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THE COMPLETION OF GENERALIZED B-METRIC SPACES AND FIXED POINTS

ŞTEFAN COBZAŞ* AND STEFAN CZERWIK**

*Babeş-Bolyai University, Department of Mathematics, Cluj-Napoca, Romania
E-mail: scobzas@math.ubbcluj.ro

**Silesian University of Technology, Institute of Mathematics, Gliwice, Poland
E-mail: steczerw@gmail.com

Abstract. We introduce the notion of generalized b-metric, as a b-metric which can take infinite values, and prove the existence and uniqueness of the completion of some particular b-metric spaces (called generalized strong b-metric spaces). Some fixed point results in b-metric spaces and their counterparts in generalized b-metric spaces are proved.
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1. Introduction

There are a lot of extensions of the notions of metric and metric space – see, for instance, the books [20], [34], [46], or the survey papers [12], [32]. In this paper we concentrate on b-metric and generalized b-metric spaces, their topological properties, the existence of the completion and some fixed point results.

A *b-metric* on a nonempty set X is a function $d : X \times X \to [0, \infty)$ satisfying the conditions

$$\begin{aligned} &\text{(i)} \quad d(x,y) = 0 \iff x = y;\\ &\text{(ii)} \quad d(x,y) = d(y,x);\\ &\text{(iii)} \quad d(x,y) \le s[d(x,z) + d(z,y)], \end{aligned} \tag{1.1}$$

for all $x, y, z \in X$, and for some fixed number $s \ge 1$. The pair (X, d) is called a b-*metric space*. Obviously, for $s = 1$ one obtains a metric on X.

Along with the inequality (iii), called the *s-relaxed triangle inequality*, one considers also the *s-relaxed polygonal inequality*

$$d(x_0, x_n) \le s[d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)], \tag{iv}$$

for all $x_0, x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$.

The relaxed triangle inequality and the corresponding spaces were rediscovered several times under various names – quasi-metric, near-metric (in [20]), metric type, etc. We mention some of these authors.

- (1970) Coifman and de Guzman [13] in connection with some problems in harmonic analysis (a b-metric is called by them "distance" function);
- (1979) the results of Coifman and de Guzman were completed by Macias and Segovia [38, 39];
- (1989) Bakhtin [6] called them "quasi-metric spaces" and proved a contraction principle for such spaces;
- (1993) Czerwik introduced them under the name "b-metric space", first for $s = 2$ in [14], and then for an arbitrary s in [15], with applications to fixed points;
- (1998,2003) Fagin *et al.* [25, 26] considered distances satisfying the s-relaxed triangle and polygonal inequalities with applications to some problems in theoretical computer science;
- (2010) Khamsi and Hussain, [31], [33] introduced them under the name "metric type spaces" and remarked that if D is a cone metric on a set X with values in a Banach space ordered by a normal cone with normality constant K, then $d(x, y) = \|D(x, y)\|$, $x, y \in X$, is a b-metric on X satisfying the K-relaxed polygonal inequality.

Some topological properties of b-metric spaces (e.g. compactness) were studied in [33]. Xia [48] studied the properties of the space $C(T, X)$ of continuous functions from a compact metric space T to a b-metric space X, and geodesics and intrinsic metrics in b-metric spaces. The results were applied to show that the optimal transport paths between atomic probability measures are geodesics in the intrinsic metric. An, Tuyen and Dung [3] extended to b-metric spaces Stone's paracompactness theorem.

One can consider also an "ultrametric" version of (iii):

$$d(x, y) \le \lambda \max\{d(x, z), d(y, z)\}\,, \tag{iii$'$}$$

for all $x, y, z \in X$. It is obvious that

$$\begin{aligned}(\text{iii}') &\implies (\text{iii}) \quad \text{with} \;\; s = \lambda;\\ (\text{iii}) &\implies (\text{iii}') \quad \text{with} \;\; \lambda = 2s.\end{aligned}$$

The condition

$$\max\{d(x, z), d(y, z)\} \le \varepsilon \implies d(x, y) \le 2\varepsilon\,, \tag{iii$''$}$$

for all $\varepsilon > 0$ and $x, y, z \in X$, is equivalent to (iii$'$) with $\lambda = 2$.

A typical example of b-normed space can be obtained from a metric space.

Example 1.1. If (X, d) is a metric space and $\beta > 1$, then $d^\beta(x, y)$ is a b-metric, satisfying the inequality

$$d^\beta(x, y) \le 2^\beta[d^\beta(x, y) + d^\beta(x, y)]\,.$$

It is obvious that the relaxed polygonal inequality implies the relaxed triangle inequality. The following example shows that the converse is not true – there exist b-metrics that do not satisfy the relaxed polygonal inequality.

