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MEANS AND CONVERGENCE OF SEMIGROUP ORBITS

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Abstract. In this paper we prove the following general theorem. Let $(E, \|\cdot\|_E)$ be a uniformly convex Banach space, and let C be a bounded, closed and convex subset of E. Assume that C has nonempty interior and is locally uniformly rotund. Let $\mathcal{F}$ be a commutative nonexpansive semigroup acting on C. If $\mathcal{F}$ has no fixed point in the interior of C, then there exists a unique point $\widetilde{x}$ on the boundary of C such that each orbit of $\mathcal{F}$ converges in norm to $\widetilde{x}$. We also establish analogous results for semigroups and mappings which are asymptotically nonexpansive in the intermediate sense.
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1. Introduction

Let $(E, \|\cdot\|_E)$ be a Banach space, C a nonempty closed subset of E and let $\mathcal{F}$ be a nonexpansive semigroup on C with a fixed point. The convergence in norm of orbits of the semigroup $\mathcal{F}$ is an important problem in metric fixed point theory because this allows us to approximate a fixed point of $\mathcal{F}$ in the simplest way ([1], [15], [17], [18], [19], [23], [24] and [25]).

The authors of the present paper have recently established the following result ([10]).

Theorem 1.1. *Let $(E, \|\cdot\|_E)$ be a uniformly convex Banach space, and let C be a bounded, closed and convex subset of E. Assume that C has nonempty interior and is locally uniformly rotund. Let $\mathcal{F}$ be a strongly measurable nonexpansive semigroup* *acting on C. If $\mathcal{F}$ has no common fixed point in the interior of C, then there exists a unique point $\widetilde{x}$ on the boundary ∂C of C such that each orbit $\{F(t)x : t \geq 0\}$ converges strongly to $\widetilde{x}$.*

In the present paper we prove similar theorems in a more general setting, namely for commutative nonexpansive semigroups and for semigroups which are asymptotically nonexpansive in the intermediate sense. Since Theorem 1.1 has a counterpart for nonexpansive mappings, we also state and prove the analogous result for mappings which are asymptotically nonexpansive in the intermediate sense. In our proofs much more sophisticated ergodic theorems, than those used in [10], are applied. Therefore, for the convenience of the reader, we recall suitable ergodic theorems at the beginning of each of the last three sections of this paper.

2. Basic notions and facts

Throughout this paper all Banach spaces are real and all topologies are Hausdorff. We denote the closed convex hull of a subset C of a Banach space $(E, \|\cdot\|_E)$ by $\overline{\mathrm{conv}}\, C$.

First we recall the notions of a strictly convex Banach space, a locally uniformly rotund set and a uniformly convex Banach space.

Definition 2.1. ([8], [9]) A Banach space $(E, \|\cdot\|_E)$ is said to be *strictly convex* if $\|\frac{x+y}{2}\|_E < 1$ whenever $x, y \in E$, $\|x\|_E \leq 1$, $\|y\|_E \leq 1$ and $x \neq y$.

Definition 2.2. ([27]; see also [22]) Let $(E, \|\cdot\|_E)$ be a Banach space, C be a nonempty, bounded, closed and convex subset of E, and let C have nonempty interior. We say that C is *locally uniformly rotund* if for each $x \in \partial C$ and for each $\varepsilon \in (0, d_x)$, where $d_x := \sup\{\|x - x'\|_E : x' \in C\}$, there exists a number $\delta(x, \varepsilon) > 0$ such that for each $y \in C$ with $\|x - y\|_E \geq \varepsilon$, we have

$$\operatorname{dist}\left(\frac{x+y}{2}, \partial C\right) := \inf\left\{\left\|\frac{x+y}{2} - x'\right\|_E : x' \in \partial C\right\} \geq \delta(x, \varepsilon).$$

Definition 2.3. ([4]) Let $(E, \|\cdot\|_E)$ be a Banach space and let

$$\overline{B_E(0,1)} = \{x \in E : \|x\|_E \leq 1\}$$

denote its closed unit ball. If for each $\varepsilon \in (0, 2]$, there exists $\delta(\varepsilon) > 0$ such that for each $x, x' \in \overline{B_E(0,1)}$ with $\|x - x'\|_E \geq \varepsilon$, we have

$$\left\|\frac{x + x'}{2}\right\|_E \leq 1 - \delta(\varepsilon),$$

then we say that the space $(E, \|\cdot\|_E)$ is *uniformly convex.*

If S is a nonempty set, then the Banach space of all bounded real-valued functions on S with the supremum norm $\|\cdot\|_\infty$ is denoted by $l^\infty(S)$.

