold.

The rest of the paper, unless otherwise specified, we assume that $\mathbb{M}$ is a finite dimensional Hadamard manifold.

Li et al. [14] derived some properties of the exponential mapping in the setting of Hadamard manifolds.

Lemma 2.1. [14] *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Hadamard manifold $\mathbb{M}$ such that $x_n \to \tilde{x} \in \mathbb{M}$. Then the following assertions hold.*

(a) *For any $y \in \mathbb{M}$, we have*

$$\exp_{x_n}^{-1} y \to \exp_{\tilde{x}}^{-1} y \ \textit{and} \ \exp_y^{-1} x_n \to \exp_y^{-1} \tilde{x}.$$

(b) *If $u_n \in T_{x_n}\mathbb{M}$ and $u_n \to \tilde{u}$, then $\tilde{u} \in T_{\tilde{x}}\mathbb{M}$.*

(c) *Given $u_n, v_n \in T_{x_n}\mathbb{M}$ and $\tilde{u}, \tilde{v} \in T_{\tilde{x}}\mathbb{M}$, if $u_n \to \tilde{u}$ and $v_n \to \tilde{v}$, then*

$$\langle u_n, v_n\rangle \to \langle \tilde{u}, \tilde{v}\rangle.$$

Proposition 2.2. [23] *For any x in a Hadamard manifold $\mathbb{M}$, the exponential mapping $\exp_x : T_x\mathbb{M} \to \mathbb{M}$ is a diffeomorphism, and for any two points $x, y \in \mathbb{M}$, there exists a unique normalized geodesic $\gamma : [0, 1] \to \mathbb{M}$ joining $x = \gamma(0)$ to $y = \gamma(1)$ which is in fact a minimal geodesic defined by*

$$\gamma(t) = \exp_x t \exp_x^{-1} y, \quad \forall t \in [0, 1].$$

A subset C of a Riemannian manifold $\mathbb{M}$ is said to be geodesic convex if for any two points x and y in C, any geodesic joining x to y is contained in C, i.e., if for all $a, b \in \mathbb{R}$, $\gamma : [a, b] \to \mathbb{M}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma(at + (1-t)b) \in C$ for all $t \in [0, 1]$.

A function $f : \mathbb{M} \to \mathbb{R}$ is said to be geodesic convex if for any geodesic $\gamma : [a, b] \to \mathbb{M}$, the composition function $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(at + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b), \quad \forall t \in [0, 1] \text{ and } \forall a, b \in \mathbb{R}.$$

Proposition 2.3. [23] *The Riemannian distance $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ is a geodesic convex function with respect to the product Riemannian metric, i.e., for any pair of geodesics $\gamma_1 : [0, 1] \to \mathbb{M}$ and $\gamma_2 : [0, 1] \to \mathbb{M}$,*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)), \quad \forall t \in [0, 1].$$

In particular, for each $x \in \mathbb{M}$, the function $d(\cdot, x) : \mathbb{M} \to \mathbb{R}$ is a geodesic convex function.

We now mention some geometric properties of finite dimensional Hadamard manifold $\mathbb{M}$ which are similar to the setting of Euclidean space $\mathbb{R}^n$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a Riemannian manifold $\mathbb{M}$ is a set consisting of three points x_1, x_2 and x_3, and three minimal geodesics γ_i joining x_i to x_{i+1}, where $i = 1, 2, 3 \pmod 3$.

Proposition 2.4. [23] *Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$. For each $i = 1, 2, 3 \pmod 3$, let $\gamma_i : [0, l_i] \to \mathbb{M}$ be the geodesic joining x_i to x_{i+1}, $l_i = L(\gamma_i)$ and α_i be the angle between tangent vectors $\dot{\gamma}_i(0)$ and $-\dot{\gamma}_{i-1}(l_{i-1})$. Then*

(a) $\alpha_1 + \alpha_2 + \alpha_3 \leq \pi$;
(b) $l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2$.

As in [14], Proposition 2.4 (b) can be written in terms of Riemannian distance and exponential mapping as

$$d^2(x_i, x_{i+1}) + d^2(x_{i+1}, x_{i+2}) - 2\left\langle \exp^{-1}_{x_{i+1}} x_i, \exp^{-1}_{x_{i+1}} x_{i+2} \right\rangle \leq d^2(x_{i-1}, x_i), \tag{2.1}$$

since

$$\left\langle \exp^{-1}_{x_{i+1}} x_i, \exp^{-1}_{x_{i+1}} x_{i+2} \right\rangle = d(x_i, x_{i+1}) d(x_{i+1}, x_{i+2}) \cos \alpha_{i+1}.$$

For further details, we refer to [12].

Lemma 2.5. [7, p. 24] (see also [15]) *Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$. Then there exists $x', y', z' \in \mathbb{R}^2$ such that*

$$d(x, y) = \|x' - y'\|, \quad d(y, z) = \|y' - z'\| \quad \text{and} \quad d(x, z) = \|x' - z'\|.$$

The triangle $\Delta(x', y', z')$ is called the comparison triangle of the geodesic triangle $\Delta(x, y, z)$, which is unique up to isometry of $\mathbb{M}$. The points x', y', z' are called comparison points to the points x, y, z, respectively.

Lemma 2.6. [15] *Let $\Delta(x, y, z)$ be a geodesic triangle in a Hadamard manifold $\mathbb{M}$ and $\Delta(x', y', z')$ be its comparison triangle.*

(a) *Let* $\alpha_1, \alpha_2, \alpha_3$ *(respectively,* $\alpha_1', \alpha_2', \alpha_3'$*) be the angles of* $\Delta(x,y,z)$ *(respectively,* $\Delta(x',y',z')$*) at the vertices* x, y, z *(respectively,* x', y', z'*). Then*

$$\alpha_1' \geq \alpha_1, \quad \alpha_2' \geq \alpha_2 \quad \textit{and} \quad \alpha_3' \geq \alpha_3.$$

(b) *Let* w *be a point on the geodesic joining* x *to* y *and* w' *its comparison point in the interval* $[x', y']$*. If* $d(w,x) = \|w' - x'\|$ *and* $d(w,y) = \|w' - y'\|$*, then* $d(w,z) \leq \|w' - z'\|$*.*

The projection of a point $x \in \mathbb{M}$ onto a closed geodesic convex subset C of a Hadamard manifold $\mathbb{M}$ is defined by

$$P_C(x) = \{z \in C : d(x,z) \leq d(x,y), \forall y \in C\}.$$

Proposition 2.7. [26] *Let* C *be a closed geodesic convex subset of a Hadamard manifold* $\mathbb{M}$*. Then for any* $x \in \mathbb{M}$*,* $P_C(x)$ *is a singleton set. Also, for any point* $x \in \mathbb{M}$*, the following assertions are equivalent:*

(a) $y = P_C(x)$*;*
(b) $\left\langle \exp_y^{-1} x, \exp_y^{-1} z \right\rangle \leq 0, \ \forall z \in C$*.*

Definition 2.8. [16] A mapping $T : \mathbb{M} \to \mathbb{M}$ is said to be firmly nonexpansive if for any $x, y \in \mathbb{M}$, the function $\varphi : [0,1] \to [0, +\infty)$ defined by

$$\varphi(t) := d\left(\exp_x t \exp_x^{-1} T(x), \exp_y t \exp_y^{-1} Ty\right), \quad \forall t \in [0,1],$$

is nonincreasing.

Li et al. [16] proved that every firmly nonexpansive mapping is nonexpansive.

Definition 2.9. A mapping $f : \mathbb{M} \to \mathbb{M}$ is said to be ϕ-contraction [6] if

$$d(f(x), f(y)) \leq \phi(d(x,y)), \quad \forall x, y \in \mathbb{M},$$

where $\phi : [0, +\infty) \to [0, +\infty)$ is a function that satisfies the following conditions:

(i) $\phi(t) < t$ for all $t > 0$;
(ii) ϕ is continuous.

Remark 2.10. (a) $\phi(t) = \frac{t}{1+t}$ for all $t \geq 0$ satisfies the conditions (i) - (ii).
(b) If $\phi(t) = kt$ for all $t \geq 0$, where $k \in (0,1)$, then f is a contraction mapping with Lipschitz constant k.
(c) Note that ϕ-contraction mappings are nonexpansive.

Note that the ϕ-contraction mappings were first introduced by Boyd and Wong [6]. They discussed the existence of a unique fixed point of a ϕ-contraction mapping in the setting of a complete metric space.

Theorem 2.11. [6, Theorem 1] *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a* ϕ*-contraction mapping, where* ϕ *is upper semicontinuous and satisfies* $\phi(t) < t$ *for all* $t > 0$*. Then* f *has a unique fixed point.*

Lemma 2.12. [1] *Let $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that $\lim\limits_{n\to\infty}\frac{\beta_n}{\mu_n}=0$ and $\sum\limits_{n=1}^{\infty}\mu_n=+\infty$. Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers satisfying the recursive inequality:*

$$w_{n+1}\le w_n-\mu_n\psi(w_n)+\beta_n,\ \forall n\in\mathbb{N},$$

where $\psi:[0,+\infty)\to[0,+\infty)$ is a continuous and nondecreasing function such that $\psi(0)=0$ and $\psi(s)>0$ for all $s>0$. Then $\lim\limits_{n\to\infty}w_n=0$.

Remark 2.13. Let $\phi:[0,+\infty)\to[0,+\infty)$ be a continuous and nonincreasing function such that $\phi(0)=0$ and $\phi(t)<t$ for all $t>0$. Then the function $\psi:[0,+\infty)\to[0,+\infty)$ defined by

$$\psi(t)=t-\phi(t),\quad \forall t\ge 0, \tag{2.2}$$

is continuous and nondecreasing such that $\psi(0)=0$ and $\psi(t)>0$ for all $t>0$.