Example 1.2. ([34], Theorem 12.10) Let $X = [0,1]$ and $d(x,y) = (x-y)^2$, $x, y \in [0,1]$. Then d is a 2-relaxed metric on X which is not polygonally s-relaxed for any $s \geq 1$.

Indeed, it is easy to check that d satisfies the 2-relaxed triangle inequality. Suppose that d satisfies the s-relaxed polygonal inequality for some $s \geq 1$. Taking $x_i = \frac{i}{n}$, $1 \leq i \leq n-1$, we obtain

$$\frac{1}{s} = \frac{1}{s} \cdot d(0,1) \leq d(0,x_1) + d(x_1,x_2) + \cdots + d(x_{n-1},1) = n \cdot \left(\frac{1}{n}\right)^2 = \frac{1}{n}\,,$$

for all $n \in \mathbb{N}$, which is impossible.

We use standard notation:

$$\mathbb{N} = \{1,2,\dots\},\ \mathbb{N}_0 = \{0,1,2,\dots\},\ \mathbb{R}_+ = [0,\infty)\,.$$

2. Topological properties of b-metric spaces and metrizability

Let (X,d) be a b-metric space. One introduces a topology on a b-metric space (X,d) in the usual way. The "open" ball $B(x,r)$ of center $x \in X$ and radius $r > 0$ is given by

$$B(x,r) = \{y \in X : d(x,y) < r\}\,.$$

A subset Y of X is called open if for every $x \in Y$ there exists a number $r_x > 0$ such that $B(x,r_x) \subseteq Y$. Denoting by τ_d the family of all open subsets of X it follows that τ_d satisfies the axioms of a topology. This topology is derived from a uniformity $\mathcal{U}_d$ on X having as basis the sets

$$U_\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}, \quad \varepsilon > 0\,.$$

The uniformity $\mathcal{U}_d$ has a countable basis $\{U_{1/n} : n \in \mathbb{N}\}$ so that, by Frink's metrization theorem ([27]), the uniformity $\mathcal{U}_d$ is derived from a metric ρ, hence the topology τ_d as well. This was remarked in the paper [38]. In [25] it is shown that the topology τ_d satisfies the hypotheses of the Nagata-Smirnov metrizability theorem.

Concerning the metrizability of uniform and topological spaces, see the treatise [24].

There exist also direct proofs of the metrizability of the topology of a b-metric space.

Let (X,d) be a b-metric space. Put

$$\rho(x,y) = \inf\Big\{\sum_{k=1}^{n} d(x_{i-1},x_i)\Big\}\,, \tag{2.1}$$

where the infimum is taken over all $n \in \mathbb{N}$ and all chains $x = x_0, x_1, \dots, x_n = y$ of elements in X connecting x and y.

As remarked Frink [27], if a b-metric d satisfies (iii′) for $\lambda = 2$, then formula (2.1) defines a metric equivalent to d. We present the result in the form given by Schroeder [47].

Theorem 2.1 (A.H. Frink [27] and V. Schroeder [47])**.** *If $d : X \times X \to [0,\infty)$ satisfies the conditions (i), (ii) from* (1.1) *and* (iii′) *for some $1 \leq \lambda \leq 2$, then the function ρ defined by* (2.1) *is a metric on X satisfying the inequalities $\frac{1}{2\lambda} d \leq \rho \leq d$.*

V. Schroeder [47] also showed that for every $\varepsilon > 0$ there exists a b-metric d satisfying (1.1).(iii) with $s = 1 + \varepsilon$ such that the mapping ρ defined by (2.1) is not a metric. Other example showing the limits of Frink's metrization method was given by An and Dung [2].

General results of metrizability were obtained in [1] and [42] by a slight modification of Frink's technique.

Let (X, d) be a b-metric space. For $0 < p \leq 1$ define

$$\rho(x, y) = \inf\left\{\sum_{k=1}^{n} d^p(x_{i-1}, x_i)\right\}, \tag{2.2}$$

where the infimum is taken over all $n \in \mathbb{N}$ and all chains $x = x_0, x_1, \ldots, x_n = y$ of elements in X.

The function ρ_p defined by (2.2) is a pseudometric satisfying the inequality

$$d^p(x, y) \geq \rho_p(x, y), \tag{2.3}$$

for all $x, y, z \in X$.

Theorem 2.2. ([42]) *Let d be a b-metric on a nonempty set X satisfying the s-relaxed triangle inequality* (1.1)*.(iii), for some $s \geq 1$. If the number $p \in (0, 1]$ is given by the equation $(2s)^p = 2$, then the mapping $\rho_p : X \times X \to [0, \infty)$ defined by* (2.2) *is a metric on X satisfying the inequalities*

$$\rho_p(x, y) \leq d^p(x, y) \leq 2\rho_p(x, y), \tag{2.4}$$

for all $x, y \in X$.