Finally, we recall a lemma which is one of the basic tools in our subsequent considerations.

Lemma 2.4. ([10]) *Let $(E, \|\cdot\|)$ be a Banach space and let C be a bounded, closed and convex subset of E. Assume that* $\operatorname{int}(C)$ *is nonempty,* $0 \in \operatorname{int}(C)$ *and C is locally uniformly rotund. Let $\widetilde{x} \in \partial C$, $x^* \in E^*$, $\|x^*\| = 1$, $k \in (0, +\infty)$ and the hyperplane $V_{k,\widetilde{x}} := \{x \in E : x^*(x) = k\}$ that supports C at the point $\widetilde{x}$ be given. If $r \in (0, +\infty)$ and the set*

$$C_r := C \cap \{x \in E : \|x - \widetilde{x}\| \geq r\}$$

is nonempty, then there exists $0 < k_1 < k$ such that

$$C_r \subset \{x \in E : x^*(x) \leq k_1\}.$$

3. Means and convergence of orbits of nonexpansive semigroups

First we recall a few facts from the fixed point theory of nonexpansive mappings.

Definition 3.1. ([8], [9]) Let $(E, \|\cdot\|_E)$ be a Banach space and let C be a noempty subset of X. If $T : C \to C$ and $\|Tx - Tx'\|_E \leq \|x - x'\|_E$ for every $x, x' \in C$), then we say that T is *nonexpansive.*

Theorem 3.2. ([2]) *If $(E, \|\cdot\|_E)$ is a uniformly convex Banach space, C is a nonempty, bounded, closed and convex subset of E, and $T : C \to C$ is nonexpansive, then the mapping T has a fixed point.*

Given a mapping T and a family of mappings $\mathcal{G}$, we denote by $\operatorname{Fix}(T)$ and by $\operatorname{Fix}(\mathcal{G})$ the fixed point set and the common fixed point set of T and $\mathcal{G}$, respectively.

Theorem 3.3. ([2]) *If C is a nonempty, closed and convex subset of a strictly convex Banach space $(E, \|\cdot\|_E)$, and if $T : C \to C$ is a nonexpansive mapping with* $\operatorname{Fix}(T) \neq \emptyset$*, then the set* $\operatorname{Fix}(T)$ *is closed and convex.*

Theorem 3.4. ([2]) *Let C be a nonempty, bounded, closed and convex subset of a uniformly convex Banach space $(E, \|\cdot\|_E)$. Then for any commuting family $\mathcal{G}$ of nonexpansive self-mappings of C, the set* $\operatorname{Fix}(\mathcal{G})$ *is nonempty.*

Remark 3.5. Observe that by Theorems 3.2, 3.3 and 3.4, in every uniformly convex Banach space, for any nonempty, bounded, closed and convex set $C \subset E$, and any commuting family $\mathcal{G}$ of nonexpansive self-mappings of C, the common fixed point set $\operatorname{Fix}(\mathcal{G})$ is nonempty, closed and convex.

Now we consider the case of semigroups. We begin with the following definitions ([5], [6], [26]). Let S be a semigroup. For $a \in S$, we define mappings $l_a : \ell^\infty(S) \to \ell^\infty(S)$ and $r_a : \ell^\infty(S) \to \ell^\infty(S)$ as follows: $(l_a f)(s) := f(as)$ and $(r_a f)(s) := f(sa)$ for $f \in \ell^\infty(S)$ and $s \in S$. Let X be a subspace of $\ell^\infty(S)$ containing constants. We say that X is left (respectively, right) translation invariant if for all $a \in S$ we have $l_a(X) \subset X$ ($r_a(X) \subset X$). The subspace X is translation invariant if X is simultaneously both left and right translation invariant.