Indeed, the continuity of ψ follows from (2.2) and continuity of ϕ. Also, $\phi(0)=0$ implies $\psi(0)=0$ and $\phi(t)<t$ gives $\psi(t)>0$ for all $t>0$. Furthermore, since ϕ is nonincreasing, for any $t_1\le t_2$, we have $-\phi(t_1)\le-\phi(t_2)$. Therefore, $t_1-\phi(t_1)\le t_2-\phi(t_2)$, and hence, $\psi(t_1)\le\psi(t_2)$. This confirms that ψ is nondecreasing.

Therefore, by replacing the function $\psi(t)=t-\phi(t)$ in Lemma 2.12, we obtain the following result.

Lemma 2.14. *Let $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that $\lim\limits_{n\to\infty}\frac{\beta_n}{\mu_n}=0$ and $\sum\limits_{n=1}^{\infty}\mu_n=+\infty$. Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers satisfying the recursive inequality:*

$$w_{n+1}\le(1-\mu_n)w_n+\mu_n\phi(w_n)+\beta_n,\quad\forall n\in\mathbb{N}, \tag{2.3}$$

where $\phi:[0,+\infty)\to[0,+\infty)$ is a continuous and nonincreasing function such that $\phi(0)=0$ and $\phi(s)<s$ for all $s>0$. Then $\lim\limits_{n\to\infty}w_n=0$.

Proof. Let $\psi(t)=t-\phi(t)$, then by Remark 2.13, all the assumptions of Lemma 2.12 are satisfied. Therefore, from (2.3), we have

$$w_{n+1}\le w_n-\mu_n\psi(w_n)+\beta_n,\quad\forall n\in\mathbb{N},$$

which implies from Lemma 2.12 that $\lim\limits_{n\to\infty}w_n=0$. □

3. Formulation of problems

Throughout the paper, we denote by $\Omega(\mathbb{M})$ the set of all single-valued vector fields $A:\mathbb{M}\to T\mathbb{M}$ such that $A(x)\in T_x\mathbb{M}$ for each $x\in\mathbb{M}$.

Definition 3.1. [20] A single-valued vector field $A\in\Omega(\mathbb{M})$ is said to be monotone if

$$\left\langle A(x),\exp_x^{-1}y\right\rangle\le\left\langle A(y),-\exp_y^{-1}x\right\rangle,\quad\forall x,y\in\mathbb{M}.$$

Let $\mathcal{X}(\mathbb{M})$ be the set of all set-valued vector fields $V:\mathbb{M}\rightrightarrows T\mathbb{M}$ such that $V(x)\subseteq T_x\mathbb{M}$ for each $x\in\mathbb{M}$. The domain $D(V)$ of V is defined by

$$D(V)=\{x\in\mathbb{M}:V(x)\ne\emptyset\}.$$

Definition 3.2. [19] Let $T : \mathbb{M} \rightrightarrows \mathbb{M}$ be a set-valued mapping. The complementary vector field $V \in \mathcal{X}(\mathbb{M})$ of T is defined by

$$V(x) = -\exp_x^{-1} T(x), \quad \forall x \in \mathbb{M},$$

where $\exp_x^{-1} T(x) = \left\{\exp_x^{-1} y \in T_x\mathbb{M} : y \in T(x)\right\}$ for all $x \in \mathbb{M}$.

Theorem 3.3. [19] *If $T : \mathbb{M} \to \mathbb{M}$ is a nonexpansive mapping, then its complementary vector field V is monotone.*

We now recall the definition of resolvent associated with a set-valued vector field and some of its well-known properties.

Definition 3.4. [16] For a given $\mu > 0$ and a set-valued vector field $V \in \mathcal{X}(\mathbb{M})$, the resolvent of the vector field V of order μ is a set-valued mapping $R_\mu^V : \mathbb{M} \rightrightarrows D(V)$ defined by

$$R_\mu^V(x) := \{z \in \mathbb{M} : x \in \exp_z \mu V(z)\}, \quad \forall x \in \mathbb{M}.$$

Remark 3.5. [16] For $\mu > 0$, the range of resolvent R_μ^V is contained in the domain of V and

$$\mathrm{Fix}(R_\mu^V) = V^{-1}(\mathbf{0}).$$

Definition 3.6. [9] A set-valued vector field $V \in \mathcal{X}(\mathbb{M})$ is said to be monotone if for any $x, y \in D(V)$,

$$\left\langle u, \exp_x^{-1} y\right\rangle \leq \left\langle v, -\exp_y^{-1} x\right\rangle, \quad \forall u \in V(x) \text{ and } \forall v \in V(y).$$

Lemma 3.7. [16] *A set-valued vector field $V \in \mathcal{X}(\mathbb{M})$ is monotone if and only if R_μ^V is single-valued and firmly nonexpansive.*

Let $\mathbb{M}$ be a Hadamard manifold, $f : \mathbb{M} \to \mathbb{M}$ be a ϕ-contraction mapping and $S, T : \mathbb{M} \to \mathbb{M}$ be nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$, where $\mathrm{Fix}(T)$ denotes the set of all fixed points of T. We consider the following variational inequality problem associated with f, T, S and $\mathrm{Fix}(T)$: Find $\bar{x} \in \mathrm{Fix}(T)$ such that

$$\left\langle \exp_{\bar{x}}^{-1} S(\bar{x}) + \frac{1}{\sigma} \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} x \right\rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{3.1}$$

where $\sigma \in (0, +\infty)$. We denote by $\mathbb{S}$ the solution set of the problem (3.1). It is considered and studied by Yao et al. [30] in the framework of Hilbert spaces by considering f to be contraction and S, T to be nonexpansive. In particular, if $S \equiv I$ the identity mapping, then problem (3.1) becomes the following problem:

$$\text{Find } \bar{x} \in \mathrm{Fix}(T) \text{ such that } \left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} x\right\rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{3.2}$$

Moudafi [18] considered and studied problem (3.2) in the framework of Hilbert spaces, and introduced the viscosity approximation method for finding its solution where f is a contraction mapping. Later, Xu [28] considered f to be a nonexpansive mapping and extended the work of Moudafi [18] in the setting of uniformly smooth Banach spaces.

Since T is nonexpansive, by [2], $\mathrm{Fix}(T)$ is a closed and geodesic convex subset of the Hadamard manifold $\mathbb{M}$. If $\mathrm{Fix}(T) \neq \emptyset$, then by Proposition 2.7, problem (3.2) can be equivalently written as the following fixed point problem:

$$\text{Find } \bar{x} \in \mathrm{Fix}(T) \quad \text{such that} \quad \bar{x} = P_{\mathrm{Fix}(T)} f(\bar{x}), \tag{3.3}$$

where $P_{\mathrm{Fix}(T)}$ is the metric projection onto the closed geodesic convex set $\mathrm{Fix}(T)$ in the Hadamard manifold $\mathbb{M}$.

Moreover, if $f \equiv I$ the identity mapping, then the problem (3.1) reduces to following variational inequality problem involving nonexpansive mappings T and S:

$$\text{Find } \bar{x} \in \mathrm{Fix}(T) \text{ such that } \left\langle \exp_{\bar{x}}^{-1} S(\bar{x}), \exp_{\bar{x}}^{-1} x \right\rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{3.4}$$

Since $\mathrm{Fix}(T)$ is nonempty closed and geodesic convex, again by using Proposition 2.7, problem (3.4) is equivalently written as the following fixed point problem:

$$\text{Find } \bar{x} \in \mathrm{Fix}(T) \quad \text{such that} \quad \bar{x} = P_{\mathrm{Fix}(T)} S(\bar{x}). \tag{3.5}$$

We also consider the following special cases of the problem (3.2).

Variational inequality problem over the set of zeros of a set-valued monotone vector field: Let $V : \mathbb{M} \rightrightarrows T\mathbb{M}$ be a monotone set-valued vector field. By Lemma 3.7, R_λ^V is single-valued, and by Remark 3.5, $V^{-1}(\mathbf{0}) = \mathrm{Fix}(R_\lambda^V)$ for all $\lambda > 0$. By considering $\mathrm{Fix}(T) = V^{-1}(\mathbf{0})$ in problem (3.2), we get the following variational inequality problem over the set of zeros of a set-valued monotone vector field:

$$\text{Find } \bar{x} \in V^{-1}(\mathbf{0}) \text{ such that } \left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle \leq 0, \quad \forall y \in V^{-1}(\mathbf{0}). \tag{3.6}$$

Hierarchical minimization problem: Let $\mathcal{G} : \mathbb{M} \to (-\infty, +\infty]$ be a proper, lower semicontinuous and geodesic convex function on a Hadamard manifold $\mathbb{M}$. The domain of the function $\mathcal{G}$, denoted by $D(\mathcal{G})$, is defined by

$$D(\mathcal{G}) := \{x \in \mathbb{M} : \mathcal{G}(x) \neq +\infty\}.$$

The directional derivative of the function $\mathcal{G}$ at x in direction $u \in T_x\mathbb{M}$ is given by

$$\mathcal{G}^D(x; u) := \lim_{t \to 0^+} \frac{\mathcal{G}(\exp_x tu) - \mathcal{G}(x)}{t}.$$

The gradient $\nabla \mathcal{G}$ of $\mathcal{G}$ at $x \in \mathbb{M}$ [11] is defined by $\langle \nabla \mathcal{G}(x), u \rangle := \mathcal{G}^D(x; u)$ for all $u \in T_x\mathbb{M}$. It is well known [25] that

$$\bar{x} \in \operatorname*{argmin}_{x \in \mathbb{M}} \mathcal{G}(x) \;\Leftrightarrow\; \nabla \mathcal{G}(\bar{x}) = \mathbf{0}, \tag{3.7}$$

where $\operatorname*{argmin}_{x \in \mathbb{M}} \mathcal{G}(x) = \{x \in \mathbb{M} : \mathcal{G}(x) \leq \mathcal{G}(y), \forall y \in \mathbb{M}\}$ is a set of minimizers of $\mathcal{G}$.