The same conclusions hold if d satisfies the conditions (i), (ii) from (1.1) *and* (iii$'$) *for some $\lambda \geq 2$. In this case $0 < p \leq 1$ is given by $\lambda^p = 2$ and the metric ρ_p satisfies the inequalities*

$$\rho_p(x, y) \leq d^p(x, y) \leq 4\rho_p(x, y), \tag{2.5}$$

for all $x, y \in X$.

The inequalities (2.4) have the following consequences.

Corollary 2.3. *Under the hypotheses of Theorem 2.2, $\tau_d = \tau_\rho$, that is the topology of any b-metric space is metrizable, and the convergence of sequences with respect toτ_d is characterized in the following way:*

$$x_n \xrightarrow{\tau_d} x \iff d(x, x_n) \longrightarrow 0,$$

for any sequence (x_n) in X and $x \in X$.

Proof. The equality of topologies follows from the inclusions

$$B_d(x, r^{1/p}) \subseteq B_\rho(x, r) \quad \text{and} \quad B_\rho\big(x, 4^{-1}r^p\big) \subseteq B_d(x, r),$$

valid for all $x \in X$ and $r > 0$.

The statement concerning sequences is a consequence of this equality and of the inequalities (2.4). □

Remark 2.4. In [1] the proof is given for a $p > 0$ satisfying the inequality $p \geq (\log_2(3s))^{-2}$. A proof of Theorem 2.2 is also given in the book by Heinonen [29, Prop. 14.5], with the evaluation $p \geq (\log_2 \lambda)^{-2}$, where λ is the constant from (iii′).

We consider now two continuity notions for b-metrics. Let (X, d) be a b-metric space. The b-metric d is called:

- *continuous* if
$$d(x_n, x) \to 0 \text{ and } d(y_n, y) \to 0 \implies d(x_n, y_n) \to d(x, y); \tag{2.6}$$
- *separately continuous* if the function $d(x, \cdot)$ is continuous on X for every $x \in X$, i.e.,
$$d(y_n, y) \to 0 \implies d(x, y_n) \to d(x, y)\,, \tag{2.7}$$

for all sequences $(x_n), (y_n)$ in X and all $x, y \in X$.

Remark 2.5. Let (X, d) be a b-metric space and $x \in X$. Then

$$B(x, r) \text{ is } \tau_d\text{-open for every } r > 0 \iff d(x, \cdot) \text{ is upper semicontinuous on } X\,.$$

Consequently, the balls $B(x, r)$ are τ_d-open, provided the b-metric is separately continuous on X.

The equivalence follows from the equality
$$B(x, r) = d(x, \cdot)^{-1}\big((-\infty, r)\big)\,.$$

The topology τ_d generated by a b-metric d has some peculiarities – a ball $B(x, r)$ need not be τ_d-open and the b-metric d could not be continuous on $X \times X$. Examples can be found in [3] and [42].

In connection to the metrizability of b-metric spaces, we mention the following notions of equivalence for b-metrics.

Let d_1, d_2 be two b-metrics on the same set X. Then d_1, d_2 are called:

- *topologically equivalent* if $\tau_{d_1} = \tau_{d_2}$;
- *uniformly equivalent* if the identity mapping I_X on X is uniformly continuous both from (X, d_1) to (X, d_2) as well as from (X, d_2) to (X, d_1), i.e.
$$\forall\varepsilon > 0, \exists\delta(\varepsilon) > 0 \text{ such that } d_1(x, y) \leq \delta(\varepsilon) \Rightarrow d_2(x, y) \leq \varepsilon,$$
$$\forall\varepsilon > 0, \exists\delta(\varepsilon) > 0 \text{ such that } d_2(x, y) \leq \delta(\varepsilon) \Rightarrow d_1(x, y) \leq \varepsilon\,.$$
- *Lipschitz equivalent* if there exist $c_1, c_2 > 0$ such that
$$c_1 d_2(x, y) \leq d_1(x, y) \leq c_2 d_2(x, y)\,,$$
for all $x, y \in X$

Of course, the above definitions applies to metrics as well, as particular cases of b-metrics.

Remark 2.6. It is obvious that, in general,

Lipschitz equivalence $\Rightarrow$ uniform equivalence $\Rightarrow$ topological equivalence.

So the expression "the topology τ_d generated by a b-metric d on a set X is metrizable" means that there exists a metric ρ on X topologically equivalent to d.