Let X be a left translation invariant (respectively, right translation invariant, translation invariant) subspace of $\ell^\infty(S)$ containing constants. If $\mu \in X^*$ and $\|\mu\|_{X^*} = \mu(1) = 1$, then the linear functional μ is called a *mean* on X. It is obvious that $\mu \in X^*$ is a mean if and only if

$$\inf_{s\in S} f(s) \le \mu(f) \le \sup_{s\in S} f(s)$$

for each $f \in X$. We also denote the value of a mean μ on X at f by $\mu_t(f(t))$. A mean μ on X is left invariant (respectively, right invariant) if $\mu(l_a f) = \mu(f)$ $(\mu(r_a f) = \mu(f))$ for each $a \in S$ and $f \in X$. An invariant mean μ is, by definition, a mean which is both left and right invariant.

A semigroup S is called *amenable* if there is a mean μ on $\ell^\infty(S)$ which is both left and right invariant. In the case where only a left (respectively, right) invariant mean exists, the semigroup S is called left-(respectively, right-)amenable.

At this point we quote the following classical theorem.

Theorem 3.6. ([6]) *If a semigroup S is both left- and right-amenable, then it is amenable.*

The next theorem is very important in our subsequent considerations.

Theorem 3.7. ([5]; see also [26]) *Every commutative semigroup S is amenable.*

Let S be a semitopological semigroup, that is, S is a semigroup with a Hausdorff topology $\mathcal{T}$ and for each $a \in S$, both the mappings $S \ni s \mapsto as \in S$ and $S \ni s \mapsto sa \in S$ are continuous. The semigroup S is called left (respectively, right) *reversible* if each two closed right (respectively, left) ideals of S have a nonempty intersection. If S is left (respectively, right) reversible, then S becomes a directed set by declaring $s \le t$ to mean $\{s\} \cup \overline{sS} \supset \{t\} \cup \overline{tS}$ (respectively, $\{s\} \cup \overline{Ss} \supset \{t\} \cup \overline{St}$) for $s, t \in S$, where $\overline{A}$ denotes the closure of A in the topology $\mathcal{T}$.

The following fact is generally known. If S is a semitopological commutative semigroup, then S is *reversible*, that is, S is both left- and right-reversible.

If $(E, \|\cdot\|_E)$ is a Banach space, then we denote the value of $x^* \in E^*$ at $x \in E$ by $x^*(x)$ or $\langle x, x^*\rangle$.

Let S be a semitopological semigroup and let $C(S) \subset \ell^\infty(S)$ be the Banach algebra of all bounded and continuous real-valued functions on S with the supremum norm $\|\cdot\|_\infty$. Let f be a bounded and continuous function on S with values in a reflexive Banach space $(E, \|\cdot\|_E)$, and let ν be an element of $C(S)^*$. Then the real-valued function ψ on E^* given by

$$\psi(x^*) := \nu(\langle f, x^*\rangle) := \nu_s\langle f(s), x^*\rangle$$

for each $x^* \in E^*$ is linear and continuous. Since $(E, \|\cdot\|_E)$ is reflexive, there exists a unique element $f(\nu) \in E$ such that

$$\langle f(\nu), x^*\rangle = \nu_s\langle f(s), x^*\rangle$$

for each $x^* \in E^*$. If $\mathcal{F} = \{F(s) : s \in S\}$ is a nonexpansive semigroup on a closed and convex subset C of E (that is, every mapping $F(s) : C \to C$ is nonexpansive),

$\{F(s)x\}_{s\in S}$ is bounded for some $x \in C$ and $f(s) := F(s)x$ for each $s \in S$, then we denote $f(\nu)$ by $F(\nu)x$.

If S is a semitopological semigroup, then $\Lambda(S)$ denotes the algebraic center of S, that is, all $s \in S$ such that $st = ts$ for all $t \in S$.

In [14] O. Kada, A. T. Lau and W. Takahashi proved the following theorem.