Proposition 3.8. [22] *Let $\mathbb{M}$ be a Riemannian manifold and $\mathcal{G} : \mathbb{M} \to \mathbb{R}$ be a differentiable function. Then $\mathcal{G}$ is geodesic convex if and only if $\nabla \mathcal{G}$ is a monotone vector field.*

The hierarchical minimization problem is to find $\bar{x} \in \mathrm{Fix}(T)$ such that

$$\bar{x} \in \operatorname*{argmin}_{x \in \mathrm{Fix}(T)} \mathcal{G}(x). \tag{3.8}$$

Let $f(z) = u$ for some $u \in \mathbb{M}$ and for all $z \in \mathrm{Fix}(T)$ and define the objective function $\mathcal{G}$ by

$$\mathcal{G}(x) := \frac{1}{2}d^2(u, x), \quad \forall x \in \mathbb{M}, \tag{3.9}$$

where $f : \mathbb{M} \to \mathbb{M}$ is a ϕ-contraction mapping. Note that $\mathcal{G}$ is geodesic convex by Proposition 2.3.

Proposition 3.9. *The element $\bar{x}$ is a solution of problem* (3.2) *if and only if it is a solution of the following hierarchical minimization problem:*

$$\bar{x} \in \underset{x \in \mathrm{Fix}(T)}{\operatorname{argmin}} \frac{1}{2}d^2(u, x), \tag{3.10}$$

where $f(z) = u$ for some $u \in \mathbb{M}$ and for all $z \in \mathrm{Fix}(T)$.

Proof. Let $\bar{x} \in \mathrm{Fix}(T)$ be a solution of the problem (3.10). Then

$$\frac{1}{2}d^2(u, \bar{x}) \leq \frac{1}{2}d^2(u, y), \quad \forall y \in \mathrm{Fix}(T).$$

Since $u = f(z)$ for all $z \in \mathrm{Fix}(T)$, therefore, for $u = f(\bar{x})$, we have

$$\frac{1}{2}d^2(f(\bar{x}), \bar{x}) \leq \frac{1}{2}d^2(f(\bar{x}), y), \quad \forall y \in \mathrm{Fix}(T).$$

By the definition of metric projection on a Hadamard manifold, we obtain

$$\bar{x} = P_{\mathrm{Fix}(T)} f(\bar{x}).$$

It follows from Proposition 2.7 that $\bar{x}$ is solution of the problem (3.2).
Conversely, let $\bar{x}$ be a solution of the problem (3.2), that is, $\bar{x} \in \mathrm{Fix}(T)$ such that

$$\left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle \leq 0, \quad \forall y \in \mathrm{Fix}(T). \tag{3.11}$$

Consider a geodesic triangle $\Delta(f(\bar{x}), \bar{x}, y)$. Then by inequality (2.1), we have

$$d^2(f(\bar{x}), \bar{x}) + d^2(\bar{x}, y) - 2\left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle \leq d^2(f(\bar{x}), y). \tag{3.12}$$

By combining (3.11) and (3.12), we obtain

$$\frac{1}{2}d^2(f(\bar{x}), \bar{x}) + \frac{1}{2}d^2(\bar{x}, y) \leq \frac{1}{2}d^2(f(\bar{x}), y),$$

and so,

$$\frac{1}{2}d^2(f(\bar{x}), \bar{x}) \leq \frac{1}{2}d^2(f(\bar{x}), y), \quad \forall y \in \mathrm{Fix}(T),$$

by replacing $u = f(\bar{x})$, we have

$$\frac{1}{2}d^2(u, \bar{x}) \leq \frac{1}{2}d^2(u, y), \quad \forall y \in \mathrm{Fix}(T),$$

that is, $\bar{x}$ is a solution of the hierarchical minimization problem (3.10). $\square$

Moreau-Yosida regularization and hierarchical minimization problem: The Moreau-Yosida regularization $\mathrm{prox}_{\lambda\Theta}(x) : \mathbb{M} \to \mathbb{M}$ of a geodesic convex function $\Theta : \mathbb{M} \to \mathbb{R}$ on a Hadamard manifold $\mathbb{M}$ is defined as follows:

$$\mathrm{prox}_{\lambda\Theta}(x) := \underset{z \in \mathbb{M}}{\operatorname{argmin}} \left\{ \Theta(z) + \frac{1}{2\lambda}d^2(x, z) : z \in \mathbb{M} \right\}, \quad \forall x \in \mathbb{M}.$$

Ferreira and Oliveira [12] showed that $\operatorname{prox}_{\lambda\Theta}$ is a single-valued mapping with $D(\operatorname{prox}_{\lambda\Theta}) = \mathbb{M}$. Moreover, $\operatorname{prox}_{\lambda\Theta}$ is also a firmly nonexpansive mapping (see, [16, Example 1]), and so $\operatorname{prox}_{\lambda\Theta}$ is nonexpansive. If $\Theta : \mathbb{M} \to (-\infty, +\infty]$ be proper, lower semicontinuous and geodesic convex function then the fixed point of Moreau-Yosida regularization has a relation (see [5, Lemma 3.2]), that is, $\operatorname{argmin}_{\mathbb{M}} \Theta = \operatorname{Fix}(\operatorname{prox}_{\lambda\Theta})$.

Since every ϕ-contraction mapping is nonexpansive, by Theorem 3.3, the complimentary vector field of ϕ-contraction mapping is monotone. Also, the gradient of a geodesic convex function is monotone. Therefore, without loss of generality we can consider a ϕ-contraction mapping $f : \mathbb{M} \to \mathbb{M}$ and a geodesic convex differentiable function $g : \mathbb{M} \to \mathbb{R}$ such that $\nabla g = -\exp_{\cdot}^{-1} f$. Then, we obtain the following result in which the problem (3.2) reduces to the hierarchical minimization problem which was considered by Cabot [8] and Solodov [24] in the setting of Euclidean spaces.

Proposition 3.10. *Let $\Theta : \mathbb{M} \to (-\infty, +\infty]$ be proper, lower semicontinuous and geodesic convex function and $g : \mathbb{M} \to \mathbb{R}$ be a geodesic convex and differentiable function. Let $f : \mathbb{M} \to \mathbb{M}$ be a ϕ-contraction mapping, and for $\lambda > 0$, let $T : \mathbb{M} \to \mathbb{M}$ be defined by*

$$T(x) := \operatorname{prox}_{\lambda\Theta}(x), \quad \forall x \in \mathbb{M}. \tag{3.13}$$

If the gradient of g is $\nabla g = -\exp_{\cdot}^{-1} f$, then the problem (3.2) *reduces to the following hierarchical minimization problem*

$$\min_{x \in \operatorname{argmin}\ \Theta} g(x). \tag{3.14}$$

Proof. Let $\bar{x}$ be a solution of the problem (3.2). Then $\bar{x} \in \operatorname{Fix}(T)$, and by the definition of T, $\bar{x} = \operatorname{prox}_{\lambda\Theta}(\bar{x})$, that is, $\bar{x} \in \operatorname{argmin}\ \Theta$. It follows from (3.2)

$$\left\langle \exp_{\bar{x}}^{-1} f(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle \leq 0, \quad \forall y \in \operatorname{argmin}\ \Theta.$$

Since $\nabla g = -\exp_{\cdot}^{-1} f$, we obtain

$$\left\langle \nabla g(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle \geq 0, \quad \forall y \in \operatorname{argmin}\ \Theta.$$

By the definition of gradient of g, we have

$$0 \leq \left\langle \nabla g(\bar{x}), \exp_{\bar{x}}^{-1} y \right\rangle = g^D \left(\bar{x}; \exp_{\bar{x}}^{-1} y\right),$$

where $g^D \left(\bar{x}; \exp_{\bar{x}}^{-1} y\right)$ is a directional derivative at $\bar{x}$ in direction $\exp_{\bar{x}}^{-1} y$, i.e.,

$$\begin{aligned} 0 \leq g^D \left(\bar{x}; \exp_{\bar{x}}^{-1} y\right) &= \lim_{t \to 0^+} \frac{g\left(\exp_{\bar{x}}(t \exp_{\bar{x}}^{-1} y)\right) - g\left(\bar{x}\right)}{t} \\ &= \lim_{t \to 0^+} \frac{g(\gamma(t)) - g(\bar{x})}{t}, \end{aligned}$$

where $\gamma(t) = \exp_{\bar{x}} t \exp_{\bar{x}}^{-1} y$ is the geodesic joining $\bar{x} = \gamma(0)$ to $y = \gamma(1)$. Then by [25, Theorem 4.2, page no. 71], the function defined by

$$\Gamma(t) := \frac{g(\gamma(t))) - g(\bar{x})}{t}$$

is nondecreasing and $g^D\left(\bar{x};\exp_{\bar{x}}^{-1} y\right) = \inf_{t>0} \Gamma(t) \leq \Gamma(t)$. Therefore, above inequality becomes

$$0 \leq \frac{g(\gamma(t))) - g(\bar{x})}{t}.$$

By the geodesic convexity of g, we obtain $0 \leq g(y) - g(x)$, that is, $g(\bar{x}) \leq g(y)$ for all $y \in \operatorname{argmin} \Theta$. Hence, $\bar{x}$ is a solution of the problem (3.14). □

4. Viscosity method and convergence results

Let $f : \mathbb{M} \to \mathbb{M}$ be a ϕ-contraction mapping and $S, T : \mathbb{M} \to \mathbb{M}$ be nonexpansive mappings, where $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous and nonincreasing function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$. We propose the following viscosity iterative algorithm for finding a solution of the problem (3.1).

Algorithm 4.1. Choose an arbitrary element $x_1 \in \mathbb{M}$, define sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ by

$$y_n := \exp_{S(x_n)}(1 - \xi_n) \exp_{S(x_n)}^{-1} T(x_n), \quad \forall n \in \mathbb{N}, \tag{4.1}$$

and

$$x_{n+1} := \exp_{f(x_n)}(1 - \mu_n) \exp_{f(x_n)}^{-1} y_n, \quad \forall n \in \mathbb{N}, \tag{4.2}$$

where $\{\mu_n\}_{n\in\mathbb{N}} \subseteq (0, 1)$ and $\{\xi_n\}_{n\in\mathbb{N}} \subseteq (0, 1)$ are the sequences of positive real numbers.

Remark 4.2. If $\mathbb{M} = \mathbb{H}$ is a Hilbert space, then Algorithm 4.1 reduces to the algorithm considered and studied by Mainge and Moudafi [17] under the assumption that f is a contraction mapping.

Now we prove the convergence of the sequence generated by Algorithm 4.1 to a solution of the problem (3.1).

Theorem 4.3. *Let $f : \mathbb{M} \to \mathbb{M}$ be a ϕ-contraction mapping and $S, T : \mathbb{M} \to \mathbb{M}$ be nonexpansive mappings such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let $\{\mu_n\}_{n\in\mathbb{N}} \subseteq (0, 1)$ and $\{\xi_n\}_{n\in\mathbb{N}} \subseteq (0, 1)$ be the sequences such that $\frac{\xi_n}{\mu_n} \leq \sigma \in (0, +\infty)$ for all $n \in \mathbb{N}$, where σ is the same as in the formulation of the problem* (3.1). *Assume that the following conditions hold:*

(i) $\lim_{n\to\infty} \mu_n = 0$;
(ii) $\sum_{n=1}^{\infty} \mu_n = +\infty$;
(iii) $\lim_{n\to\infty} \frac{\xi_n}{\mu_n} = 0$;
(iv) $\lim_{n\to\infty} \left|1 - \frac{\mu_{n-1}}{\mu_n}\right| = 0$.

If $0 < \tau = \sup\{\phi(d(x_n, \bar{a}))/d(x_n, \bar{a}) : x_n \neq \bar{a}, n \in \mathbb{N}\} < 1$ for all $\bar{a} \in$ $\mathrm{Fix}(T)$, *then*

(a) *the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by Algorithm* 4.1 *is bounded*;
(b) $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$;
(c) *the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by Algorithm* 4.1 *converges to a solution of the problem* (3.1).

Proof. (a) Clearly, from (4.2) and (4.1), x_{n+1} and y_n are on the geodesics joining y_n and $f(x_n)$, and $T(x_n)$ and $S(x_n)$, respectively. Let $\gamma_n^1 : [0,1] \to \mathbb{M}$ and $\gamma_n^2 : [0,1] \to \mathbb{M}$ be the geodesics such that $\gamma_n^1(0) = y_n$, $\gamma_n^1(1) = f(x_n)$ and $\gamma_n^2(0) = T(x_n)$, $\gamma_n^2(1) = S(x_n)$. Therefore, $x_{n+1} = \gamma_n^1(\mu_n)$ and $y_n = \gamma_n^2(\xi_n)$. Let $\bar{a}$ be a solution of the problem (3.1). Then, $\bar{a} \in \mathrm{Fix}(T)$. By Proposition 2.3, we have

$$\begin{aligned} d(y_n, \bar{a}) &= d(\gamma_n^2(\xi_n), \bar{a}) \\ &\leq (1-\xi_n) d(\gamma_n^2(0), \bar{a}) + \xi_n d(\gamma_n^2(1), \bar{a}) \\ &= (1-\xi_n) d(T(x_n), T(\bar{a})) + \xi_n d(S(x_n), \bar{a}) \\ &\leq (1-\xi_n) d(x_n, \bar{a}) + \xi_n \big(d(S(x_n), S(\bar{a})) + d(S(\bar{a}), \bar{a}) \big) \\ &\leq (1-\xi_n) d(x_n, \bar{a}) + \xi_n \big(d(x_n, \bar{a}) + d(S(\bar{a}), \bar{a}) \big) \\ &= d(x_n, \bar{a}) + \xi_n d(S(\bar{a}), \bar{a}). \end{aligned} \tag{4.3}$$

This together with (4.2) implies that

$$\begin{aligned} d(x_{n+1}, \bar{a}) &= d(\gamma_n(\mu_n), \bar{a}) \\ &\leq (1-\mu_n) d(\gamma_n^1(0), \bar{a}) + \mu_n d(\gamma_n^1(1), \bar{a}) \\ &= (1-\mu_n) d(y_n, \bar{a}) + \mu_n d(f(x_n), \bar{a}) \\ &\leq (1-\mu_n)\big(d(x_n, \bar{a}) + \xi_n d(S(\bar{a}), \bar{a}) \big) + \mu_n \left(d(f(x_n), f(\bar{a})) + d(f(\bar{a}), \bar{a}) \right) \\ &\leq (1-\mu_n) d(x_n, \bar{a}) + \xi_n d(S(\bar{a}), \bar{a}) + \mu_n \left(\phi(d(x_n, \bar{a})) + d(f(\bar{a}), \bar{a}) \right) \\ &= (1-\mu_n) d(x_n, \bar{a}) + \mu_n \phi(d(x_n, \bar{a})) + \mu_n \left(d(f(\bar{a}), \bar{a}) + \frac{\xi_n}{\mu_n} d(S(\bar{a}), \bar{a}) \right). \end{aligned} \tag{4.4}$$

Since $0 < \tau = \sup\{\phi(d(x_n, \bar{a}))/d(x_n, \bar{a}) : x_n \neq \bar{a}, n \in \mathbb{N}\} < 1$ and $\frac{\xi_n}{\mu_n} \leq \sigma$ for all $n \in \mathbb{N}$, then it follows from (4.4) that

$$\begin{aligned} d(x_{n+1}, \bar{a}) &\leq (1 - \mu_n(1-\tau)) d(x_n, \bar{a}) + \mu_n \left(d(f(\bar{a}), \bar{a}) + \sigma d(S(\bar{a}), \bar{a}) \right) \\ &\leq \max\left\{ d(x_n, \bar{a}), \frac{1}{1-\tau}\big(d(f(\bar{a}), \bar{a}) + \sigma d(S(\bar{a}), \bar{a}) \big) \right\} \\ &\;\;\vdots \\ &\leq \max\left\{ d(x_1, \bar{a}), \frac{1}{1-\tau}\left(d(f(\bar{a}), \bar{a}) + \sigma d(S(\bar{a}), \bar{a}) \right) \right\}. \end{aligned} \tag{4.5}$$

Therefore, $\{x_n\}_{n\in\mathbb{N}}$ is bounded, and hence there exists a constant $K > 0$ such that $d(x_n, \bar{a}) \leq K$. Since $\mu_n \to 0$ as $n \to \infty$, $\{\mu_n\}_{n\in\mathbb{N}}$ is bounded. Therefore, there exists a constant $\eta > 0$ such that $\mu_n \leq \eta$ for all $n \in \mathbb{N}$. Thus, From (4.3), we have

$$\begin{aligned} d(y_n, \bar{a}) &\leq d(x_n, \bar{a}) + \xi_n d(S(\bar{a}), \bar{a}) \\ &= d(x_n, \bar{a}) + \mu_n \frac{\xi_n}{\mu_n} d(S(\bar{a}), \bar{a}) \\ &\leq K + \eta\sigma d(S(\bar{a}), \bar{a}), \end{aligned}$$

and therefore, $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Since f is ϕ-contraction, and T and S are non-expansive mappings, by condition (ii) in Definition 2.9, we have

$$\begin{aligned} d(f(x_n),\bar{a}) &\leq d(f(x_n),f(\bar{a}))+d(f(\bar{a}),\bar{a}) \\ &\leq \phi(d(x_n,\bar{a}))+d(f(\bar{a}),\bar{a}) \\ &< d(x_n,\bar{a})+d(f(\bar{a}),\bar{a}) \\ &\leq K+d(f(\bar{a}),\bar{a}), \end{aligned}$$

$$\begin{aligned} d(S(x_n),\bar{a}) &\leq d(S(x_n),S(\bar{a}))+d(S(\bar{a}),\bar{a}) \\ &\leq d(x_n,\bar{a})+d(S(\bar{a}),\bar{a}) \\ &\leq K+d(S(\bar{a}),\bar{a}), \end{aligned}$$

and

$$d(T(x_n),\bar{a})=d(T(x_n),T(\bar{a}))\leq d(x_n,\bar{a})\leq K,$$

which means that the sequences $\{f(x_n)\}_{n\in\mathbb{N}}$, $\{S(x_n)\}_{n\in\mathbb{N}}$ and $\{T(x_n)\}_{n\in\mathbb{N}}$ are bounded.

(b) From convexity of Riemannian distance d, we obtain

$$\begin{aligned} d(y_n,y_{n-1}) &= d(\gamma_n^2(\xi_n),\gamma_{n-1}^2(\xi_{n-1})) \\ &\leq d(\gamma_n^2(\xi_n),\gamma_{n-1}^2(\xi_n))+d(\gamma_{n-1}^2(\xi_n),\gamma_{n-1}^2(\xi_{n-1})) \\ &\leq (1-\xi_n)d(\gamma_n^2(0),\gamma_{n-1}^2(0))+\xi_n d(\gamma_n^2(1),\gamma_{n-1}^2(1)) \\ &\quad +|\xi_n-\xi_{n-1}|d(S(x_{n-1}),T(x_{n-1})) \\ &= (1-\xi_n)d(T(x_n),T(x_{n-1}))+\xi_n d(S(x_n),S(x_{n-1})) \\ &\quad +|\xi_n-\xi_{n-1}|d(S(x_{n-1}),T(x_{n-1})) \\ &\leq (1-\xi_n)d(x_n,x_{n-1})+\xi_n d(x_n,x_{n-1})+|\xi_n-\xi_{n-1}|d(S(x_{n-1}),T(x_{n-1})) \\ &\leq d(x_n,x_{n-1})+|\xi_n-\xi_{n-1}|K_1, \end{aligned}$$

where K_1 is a constant such that $K_1=\sup\limits_{n\in\mathbb{N}}\{d(S(x_{n-1}),T(x_{n-1}))\}$. Again by convexity of Riemannian distance d, we have