Theorem 3.8. *Let S be a right reversible semitopological semigroups, C be a nonempty, closed and convex subset of a uniformly convex Banach space $(E, \|\cdot\|_E$, and let $\mathcal{F} = \{F(s) : s \in S\}$ be a nonexpansive semigroup on C. Let μ be a right invariant mean on $C(S)$. Then*

(1) *$F(\mu)$ is nonexpansive on C,*
(2) *if $F(\mu)x \in \mathrm{Fix}(\mathcal{F})$ for each $x \in C$, then $F(\mu)^2 = F(\mu)$, that is, $F(\mu)$ is a retraction on C,*
(3) *$F(\mu)F(s) = F(\mu)$ for each $s \in S$,*
(4) *$F(s)F(\mu) = F(\mu)$ for each $s \in \Lambda(S)$,*
(5) *$F(\mu)x \in \bigcap_{s\in S} \overline{\mathrm{conv}}\,\{F(t)x : t \geq s\}$ for each $x \in C$.*

Directly from this theorem they obtain the following corollary.

Corollary 3.9. ([11]) *Let S be a commutative semitopological semigroup, C be a nonempty, bounded and closed convex subset of a uniformly convex Banach space, and let $\mathcal{F} = \{F(s) : s \in S\}$ be a nonexpansive semigroup on C. Then for each invariant mean μ on $C(S)$, the mapping $F(\mu)$ is a nonexpansive retraction from C onto* $\mathrm{Fix}(\mathcal{F})$ *and $F(\mu)x \in \bigcap_{s\in S} \overline{\mathrm{conv}}\,\{F(t)x : t \geq s\}$ for each $x \in C$.*

Now we are able to prove the following theorem.

Theorem 3.10. *Let $(E, \|\cdot\|_E)$ be a uniformly convex Banach space, and let C be a bounded, closed and convex subset of E. Assume that C has nonempty interior and is locally uniformly rotund. Let S be a commutative semitopological semigroup and let $\mathcal{F} = \{F(s) : s \in S\}$ be a commutative nonexpansive semigroup on C. If $\mathcal{F}$ has no common fixed point in the interior of C, then there exists a unique point $\widetilde{x}$ on the boundary ∂C of C such that each orbit $\{F(s)x : s \in S\}$ converges strongly to $\widetilde{x}$.*

Proof. Without any loss of generality we may assume that $0 \in \mathrm{int}(C)$. By Theorems 3.3 and 3.4, and Remark 3.5, the semigroup $\mathcal{F}$ has exactly one common fixed point $\widetilde{x}$ and this point lies on the boundary ∂C of C. We claim that each orbit $\{F(s)x : s \in S\}$ converges strongly to $\widetilde{x}$. To see this, we first observe that for each point $x \in C$, the function $\{\|F(s)x - \widetilde{x}\|_E\}$, $s \in S$, is decreasing and therefore there exists $\lim_{s\in S} \|F(s)x - \widetilde{x}\|_E$. Suppose to the contrary that there exists a point $y \in C$ such that $r := \lim_{s\in S} \|F(s)y - \widetilde{x}\|_E > 0$. Then we have

$$F(s)y \in C_r = C \cap \{x \in E : \|x - \widetilde{x}\|_E \geq r\}$$

for all $s \in S$. Now let $x^* \in E^*$, $\|x^*\|_{E^*} = 1$, and $0 < k \in \mathbb{R}$ be such that the hyperplane $V_{k,\widetilde{x}} = \{x \in E : x^*(x) = k\}$ supports C at $\widetilde{x}$. By Lemma 2.4, there exists a number $0 < k_1 < k$ such that

$$C_r \subset \{x \in E : x^*(x) \leq k_1\}.$$

This implies that

$$\overline{\text{conv}}\, C_r \subset \{x \in E : x^*(x) \leq k_1\}.$$

Now we take an invariant mean μ on $C(S)$. Then by Corollary 3.9, the mapping $F(\mu)$ is a nonexpansive retraction from C onto $\text{Fix}(\mathcal{F}) = \{\widetilde{\text{x}}\}$ and

$$F(\mu)y \in \bigcap_{s \in S} \overline{\text{conv}}\, \{F(t)y : t \geq s\}.$$

Hence we get

$$\widetilde{x} \in \bigcap_{s \in S} \overline{\text{conv}}\, \{F(t)y : t \geq s\} \subset \overline{\text{conv}}\, \{F(s)y : s \in S\} \subset \overline{\text{conv}}\, C_r$$

but this is impossible because then we have

$$x^*(\widetilde{x}) \leq k_1 < k = x^*(\widetilde{x}).$$

The contradiction we have reached completes the proof.