$$\begin{aligned} & d(x_{n+1},x_n) \\ &= d(\gamma_n^1(\mu_n),\gamma_{n-1}^1(\mu_{n-1})) \\ &\leq d(\gamma_n^1(\mu_n),\gamma_{n-1}^1(\mu_n))+d(\gamma_{n-1}^1(\mu_n),\gamma_{n-1}^1(\mu_{n-1})) \\ &\leq (1-\mu_n)d(\gamma_n^1(0),\gamma_{n-1}^1(0))+\mu_n d(\gamma_n^1(1),\gamma_{n-1}^1(1))+|\mu_n-\mu_{n-1}|d(f(x_{n-1}),y_{n-1}) \\ &= (1-\mu_n)d(y_n,y_{n-1})+\mu_n d(f(x_n),f(x_{n-1}))+|\mu_n-\mu_{n-1}|d(f(x_{n-1}),y_{n-1}) \\ &\leq (1-\mu_n)\left(d(x_n,x_{n-1})+|\xi_n-\xi_{n-1}|K_1\right)+\mu_n\phi(d(x_n,x_{n-1}))+|\mu_n-\mu_{n-1}|K_2 \\ &\leq (1-\mu_n)d(x_n,x_{n-1})+\mu_n\phi(d(x_n,x_{n-1})+|\xi_n-\xi_{n-1}|K_1+|\mu_n-\mu_{n-1}|K_2, \end{aligned}$$

where K_2 is a constant such that $K_2=\sup\limits_{n\in\mathbb{N}}\{d(f(x_{n-1}),y_{n-1})\}$. Since $\lim\limits_{n\to\infty}\frac{\xi_n}{\mu_n}=0$, we have

$$\lim_{n\to\infty}\frac{1}{\mu_n}|\xi_n-\xi_{n-1}|=0,$$

and from assumption (iii) we have

$$\lim_{n\to\infty} \frac{1}{\mu_n}|\mu_n - \mu_{n-1}| = 0.$$

Therefore, from Lemma 2.14, we conclude that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

(c) We first prove that

$$\limsup_{n\to\infty} \left\{ \langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n)\rangle + \frac{1}{\sigma}\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n\rangle \right\} \leq 0,$$

where $\bar{a}$ is a solution of the problem (3.1).

Since the sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\{T(x_n)\}_{n\in\mathbb{N}}$ are bounded, so is

$$\left\{ \langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n)\rangle + \frac{1}{\sigma}\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n\rangle \right\}_{n\in\mathbb{N}},$$

and hence, its upper limit exists. We may assume a subsequence $\{x_{n_j}\}_{j\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} &\limsup_{n\to\infty} \left\{ \langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n)\rangle + \frac{1}{\sigma}\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n\rangle \right\} \\ &= \lim_{j\to\infty} \left\{ \langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_{n_j})\rangle + \frac{1}{\sigma}\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_{n_j}\rangle \right\}, \end{aligned}$$

and $x_{n_j} \to x'$ for some $x' \in \mathbb{M}$. By convexity of Riemanninan distance d, we have

$$\begin{aligned} d(x_{n_j+1}, y_{n_j}) &= d(\gamma^1_{n_j}(\mu_{n_j}), y_{n_j}) \\ &\leq (1-\mu_{n_j}) d(\gamma^1_{n_j}(0), y_{n_j}) + \mu_{n_j} d(\gamma^1_{n_j}(1), y_{n_j}) \\ &= (1-\mu_{n_j}) d(y_{n_j}, y_{n_j}) + \mu_{n_j} d(f(x_{n_j}), y_{n_j}) \\ &= \mu_{n_j} d(f(x_{n_j}), y_{n_j}), \end{aligned}$$

and

$$\begin{aligned} d(y_{n_j}, T(x_{n_j})) &= d(\gamma^2_{n_j}(\xi_{n_j}), T(x_{n_j})) \\ &\leq (1-\xi_{n_j}) d(\gamma^2_{n_j}(0), T(x_{n_j})) + \xi_{n_j} d(\gamma^2_{n_j}(1), T(x_{n_j})) \\ &= (1-\xi_{n_j}) d(T(x_{n_j}), T(x_{n_j})) + \xi_{n_j} d(S(x_{n_j}), T(x_{n_j})) \\ &= \xi_{n_j} d(S(x_{n_j}), T(x_{n_j})). \end{aligned}$$

Since $\{y_n\}_{n\in\mathbb{N}}$, $\{T(x_n)\}_{n\in\mathbb{N}}$, $\{S(x_n)\}_{n\in\mathbb{N}}$ and $\{f(x_n)\}_{n\in\mathbb{N}}$ are bounded, and so are $\{d(f(x_{n_j}), y_{n_j})\}_{n\in\mathbb{N}}$ and $\{d(S(x_{n_j}), T(x_{n_j}))\}_{n\in\mathbb{N}}$. Since $\xi_n \leq \sigma\mu_n$, from condition (iii), we have $\xi_n \to 0$ as $n \to \infty$. Therefore, we have

$$\lim_{j\to\infty} d(x_{n_j+1}, y_{n_j}) = 0,$$

and

$$\lim_{j\to\infty} d(y_{n_j}, T(x_{n_j})) = 0.$$

Since $d(x_{n_j}, y_{n_j}) \leq d(x_{n_j+1}, x_{n_j}) + d(x_{n_j+1}, y_{n_j})$, we obtain $\lim_{j\to\infty} d(x_{n_j}, y_{n_j}) = 0$, and therefore,

$$d(x', T(x')) \leq d(x', x_{n_j}) + d(x_{n_j}, y_{n_j}) + d(y_{n_j}, T(x_{n_j})) + d(T(x'), T(x_{n_j})) \to 0 \quad \text{as } j \to \infty.$$

This implies that $x' \in \mathrm{Fix}(T)$. Since $\bar{a}$ is the solution of the problem (3.1), we have

$$\left\langle \exp_{\bar{a}}^{-1} S(\bar{a}) + \frac{1}{\sigma} \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y \right\rangle \leq 0, \quad \forall y \in \mathrm{Fix}(T).$$

Since $x_{n_j} \to x' \in \mathrm{Fix}(T)$ and $\lim_{j\to\infty} d(x_{n_j}, y_{n_j})$, hence, $y_{n_j} \to x'$. Therefore,

$$\begin{aligned} \limsup_{n\to\infty} & \left\{ \left\langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n) \right\rangle + \frac{1}{\sigma} \left\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n \right\rangle \right\} \\ &= \lim_{j\to\infty} \left\{ \left\langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_{n_j}) \right\rangle + \frac{1}{\sigma} \left\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_{n_j} \right\rangle \right\} \\ &= \left\langle \exp_{\bar{a}}^{-1} S(\bar{a}) + \frac{1}{\sigma} \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} x' \right\rangle \leq 0. \end{aligned} \tag{4.6}$$

It follows that there exists a sequence $\{c_n\}_{n\in\mathbb{N}}$ in $(0, +\infty)$ with $\lim_{n\to\infty} c_n = 0$ such that

$$\left\{ \left\langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n) \right\rangle + \frac{1}{\sigma} \left\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n \right\rangle \right\} \leq c_n, \quad \forall n \in \mathbb{N}.$$

Next, we prove that $\lim_{n\to\infty} d(x_n, \bar{a}) = 0$. For each $n \in \mathbb{N}$, let $p_n = f(x_n)$, $p = f(\bar{a})$, $q_n = y_n$, $l_n = S(x_n)$, $l = S(\bar{a})$ and $m_n = T(x_n)$. Consider the geodesic triangles $\Delta(p_n, q_n, \bar{a})$, $\Delta(p, q_n, \bar{a})$, $\Delta(p_n, q_n, p)$, $\Delta(l, m_n, \bar{a})$, $\Delta(l_n, m_n, \bar{a})$ and $\Delta(l_n, m_n, l)$. Then by Lemma 2.5, there exist comparison triangles $\Delta(p_n', q_n', \bar{a}')$, $\Delta(p', q_n', \bar{a}')$, $\Delta(p_n', q_n', p')$, $\Delta(l', m_n', \bar{a}')$, $\Delta(l_n', m_n', \bar{a}')$ and $\Delta(l_n', m_n', l')$ such that

$$d(p_n, q_n) = \|p_n' - q_n'\|, \quad d(q_n, \bar{a}) = \|q_n' - \bar{a}'\| \quad \text{and} \quad d(p_n, \bar{a}) = \|p_n' - \bar{a}'\|,$$

$$d(p, \bar{a}) = \|p' - \bar{a}'\|, \quad d(q_n, \bar{a}) = \|q_n' - \bar{a}'\| \quad \text{and} \quad d(p_n, p) = \|p_n' - p'\|,$$

$$d(l_n, m_n) = \|l_n' - m_n'\|, \quad d(m_n, \bar{a}) = \|m_n' - \bar{a}'\| \quad \text{and} \quad d(l_n, \bar{a}) = \|l_n' - \bar{a}'\|$$

and

$$d(l, \bar{a}) = \|l' - \bar{a}'\|, \quad d(m_n, \bar{a}) = \|m_n' - \bar{a}'\| \quad \text{and} \quad d(l_n, l) = \|l_n' - l'\|.$$