Remark 3.11. Theorem 3.10 is a generalization of Theorem 5.1 in [10].

4. Convergence of orbits of semigroups which are asymptotically nonexpansive in the intermediate sense

In this section we show that a result analogous to that in Section 3 is also valid for orbits of semigroups which are asymptotically nonexpansive in the intermediate sense. We first recall several definitions and facts which we need in our subsequent considerations.

Let C be a nonempty, bounded, closed and convex subset of a Banach space $(E, \|\cdot\|_E)$. If $T : C \to C$ is continuous and

$$\limsup_{k \to \infty} \sup_{x,y \in C} \left(\left\| T^k x - T^k y \right\|_E - \|x - y\|_E \right) \leq 0,$$

then T is called *asymptotically nonexpansive in the intermediate sense* ([3]).

Recall that if T is a self-mapping of C, then $\text{Fix}\,(\text{T})$ always denotes the set of fixed points of T.

Next we recall the notion of a semigroup which is asymptotically nonexpansive in the intermediate sense ([12]).

Once more, let $(E, \|\cdot\|_E)$ be a Banach space and C a nonempty subset of E. Let $\mathcal{F} = \{F\,(t)\}_{t \geq 0}$ be a family of self-mappings of C. Recall that $\mathcal{F}$ is said to be a semigroup which is asymptotically nonexpansive in the intermediate sense if the following five conditions are satisfied:

(1) $F\,(t) : C \to C$ is continuous for each $t \in [0, \infty)$;
(2) $F\,(s + t)\,x = F\,(s)\,F\,(t)\,x$ for all $s, t \in [0, \infty)$ and $x \in C$;
(3) $F\,(0) = I$, where I is the identity mapping;
(4) $\limsup_{t \to \infty} \sup_{x,y \in C} \left(\|F\,(t)\,x - F\,(t)\,y\|_E - \|x - y\|_E\right) \leq 0$;
(5) the orbit $\{F\,(t)\,x\}_{t \geq 0}$ is continuous in $t \in [0, \infty)$ for each $x \in C$.

We continue to denote the set of common fixed points of a semigroup $\mathcal{F}$ by Fix$(\mathcal{F})$.

Remark 4.1. It is generally known that using the asymptotic center method ([16]; see also [7], [8] and [9]), we get that if $(E, \|\cdot\|_E)$ is a uniformly convex Banach space, C is a nonempty, bounded, closed and convex subset of E, and T is a self-mapping of C which is asymptotically nonexpansive in the intermediate sense, then the fixed point set of T is nonempty, closed and convex. Hence we can conclude that if $\mathcal{G}$ is a commuting family of self-mappings of C which are asymptotically nonexpansive in the intermediate sense, then the common fixed point set of $\mathcal{G}$ is nonempty, closed and convex (see the proof of Theorem 2 in [2]).

Next we prove the following auxiliary lemma.

Lemma 4.2. *Let $(E, \|\cdot\|_E)$ be a Banach space and C be a nonempty subset of E. Let $\mathcal{F} = \{F(t)\}_{t\geq 0}$ be a semigroup which acts on C and is asymptotically nonexpansive in the intermediate sense, and let $\widetilde{x} \in C$ be a fixed point of $\mathcal{F}$. If the orbit $\{F(t)\,x\}_{t\geq 0}$ of a point $x \in C \setminus \{\widetilde{x}\}$ does not tend to $\widetilde{x}$, then $\inf_{t\geq 0} \|F(t)\,x - \widetilde{x}\|_E > 0$.*

Proof. Suppose to the contrary that $\inf_{t\geq 0} \|F(t)\,x - \widetilde{x}\|_E = 0$. Then there exists a sequence $\{t_n\}_n$ in $(0,\infty)$ such that $\lim_n \|F(t_n)\,x - \widetilde{x}\|_E = 0$. Given an $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|F(t_{n_0})\,x - \widetilde{x}\|_E < \frac{\varepsilon}{2}.$$

Since $\widetilde{x}$ is a common fixed point of $\mathcal{F}$ and $\mathcal{F}$ is a semigroup of mappings which are asymptotically nonexpansive in the itermediate sense, there is a number $t_\varepsilon \in (0,\infty)$ such that

$$\|F(t)\,(F(t_{n_0})\,x) - \widetilde{x}\|_E < \|F(t_{n_0})\,x - \widetilde{x}\|_E + \frac{\varepsilon}{2}$$

for each $t \geq t_\varepsilon$. Hence for $t \geq t_{n_0} + t_\varepsilon$, we obtain

$$\|F(t)\,x - \widetilde{x}\|_E < \varepsilon.$$

This in its turn means that the orbit $\{F(t)\,x\}_{t\geq 0}$ does tend to $\widetilde{x}$, contrary to our assumption that the orbit $\{F(t)\,x\}_{t\geq 0}$ does not converge to $\widetilde{x}$. Thus the assertion of the theorem holds.

We also recall the following fact.

Theorem 4.3. ([12]) *Let $(E, \|\cdot\|_E)$ be a uniformly convex Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let $\mathcal{F} = \{F(t)\}_{t\geq 0}$ be a semigroup which acts on C and is asymptotically nonexpansive in the intermediate sense. Let $x \in C$. Then for any $\varepsilon > 0$ and $t > 0$, there exists a number $R_{\varepsilon,t} > 0$ such that for all $h \geq R_{\varepsilon,t}$, $r \geq R_{\varepsilon,t}$ and $t > 0$, we have*

$$\left\| F(h)\left(\frac{1}{t}\int_0^t F(r+\tau)\,x\,d\tau\right) - \frac{1}{t}\int_0^t F(h+r+\tau)\,x\,d\tau \right\|_E < \varepsilon.$$

In particular, for each $t > 0$, there exists $r_t > 0$ such that

$$\left\| F(h)\left(\frac{1}{t}\int_0^t F(r+\tau)\,x\,d\tau\right) - \frac{1}{t}\int_0^t F(h+r+\tau)\,x\,d\tau \right\|_E < \frac{1}{t}$$

for all $h, r \geq r_t$.

Remark 4.4. Without loss of generality (see the proof of Theorem 3.2 in [12]) we may assume in the sequel that

$$\lim_{t \to \infty} r_t = \infty.$$

The next theorem, which actually follows from a part of the proof of Theorem 4.1 in [12], is crucial in the proof of the main theorem of this section.

Theorem 4.5. *Let* $(E, \|\cdot\|_E)$ *be a uniformly convex Banach space and let* C *be a nonempty, bounded, closed and convex subset of* E. *Let* $\mathcal{F} = \{F(t)\}_{t \geq 0}$ *be a semigroup which acts on* C *and is asymptotically nonexpansive in the intermediate sense. Let* $x \in C$. *Then each subsequential weak limit of* $\left\{\frac{1}{t}\int_0^t F(r_t + \tau)\, x\, d\tau\right\}_{t>0}$ *is a fixed point of* $\mathcal{F}$, *that is, if* $0 < t_j \to \infty$ *and* $\overline{x} = w - \lim_j \frac{1}{t_j}\int_0^{t_j} F\left(r_{t_j} + \tau\right) x\ d\tau$, *then* $\overline{x} \in \mathrm{Fix}\,(\mathcal{F})$.

Now we can state and prove the main theorem of this section.

Theorem 4.6. *Let* $(E, \|\cdot\|_E)$ *be a uniformly convex Banach space and let* C *be a nonempty, bounded, closed and convex subset of* E. *Assume that* C *has nonempty interior and is locally uniformly rotund. Let* $\mathcal{F} = \{F(t)\}_{t \geq 0}$ *be a semigroup which acts on* C *and is asymptotically nonexpansive in the intermediate sense. If* $\mathcal{F}$ *has no common fixed point in the interior of* C, *then there exists a unique point* $\widetilde{x}$ *on the boundary* ∂C *of* C *such that each orbit* $\{F(t)x\}_{t\geq 0}$ *converges strongly to* $\widetilde{x}$.