Let α and β denote the angles at $\bar{a}$ in triangles $\Delta(p, q_n, \bar{a})$ and $\Delta(l, m_n, \bar{a})$, respectively, and α' and β' be the comparison angles at $\bar{a}'$ in triangles $\Delta(p', q_n', \bar{a}')$ and $\Delta(l', m_n', \bar{a}')$, respectively. Therefore, $\alpha \leq \alpha'$ and $\beta \leq \beta'$ by Lemma 2.6 (a), and so, $\cos\alpha' \leq \cos\alpha$ and $\cos\beta' \leq \cos\beta$, respectively. Let

$$x_{n+1}' := \mu_n p_n' + (1 - \mu_n) q_n' \text{ and } q_n' := \xi_n l_n' + (1 - \xi_n) m_n'$$

be the comparison point of x_{n+1} and y_n, respectively. Then by Lemma 2.6 (b), we have

$$\begin{aligned}
&d^2(x_{n+1}, \bar{a})\\
&\leq \|x'_{n+1} - \bar{a}'\|^2\\
&= \|\mu_n p'_n + (1-\mu_n) q'_n - \bar{a}'\|^2\\
&= \|\mu_n (p'_n - \bar{a}') + (1-\mu_n)(q'_n - \bar{a}')\|^2\\
&= \mu_n^2 \left\|p'_n - \bar{a}'\right\|^2 + (1-\mu_n)^2 \left\|q'_n - \bar{a}'\right\|^2 + 2\mu_n(1-\mu_n) \langle p'_n - \bar{a}', q'_n - \bar{a}'\rangle\\
&= \mu_n^2 \left\|p'_n - \bar{a}'\right\|^2 + (1-\mu_n)^2 \left\|q'_n - \bar{a}'\right\|^2 + 2\mu_n(1-\mu_n)\big(\langle p'_n - p', q'_n - \bar{a}'\rangle\\
&\quad + \langle p' - \bar{a}', q'_n - \bar{a}'\rangle \big)\\
&\leq \mu_n^2 \left\|p'_n - \bar{a}'\right\|^2 + (1-\mu_n)^2 \|q'_n - \bar{a}'\|^2 + 2\mu_n(1-\mu_n)\big(\|p'_n - p'\| \|q'_n - \bar{a}'\|\\
&\quad + \|p' - \bar{a}'\| \|q'_n - \bar{a}'\| \cos\alpha'\big)\\
&\leq \mu_n^2 d^2(p_n, \bar{a}) + (1-\mu_n)^2 d^2(q_n, \bar{a}) + 2\mu_n(1-\mu_n)\big(d(p_n, p) d(q_n, \bar{a})\\
&\quad + d(p, \bar{a}) d(q_n, \bar{a}) \cos\alpha\big)\\
&= \mu_n^2 d^2(f(x_n), \bar{a}) + (1-\mu_n)^2 d^2(q_n, \bar{a}) + 2\mu_n(1-\mu_n)\big(d(f(x_n), f(\bar{a})) d(y_n, \bar{a})\\
&\quad + d(f(\bar{a}), \bar{a}) d(y_n, \bar{a}) \cos\alpha\big)\\
&\leq \mu_n^2 d^2(f(x_n), \bar{a}) + (1-\mu_n)^2 d^2(q_n, \bar{a}) + 2\mu_n(1-\mu_n)\phi(d(x_n, \bar{a})) d(y_n, \bar{a})\\
&\quad + 2\mu_n(1-\mu_n) d(f(\bar{a}), \bar{a}) d(y_n, \bar{a}) \cos\alpha.
\end{aligned}$$

Since $\{f(x_n)\}_{n\in\mathbb{N}}$ is bounded then we may assume a constant K_3 such that

$$K_3 = \sup_{n\in\mathbb{N}} \{d^2(f(x_n), \bar{a})\}.$$

It follows from (4.3) that

$$\begin{aligned}
&d^2(x_{n+1}, \bar{a})\\
&\leq \mu_n^2 K_3 + (1-\mu_n)^2 d^2(q_n, \bar{a}) + 2\mu_n(1-\mu_n)\phi(d(x_n, \bar{a}))\left(d(x_n, \bar{a}) + \xi_n d(S(\bar{a}), \bar{a})\right)\\
&\quad + 2\mu_n(1-\mu_n) d(f(\bar{a}), \bar{a}) d(y_n, \bar{a}) \cos\alpha.
\end{aligned}$$

Define a sequence $\{w_n\}_{n\in\mathbb{N}}$ by $w_n = d^2(x_n, \bar{a})$. Let

$$\delta = d(S(\bar{a}), \bar{a}).$$

Since

$$d(f(\bar{a}), \bar{a}) d(y_n, \bar{a}) \cos\alpha = \left\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n \right\rangle,$$

we have

$$\begin{aligned}
w_{n+1} &\leq \mu_n^2 K_3 + (1-\mu_n)^2 d^2(q_n, \bar{a}) + 2\mu_n(1-\mu_n)\phi(\sqrt{w_n})\sqrt{w_n}\\
&\quad + 2\mu_n(1-\mu_n) \left\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n \right\rangle + 2\mu_n \xi_n (1-\mu_n)\phi(\sqrt{w_n})\delta. \qquad (4.7)
\end{aligned}$$

Since $q_n' := \xi_n l_n' + (1-\xi_n)m_n'$ is the comparison point of y_n and $d(q_n,\bar{a}) = \|q_n' - \bar{a}'\|$, then we have

$$\begin{aligned}
&d^2(q_n,\bar{a}) = \|q_n' - \bar{a}'\|^2 \\
&= \|\xi_n l_n' + (1-\xi_n)m_n' - \bar{a}'\|^2 \\
&= \xi_n^2 \|l_n' - \bar{a}'\|^2 + (1-\xi_n)^2 \|m_n' - \bar{a}'\|^2 + 2\xi_n(1-\xi_n)\langle l_n' - \bar{a}', m_n' - \bar{a}'\rangle \\
&= \xi_n^2 \|l_n' - \bar{a}'\|^2 + (1-\xi_n)^2 \|m_n' - \bar{a}'\|^2 + 2\xi_n(1-\xi_n)\big(\langle l_n' - l', m_n' - \bar{a}'\rangle \\
&\quad + \langle l' - \bar{a}', m_n' - \bar{a}'\rangle\big) \\
&\le \xi_n^2 \|l_n' - \bar{a}'\|^2 + (1-\xi_n)^2 \|m_n' - \bar{a}'\|^2 + 2\xi_n(1-\xi_n)\big(\|l_n' - l'\|\|m_n' - \bar{a}'\| \\
&\quad + \|l' - \bar{a}'\|\|m_n' - \bar{a}'\|\cos\beta'\big) \\
&\le \xi_n^2 d^2(l_n,\bar{a}) + (1-\xi_n)^2 d^2(m_n,\bar{a}) + 2\xi_n(1-\xi_n)\big(d(l_n,l)d(m_n,\bar{a}) \\
&\quad + d(l,\bar{a})d(m_n,\bar{a})\cos\beta) \\
&= \xi_n^2 d^2(S(x_n),\bar{a}) + (1-\xi_n)^2 d^2(T(x_n),\bar{a}) + 2\xi_n(1-\xi_n)\big(d(S(x_n),S(\bar{a}))d(T(x_n),\bar{a}) \\
&\quad + d(S(\bar{a}),\bar{a})d(T(x_n),\bar{a})\cos\beta\big) \\
&\le \xi_n^2 d^2(S(x_n),\bar{a}) + (1-\xi_n)^2 d^2(x_n,\bar{a}) + 2\xi_n(1-\xi_n)\big(d^2(x_n,\bar{a}) \\
&\quad + d(S(\bar{a}),\bar{a})d(T(x_n),\bar{a})\cos\beta\big) \\
&= (1-\xi_n^2)d^2(x_n,\bar{a}) + 2\xi_n(1-\xi_n)\left\langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n)\right\rangle + \xi_n^2 d^2(S(x_n),\bar{a}).
\end{aligned}$$

Since $\bar{a} \in \mathrm{Fix}(T)$ and $w_n = d^2(x_n,\bar{a})$,

$$d(S(\bar{a}),\bar{a})d(T(x_n),\bar{a})\cos\beta = \left\langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n)\right\rangle,$$

$d(x_n,\bar{a}) \le K$ and $\{S(x_n)\}_{n\in\mathbb{N}}$ is bounded, there exists is a constant K_4 such that

$$K_4 = \sup_{n\in\mathbb{N}}\{d^2(S(x_n),\bar{a})\},$$

and therefore, we have

$$d^2(q_n,\bar{a}) \le (1-\xi_n^2)w_n + 2\xi_n(1-\xi_n)\left\langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n)\right\rangle + \xi_n^2 K_4. \tag{4.8}$$

This together with inequality (4.7) gives

$$\begin{aligned}
w_{n+1} \le \mu_n^2 K_3 &+ (1-\mu_n)^2\big((1-\xi_n^2)w_n + 2\xi_n(1-\xi_n)\left\langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n)\right\rangle \\
&+ \xi_n^2 K_4\big) + 2\mu_n(1-\mu_n)\phi(\sqrt{w_n})\sqrt{w_n} + 2\mu_n(1-\mu_n)\left\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n\right\rangle \\
&+ 2\mu_n\xi_n(1-\mu_n)\phi(\sqrt{w_n})\delta.
\end{aligned} \tag{4.9}$$

Since $0 < \mu_n, \xi_n < 1$ and $\xi_n \le \sigma\mu_n$, we have

$$(1-\mu_n)^2 \le (1-\mu_n), \quad \mu_n(1-\mu_n) \le \mu_n, \quad \xi_n^2 \le \xi_n \quad \text{and} \quad \xi_n(1-\xi_n) \le \xi_n.$$

Therefore,

$$\begin{aligned}
w_{n+1} &\le \mu_n^2 K_3 + (1-\mu_n)w_n + \xi_n^2 K_4 + 2\sigma\mu_n\left\langle \exp_{\bar{a}}^{-1} S(\bar{a}), \exp_{\bar{a}}^{-1} T(x_n)\right\rangle \\
&\quad + 2\mu_n\phi(\sqrt{w_n})\sqrt{w_n} + 2\mu_n\left\langle \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y_n\right\rangle + 2\mu_n\xi_n\phi(\sqrt{w_n})\delta \\
&= (1-\mu_n)w_n + 2\mu_n\phi(\sqrt{w_n})\sqrt{w_n} + \mu_n^2 K_3 + \xi_n^2 K_4 + 2\sigma\mu_n c_n + 2\mu_n\xi_n\phi(\sqrt{w_n})\delta.
\end{aligned}$$

Thus, we have

$$w_{n+1} \leq (1-\mu_n)w_n + \mu_n\psi(w_n) + \beta_n, \quad \forall n \in \mathbb{N},$$

where $\beta_n = \mu_n^2 K_3 + \xi_n^2 K_4 + 2\sigma\mu_n c_n + 2\mu_n\xi_n\phi(\sqrt{w_n})\delta$ and $\psi(t) = 2\sqrt{t}\phi(\sqrt{t})$. Since $w_n = d^2(x_n, \bar{a}) \leq K^2$ and ϕ is nondecreasing, we have $\phi(\sqrt{w_n}) \leq \phi(K^2)$. Therefore,

$$\frac{\beta_n}{\mu_n} \leq \mu_n K_3 + \frac{\xi_n^2}{\mu_n} K_4 + 2\sigma c_n + 2\xi_n\phi(K)\delta.$$

Since $\lim_{n\to\infty} c_n = 0$ and $\xi_n = \frac{\xi_n}{\mu_n}\mu_n \to 0$ as $n \to \infty$ because of assumptions (i) and (iii), by conditions (i) - (iii), we have

$$\lim_{n\to\infty} \frac{\beta_n}{\mu_n} = 0.$$

Hence by Lemma 2.14, $\{x_n\}_{n\in\mathbb{N}}$ converges to $\bar{a}$. □

Remark 4.4. In the absence of the projection mapping and by using the properties of geodesic convexity, the above algorithm and convergence result improve and extend the corresponding results in [30, 18, 17, 28, 29] from linear space setting to nonlinear spaces, more precisely, to Hadamard manifolds.