Proof. There is no loss of generality in assuming that $0 \in \mathrm{int}(C)$. By Remark 4.1, the semigroup $\mathcal{F}$ has exactly one common fixed point $\widetilde{x}$ which, in addition, lies on the boundary ∂C of C. We claim that each orbit $\{F(t)x\}_{t\geq 0}$ of $\mathcal{F}$ converges strongly to $\widetilde{x}$. Indeed, suppose to the contrary that there exists a point $y \in C$ such that its orbit $\{F(t)y\}_{t\geq 0}$ does not converge strongly to $\widetilde{x}$. Then by Lemma 4.2, we have $r := \inf_{t \geq 0} \|F(t)\, y - \widetilde{x}\|_E > 0$. Hence we get

$$F(t)y \in C_r = C \cap \{x \in E : \|x - \widetilde{x}\|_E \geq r\}$$

for all $t \in [0, \infty)$. Now let $x^* \in E^*$, $\|x^*\|_{E^*} = 1$, and $0 < k \in \mathbb{R}$ be such that the hyperplane $V_{k,\widetilde{x}} = \{x \in E : x^*(x) = k\}$ supports C at the point $\widetilde{x}$. By Lemma 2.4, there is a number $0 < k_1 < k$ such that

$$C_r \subset \{x \in E : x^*(x) \leq k_1\}.$$

This implies, in its turn, that

$$\overline{\mathrm{conv}}\, C_r \subset \{x \in E : x^*(x) \leq k_1\}.$$

Now consider a weakly convergent sequence $\left\{\frac{1}{t_j}\int_0^{t_j} F\left(r_{t_j} + \tau\right) y\ d\tau\right\}_j$, where $t_j \to \infty$. Observe that by Theorem 4.5, this sequence tends to the fixed point of $\mathcal{F}$, that is, to $\widetilde{x}$. But this is impossible because then we get

$$x^*(\widetilde{x}) \leq k_1 < k = x^*(\widetilde{x}).$$

The contradiction we have reached completes the proof of the theorem.

5. Convergence of iterates of mappings which are asymptotically nonexpansive in the intermediate sense

In this section we are concerned with a result for iterates of mappings which are asymptotically nonexpansive in the intermediate sense. This result is analogous to Theorem 4.6. It is a consequence of the following theorems, which are analogous to Theorems 4.3 and 4.5.

Theorem 5.1. ([13]) *Let $(E, \|\cdot\|_E)$ be a uniformly convex Banach space and let T be a self-mapping of a nonempty, bounded, closed and convex subset C of E, which is asymptotically nonexpansive in the intermediate sense. If $x \in C$, then for each $\varepsilon > 0$ and $n \geq 1$, there exists a number $M_{\varepsilon,n} \geq 1$ such that for all $k \geq M_{\varepsilon,n}$ and $m \geq M_{\varepsilon,n}$, we have*

$$\left\| T^k \left(\frac{1}{n} \sum_{i=0}^{n-1} T^{i+m+1} x \right) - \frac{1}{n} \sum_{i=0}^{n-1} T^{k+i+m+1} x \right\|_E < \varepsilon.$$

In particular, for each $n \geq 1$, there exists a natural number $m_n \geq 1$ such that

$$\left\| T^k \left(\frac{1}{n} \sum_{i=0}^{n-1} T^{i+m+1} x \right) - \frac{1}{n} \sum_{i=0}^{n-1} T^{k+i+m+1} x \right\|_E < \frac{1}{n}$$

for all $k, m \geq m_n$.

Remark 5.2. Without loss of generality (see the proof of Theorem 5.2 in [13]) we may assume in the sequel that

$$\lim_{n \to \infty} m_n = \infty.$$

Theorem 5.3. *Let $(E, \|\cdot\|_E)$ be a uniformly convex Banach space, C a nonempty, bounded, closed and convex subset of E, T a self-mapping of C which is asymptotically nonexpansive in the intermediate sense, $x \in C$, and let $\{m_n\}_{n \geq 1}$ be the sequence of positive real numbers appearing in Theorem 5.1. Then each weak subsequential limit of $\left\{ \frac{1}{n} \sum_{i=0}^{n-1} T^{m_n+i+1} x \right\}_n$ is a fixed point of T.*

We also need the following auxiliary lemma which is analogous to Lemma 4.2.