If $S \equiv I$ the identity mapping, then we have the following result.

Corollary 4.5. *Let $f : \mathbb{M} \to \mathbb{M}$ be a ϕ-contraction mapping and $T : \mathbb{M} \to \mathbb{M}$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let $\{\mu_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ and $\{\xi_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ be the sequences such that the conditions* (i) - (iv) *of Theorem* 4.3 *hold. If $0 < \tau = \sup\{\phi(d(x_n, \bar{a}))/d(x_n, \bar{a}) : x_n \neq \bar{a}, n \in \mathbb{N}\} < 1$ for all $\bar{a} \in \mathrm{Fix}(T)$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by*

$$y_n := \exp_{x_n}(1-\xi_n)\exp_{x_n}^{-1} T(x_n), \quad \forall n \in \mathbb{N},$$

and

$$x_{n+1} := \exp_{f(x_n)}(1-\mu_n)\exp_{f(x_n)}^{-1} y_n, \quad \forall n \in \mathbb{N},$$

converges to a solution of the problem (3.2).

From the above result, we can easily derive the following result related to the monotone inclusion problem.

Corollary 4.6. *Let $f : \mathbb{M} \to \mathbb{M}$ be a ϕ-contraction mapping and $V \in \mathcal{X}(\mathbb{M})$ be a monotone set-valued vector field such that $V^{-1}(\mathbf{0}) \neq \emptyset$. Let $\{\mu_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ and $\{\xi_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ be a sequence such that conditions* (i) - (iv) *of Theorem* 4.3 *hold. If $0 < \tau = \sup\{\phi(d(x_n, \bar{a}))/d(x_n, \bar{a}) : x_n \neq \bar{a}, n \in \mathbb{N}\} < 1$ for all $\bar{a} \in V^{-1}(\mathbf{0})$, then*

(a) *for $\lambda > 0$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by*

$$y_n := \exp_{x_n}(1-\xi_n)\exp_{x_n}^{-1} R_\lambda^V(x_n), \quad \forall n \in \mathbb{N}, \tag{4.10}$$

and

$$x_{n+1} := \exp_{f(x_n)}(1-\mu_n)\exp_{f(x_n)}^{-1} y_n, \quad \forall n \in \mathbb{N}, \tag{4.11}$$

is bounded;

(b) $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$;

(c) *the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by* (4.11) *converges to a solution of the problem* (3.6).

In view of Proposition 3.9 and Proposition 3.10, and by taking $S \equiv I$ identity mapping in Theorem 4.3, we get following consequences of Corollary 4.5.

Corollary 4.7. *Let* $\mathcal{G} : \mathbb{M} \to \mathbb{R}$ *be a geodesic convex function on a Hadamard manifold* $\mathbb{M}$ *defined by* (3.9) *and* $T : \mathbb{M} \to \mathbb{M}$ *be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let* $\{\mu_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ *and* $\{\xi_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ *be the sequences such that the conditions* (i) - (iv) *of Theorem* 4.3 *hold. If* $0 < \tau = \sup\{\phi(d(x_n,\bar{a}))/d(x_n,\bar{a}) : x_n \neq \bar{a}, n \in \mathbb{N}\} < 1$ *for all* $\bar{a} \in \mathrm{Fix}(T)$. *Then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by*

$$y_n := \exp_{x_n}(1-\xi_n)\exp_{x_n}^{-1} T(x_n), \quad \forall n \in \mathbb{N},$$

and

$$x_{n+1} := \exp_{f(x_n)}(1-\mu_n)\exp_{f(x_n)}^{-1} y_n, \quad \forall n \in \mathbb{N},$$

converges to a solution of the problem (3.8).

Corollary 4.8. *Let* $\Theta : \mathbb{M} \to (-\infty, +\infty]$ *be a proper, lower semicontinuous and geodesic convex function and* $g : \mathbb{M} \to \mathbb{R}$ *be a geodesic convex and differentiable function such that* $\nabla g = -\exp_{\cdot}^{-1} f$, *and* $\operatorname{argmin} \Theta \neq \emptyset$ *and a nonexpansive mapping* $T : \mathbb{M} \to \mathbb{M}$ *be defined by* (3.13) *such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let* $\{\mu_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ *and* $\{\xi_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ *be the sequences such that the conditions* (i) - (iv) *of Theorem* 4.3 *hold. If* $0 < \tau = \sup\{\phi(d(x_n,\bar{a}))/d(x_n,\bar{a}) : x_n \neq \bar{a}, n \in \mathbb{N}\} < 1$ *for all* $\bar{a} \in \mathrm{Fix}(T)$. *Then the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *defined by*

$$y_n := \exp_{x_n}(1-\xi_n)\exp_{x_n}^{-1} T(x_n), \quad \forall n \in \mathbb{N},$$

and

$$x_{n+1} := \exp_{f(x_n)}(1-\mu_n)\exp_{f(x_n)}^{-1} y_n, \quad \forall n \in \mathbb{N},$$

converges to a solution of the problem (3.14).

5. Numerical example

Example 5.1. Let $\mathbb{M} = (\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold with Riemannian metric $\langle u, v\rangle := u^\top G(x) v$ for all $u, v \in T_x\mathbb{M}$ and all $x = (x_1, x_2, x_3) \in \mathbb{M}$, where $G(x)$ is a 3×3 matrix defined by

$$G(x) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+4x_2^2 & -2x_2 \\ 0 & -2x_2 & 1 \end{pmatrix}, \quad \forall x \in \mathbb{M}.$$

Define a mapping $\Phi : \mathbb{R}^3 \to \mathbb{M}$ on the Euclidean space $\mathbb{R}^3$ by

$$\Phi(x) := \left(x_1, x_2, x_2^2 - x_3\right), \quad \forall x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Then it is an isometry and its inverse Φ^{-1} is given by

$$\Phi^{-1}(x) := \left(x_1, x_2, x_2^2 - x_3\right), \quad \forall x = (x_1, x_2, x_3) \in \mathbb{M}.$$

The Riemannian distance between for any x and y in $\mathbb{M}$ is defined by

$$d^2(x,y) := \left\|\Phi^{-1}(x) - \Phi^{-1}(y)\right\|^2 = \sum_{i=1}^{2}(x_i - y_i)^2 + (x_2^2 - x_3 - y_2^2 + y_3)^2. \tag{5.1}$$

For further details, see [10]. The geodesic joining the points $\gamma(0) = x$ and $\gamma(1) = y$ is

$$\gamma(t) := (\gamma_1(t), \gamma_2(t), \gamma_3(t)), \quad \forall t \in [0,1], \tag{5.2}$$

where $\gamma_i(t) = x_i + t(y_i - x_i)$ for all $i = 1, 2$ and

$$\gamma_3(t) = x_3 + t((y_3 - x_3) - 2(y_2 - x_2)^2) + 2t^2(y_2 - x_2)^2.$$

Let $\beta : (0,1) \to \mathbb{R}^3$ be a geodesic on $\mathbb{R}^3$ define by

$$\beta(t) = vt + x$$

for all $v = (v_1, v_2, v_3), x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $\beta(0) = x$ and $\beta'(0) = v$. Since Φ is an isometry between $\mathbb{R}^3$ and $\mathbb{M}$, Φ preserves the geodesics between $\mathbb{R}^3$ and $\mathbb{M}$, i.e., β is a geodesic in $\mathbb{R}^3$ if and only if $\gamma = \Phi \circ \beta$ is a geodesic in $\mathbb{M}$ (For details, see [10, 11, 23]). Then $\gamma = \Phi \circ \beta$ is given by

$$\gamma(t) := (w_1(t), w_2(t), w_3(t)),$$

where $w_i(t) = x_i + v_i t$ for all $i = 1, 2$ and

$$w_3(t) = x_3 + v_3 t + v_2^2 t^2$$

such that $\gamma(0) = x \in \mathbb{R}^3$ and $\gamma'(0) = v = (v_1, v_2, v_3)$. Clearly, $\exp_x(tv) = \gamma(t)$.
To obtain the inverse exponential mapping, we may write

$$y = \exp_x\left(d(x,y)\frac{\exp_x^{-1} y}{d(x,y)}\right), \quad \forall x, y \in \mathbb{M}.$$

Therefore, after simplifying, we get

$$\exp_x^{-1} y = (y_1 - x_1, y_2 - x_2, y_3 - x_3 - (y_2 - x_2)^2).$$

Define a mapping

$$f(x) = (x_1/2, x_2/2, x_3/2 + x_2^2/2), \quad \forall x \in \mathbb{M}.$$

It clear that f is a ϕ-contraction mapping with a continuous function $\phi(t) = \frac{1}{4}t$.
We define two nonexpansive mappings S and T by

$$S(x) = (-x_1, x_2, x_3) \quad \text{and} \quad T(x) = (-x_1, -x_2, x_3), \quad \forall x \in \mathbb{M}.$$

Then $\mathrm{Fix}(S) = \{(x_1, x_2, x_3) \in \mathbb{M} : x_1 = 0\}$ and

$$\mathrm{Fix}(T) = \{(x_1, x_2, x_3) \in \mathbb{M} : x_1 = x_2 = 0\}.$$

Therefore, the solution set of problem (3.1) is $\mathbb{S} = \{(0,0,0)\}$.
Indeed, choose $\bar{x} = (0,0,p) \in \mathrm{Fix}(T)$ and for any $y = (0,0,q) \in \mathrm{Fix}(T)$, we have

$$\exp_{\bar{a}}^{-1} f(\bar{a}) = (0,0,p/2),\ \exp_{\bar{a}}^{-1} S(\bar{a}) = (0,0,0) \text{ and } \exp_{\bar{a}}^{-1} y = (0,0,q-p),$$

and

$$G(\bar{a}) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, for $\sigma > 0$, we have

$$\left\langle \exp_{\bar{a}}^{-1} S(\bar{a}) + \frac{1}{\sigma} \exp_{\bar{a}}^{-1} f(\bar{a}), \exp_{\bar{a}}^{-1} y \right\rangle = \frac{p}{2\sigma}(q-p) = 0, \quad \forall y \in \mathrm{Fix}(T)$$
$$\Leftrightarrow p = 0.$$

Under the above constructions, Algorithm 4.2 has the following form:

$$x_{n+1} = \gamma_n^1 (1 - \mu_n), \quad \forall\, n \in \mathbb{N},$$

where $\gamma_n^1(0) = f(x_n) = (a_n/2, b_n/2, c_n/2 + b_n^2/2)$ and $\gamma_n^1(1) = y_n = \gamma_n^2(1 - \xi_n)$ such that $\gamma_n^2(0) = T(x_n) = (-a_n, -b_n, c_n)$, $\gamma_n^2(1) = S(x_n) = (-a_n, b_n, c_n)$ for all $x_n = (a_n, b_n, c_n) \in \mathbb{M}$ and all $n \in \mathbb{N}$.

Let $\mu_n = \frac{1}{n}$ and $\xi_n = \frac{1}{n^2}$. Then clearly μ_n and ξ_n satisfy the condition (i)-(iv) of Theorem 4.3. Therefore, form (5.2), we get

$$y_n = \Big(-a_n, b_n\big(2s_n - 1\big), c_n + 4b_n^2 s_n^2 \big(2s_n^2 - 1\big)\Big),$$
$$x_{n+1} = \Big(\frac{a_n}{2}\big(1 - 3t_n\big), \frac{b_n}{2}\big(1 + t_n(4s_n - 3)\big),$$
$$c_n/2 + b_n^2/2 + t_n\big((c_n/2 + b_n^2(8s_n^3 - 12s_n^2 + 16s_n - 17/2) + 8b_n^2 t_n^2 (s_n - 1)^2)\big)\Big),$$

where $s_n = 1 - \xi_n$ and $t_n = 1 - \mu_n$. By initial choice $x_1 = (0,0,1)$, we get the following table of the convergence and graph of the error term $d(x_{n+1}, x_n)$ of Theorem 4.2 by using GNU Octave program version 4.2.2-1ubuntu1 and performed on a PC desktop Intel(R) Core(TM) i5-5200U CPU @ 2.20 GHz, RAM 2.00 GB.

n	x_n	error term $d(x_{n+1}, x_n)$
1	(0,0,1.0000000000000000)	
2	(0,0,0.5000000000000000)	0.25
3	(0,0,0.3750000000000000)	0.15625
4	(0,0,0.3125000000000000)	3.90×10^{-3}
5	(0,0,0.2734375000000000)	1.52×10^{-3}
6	(0,0,0.2460937500000000)	7.47×10^{-4}
7	(0,0,0.2255859375000000)	4.20×10^{-4}
8	(0,0,0.2094726562500000)	2.59×10^{-4}
9	(0,0,0.1963806152343750)	1.71×10^{-4}
10	(0,0,0.1854705810546875)	1.19×10^{-4}
11	(0,0,0.1761970520019531)	8.59×10^{-5}
12	(0,0,0.1681880950927734)	6.41×10^{-5}
13	(0,0,0.1611802577972412)	4.91×10^{-5}
...	...	...
...	...	...

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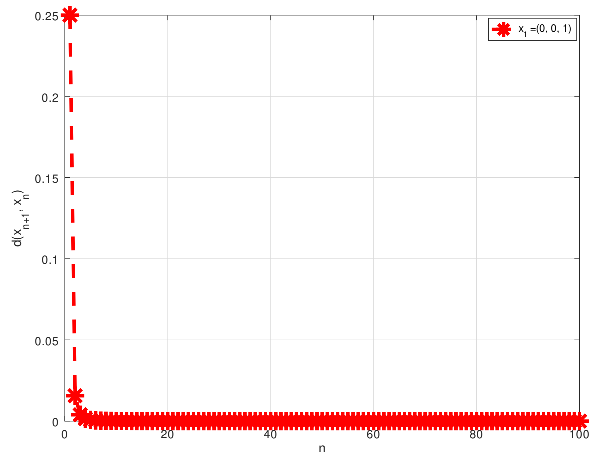

FIGURE 1. $d(x_{n+1}, x_n)$ converging to zero

REFERENCES

[1] Y.I. Alber, S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces*, Oper. Theory Adv. Appl., **98**(1997), 7-22.

[2] S. Al-Homidan, Q.H. Ansari, F. Babu, *Halpern and Mann type algorithms for fixed points and inclusion problems on Hadamard manifolds*, Numer. Funct. Anal. Optim., **40**(6)(2019), 621-653.

[3] Q.H. Ansari, F. Babu, *Proximal point algorithm for inclusion problems in Hadamard manifolds with applications*, Optim. Lett., (2019), DOI: 10.1007/s11590-019-01483-0.

[4] Q.H. Ansari, F. Babu, X.-B. Li, *Variational inclusion problems in Hadamard manifolds*, J. Nonlinear Convex Anal., **19**(2)(2018), 219-237.

[5] M. Bačák, *Computing medians and means in Hadamard spaces*, SIAM J. Optim., **24**(2014), 1542-1566.

[6] D.W. Boyd, J.S. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**(1969), 335-341.

[7] M. Bridson, A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer-Verlag, Berlin, Heidelberg, New York, 1999.

[8] A. Cabot, *Proximal point algorithm controlled by a slowly vanishing term: applications to hierarchical minimization*, SIAM J. Optim., **15**(2005), 555-572.

[9] J.X. da Cruz Neto, O.P. Ferreira, L.R. Lucambio Pérez, *Monotone point-to-set vector fields*, Balkan J. Geometry Appl., **5**(1)(2000), 69-79.

[10] J.X. da Cruz Neto, O.P. Ferreira, L.R. Lucambio Pérez, S.Z. Németh, *Convex- and monotone-tansformable mathematical programming problems and a proximal-like point method*, J. Global Optim., **35**(2006), 53-69.

[11] M.P. do Carmo, *Riemannian Geometry*, Birkhäuser, Boston, Basel, Berlin, 1992.

[12] O.P. Ferreira, P.R. Oliveira, *Proximal point algorithms on Riemannian manifolds*, Optimization, **51**(2002), 257-270.

[13] S. Huang, *Approximations with weak contractions in Hadamard manifolds*, Linear Nonlinear Anal., **1**(2)(2015), 317-328.

[14] C. Li, G. López, V. Martín-Márquez, *Monotone vector fields and the proximal point algorithm on Hadamard manifolds*, J. Lond. Math. Soc., **79**(2009), 663-683.

[15] C. Li, G. López, V. Martín-Márquez, *Itetive algorithms for nonexpansive mappings on Hadamard manifolds*, Taiwanese J. Math., **14**(2)(2010), 541-559.

[16] C. Li, G. López, V. Martín-Márquez, *Resolvents of set-valued monotone vector fields in Hadamard manifolds*, Set-valued Anal., **19**(2011), 361-383.
[17] P.-E. Mainge, A. Moudafi, *Strong convergence of an iterative method for hierarchical fixed-points problems*, Pacific J. Optim., **3**(2007), 529-538.
[18] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl., **241**(2000), 46-55.
[19] S.Z. Németh, *Monotonicity of the complementary vector field of a nonexpansive map*, Acta Math. Hungar., **84**(3)(1999), 189-197.
[20] S.Z. Németh, *Monotone vector fields*, Publ. Math. Debrecen, **54**(1999), 437-449.
[21] S.Z. Németh, *Variational inequalities on Hadamard manifolds*, Nonlinear Anal., **52**(2003), 1491-1498.
[22] T. Rapcsák, *Smooth Nonlinear Optimization in* $\mathbb{R}^n$, Kluwer Academic Publishers, Dordrecht, 1997.
[23] T. Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs, Amer. Math. Soc., Providence, RI, 1996.
[24] M. Solodov, *An explicit descent method for bilevel convex optimization*, J. Convex Anal., **14**(2007), 227-237.
[25] C. Udriste, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1994.
[26] R. Walter, *On the metric projections onto convex sets in Riemannian spaces*, Arch. Math., **XXV**(1974), 91-98.
[27] F. Wang, H. Pham, *On a new algorithm for solving variational inequality and fixed point problems*, J. Nonlinear Var. Anal., **3**(2)(2019), 225-233.
[28] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279-291.
[29] H.K. Xu, *Viscosity method for Hierarchical fixed point approach to variational inequalities*, Taiwanese J. Math., **14**(2)(2010), 463-478.
[30] Y. Yao, Y.J. Cho, Y.C. Liou, *Iterative algorithms for hierarchical fixed points problems and variational inequalities*, Math. Comput. Model., **52**(2010), 1697-1705.

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