Lemma 5.4. *Let $(E, \|\cdot\|_E)$ be a Banach space, C a nonempty subset of E, $T : C \to C$ a mapping which is an asymptotically nonexpansive in the itermediate sense and let $\widetilde{x} \in C$ be a fixed point of T. If for a point $x \in C \setminus \{\widetilde{x}\}$, its sequence of iterates $\{T^n x\}_n$ does not tend to $\widetilde{x}$, then $\inf_n \|T^n x - \widetilde{x}\|_E > 0$.*

Proof. Suppose to the contrary that $\inf_n \|T^n x - \widetilde{x}\|_E = 0$. Then there exists a subsequence $\{T^{n_l} x\}_l$ of the sequence of iterates $\{T^n x\}_n$ such that

$$\lim_l \|T^{n_l} x - \widetilde{x}\|_E = 0.$$

Given an $\varepsilon > 0$, there exists $l_\varepsilon \in \mathbb{N}$ such that

$$\|T^{n_{l_\varepsilon}} x - \widetilde{x}\|_E < \frac{\varepsilon}{2}.$$

By assumption, $\widetilde{x}$ is a fixed point of T and therefore directly from the definition of a mapping which is asymptotically nonexpansive in the itermediate sense, we obtain $k_0 \in \mathbb{N}$ such that

$$\|T^k(T^{n_{l_\varepsilon}}x) - \widetilde{x}\|_E < \|T^{n_{l_0}}x - \widetilde{x}\|_E + \frac{\varepsilon}{2}$$

for each $k \geq k_0$. Hence for $n \geq n_{l_\varepsilon} + k_0$, we have

$$\|T^n x - \widetilde{x}\|_E < \varepsilon.$$

This, in its turn, means that the sequence of iterates $\{T^n x\}_n$ does tend to $\widetilde{x}$, contrary to the assumption of the theorem. The contradiction we have reached completes the proof.

Now it is evident that slight changes (see below) in the proof of Theorem 4.6 yield a proof of the following theorem.

Theorem 5.5. *Let $(E, \|\cdot\|_E)$ be a uniformly convex Banach space, and let C be a bounded, closed and convex subset of E. Assume that C has nonempty interior and is locally uniformly rotund. Let $T : C \to C$ be a mapping which is asymptotically nonexpansive in the itermediate sense. If T has no fixed point in the interior of C, then there exists a unique point $\widetilde{x}$ on the boundary ∂C of C such that each sequence of iterates $\{T^n x\}_n$ converges strongly to $\widetilde{x}$.*

Proof. Without any loss of generality we may assume that $0 \in \operatorname{int}(C)$. By Remark 4.1, the mapping T has exactly one fixed point $\widetilde{x}$ which, in addition, lies on the boundary ∂C of C. We claim that each orbit $\{T^n x\}_n$ converges strongly to $\widetilde{x}$. Suppose to the contrary that there exists a point $y \in C$ such that $\{T^n y\}_{t\geq 0}$ does not converge in norm to $\widetilde{x}$. Then by Lemma 5.4, we have $r := \inf_n \|T^n y - \widetilde{x}\|_E > 0$. Hence we get

$$T^n y \in C_r = C \cap \{x \in E : \|x - \widetilde{x}\|_E \geq r\}$$

for all $n \in \mathbb{N}$. Now let $x^* \in E^*$, $\|x^*\|_{E^*} = 1$, and $0 < k \in \mathbb{R}$ be such that the hyperplane $V_{k,\widetilde{x}} = \{x \in E : x^*(x) = k\}$ supports C at $\widetilde{x}$. By Lemma 2.4, there is a number $0 < k_1 < k$ such that

$$C_r \subset \{x \in X : x^*(x) \leq k_1\}.$$

This implies that

$$\overline{\operatorname{conv}}\, C_r \subset \{x \in E : x^*(x) \leq k_1\}.$$

Now consider a weakly convergent subsequence $\left\{\frac{1}{n_j}\sum_{i=0}^{n-1} T^{m_{n_j}+i+1}y\right\}_j$, where $m_{n_j} \to \infty$. Note that by Theorem 5.3, this subsequence tends to the fixed point of T, that is, to $\widetilde{x}$. But this is impossible because then we have

$$x^*(\widetilde{x}) \leq k_1 < k = x^*(\widetilde{x}).$$

The contradiction we have reached completes the proof of the theorem.

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