The problem of the existence of a metric that is Lipschitz equivalent to a b-metric was solved in [25], where this property was called *metric boundedness.*

Theorem 2.7. ([25], see also [34], Theorem 12.9) *Let (X, d) be a b-metric space. Then d is Lipschitz equivalent to a metric if and only if d satisfies the s-relaxed polygonal inequality* (iv) *for some $s \geq 1$.*

Concerning the openness of balls in b-metric spaces we mention the following result.

Theorem 2.8. ([38]) *Let (X, d) be a b-metric space. Then there exist a b-metric d' on X, Lipschitz equivalent to d, and the constants $C > 0$ and $0 < \alpha < 1$ such that*

$$|d'(x,z) - d'(y,z)| \leq Cr^{1-\alpha} \left(d'(x,y)\right)^{\alpha}, \tag{2.8}$$

whenever $\max\{d'(x,z), d'(y,z)\} < r$.

Remark 2.9. The inequality (2.8) can be written in the equivalent form

$$|d'(x,z) - d'(y,z)| \leq C \left(d'(x,y)\right)^{\alpha} \left(\max\{d'(x,z), d'(y,z)\}\right)^{1-\alpha}, \tag{2.9}$$

and it is easy to check that the balls corresponding to a b-metric d' satisfying (2.9) are $\tau_{d'}$-open.

2.1. **Strong b-metric spaces and completion.** Let (X, d) be a b-metric space. As we have seen, the topology τ_d generated by the b-metric d has some drawbacks in what concerns the continuity property of d and the topological openness of the "open" balls. To remedy these shortcomings Kirk and Shahzad [34, §12.4] introduced a special class of b-metrics. A mapping $d : X \times X \to [0, \infty)$ is called a *strong b-metric* if it satisfies the conditions (i) and (ii) from (1.1) and

$$d(x,y) \leq d(x,z) + sd(y,z), \tag{v}$$

for some $s \geq 1$ and all $x, y, z \in X$. Taking into account the symmetry of d, the inequality (v) is equivalent to

$$d(x,y) \leq \min\{sd(x,z) + d(y,z), d(x,z) + sd(y,z)\}, \tag{v$'$}$$

for all $x, y, z \in X$. Also (v) implies the s-relaxed triangle inequality.

The topology generated by a strong b-metric has good properties as, for instance, the openness of the balls $B(x, r)$. Indeed, if $y \in B(x, r)$, then

$$d(y,z) \leq d(x, y + sd(y,z) < \varepsilon,$$

provided $sd(y,z) < \varepsilon - d(x,y)$, that is $B(y, r') \subseteq B(x, r)$, where $r' = (\varepsilon - d(x,y))/s$.

Also the following inequality

$$|d(x,y) - d(x',y')| \leq s[d(x,x') + d(y,y')], \tag{2.10}$$

holds for all $x, y, x', y' \in X$, implying the continuity of the b-metric: if $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$, then the relations

$$|d(x_n, y_n) - d(x,y)| \leq s[d(x_n, x) + d(y_n, y)] \longrightarrow 0 \text{ as } n \to \infty,$$

show that $d(x_n, y_n) \longrightarrow d(x,y)$ as $n \to \infty$.

It is easy to check that a strong b-metric satisfies the s-relaxed polygonal inequality.

A *Cauchy sequence* in a b-metric space (X, d) is a sequence (x_n) in X such that $\lim_{m,n\to\infty} d(x_n, x_m) = 0$. The inequality

$$d(x_n, x_m) \le s\left[d(x_n, x) + d(x, x_m)\right], \ m, n \in \mathbb{N},$$

shows that every convergent sequence is Cauchy. The b-metric space (X, d) is called *complete* if every Cauchy sequence converges to some $x \in X$. The completeness is preserved by the uniform equivalence of b-metrics, but not by the topological equivalence.

By a *completion* of a b-metric space (X, d) one understands a complete b-metric space (Y, ρ) such that there exists an isometric embedding $j : X \to Y$ with $j(X)$ dense in Y.

By an *isometric embedding* of a b-metric space (X_1, d_1) into a b-metric space (X_2, d_2) one understands a mapping $f : X_1 \to X_2$ such that

$$d_2(f(x), f(y)) = d_1(x, y)\,,$$

for all $x, y \in X_1$. Two b-metric spaces (X_1, d_1), (X_2, d_2) are called *isometric* if there exists a surjective isometric embedding $f : X_1 \to X_2$.

A question raised in [34, p. 128] is:

Does every strong b-metric space admit a completion?

This question was answered in the affirmative in [4].

Theorem 2.10. *Let (X, d) be a strong b-metric space.*

1. *There exists a complete strong b-metric space $(\tilde{X}, \tilde{d}\,)$ which is a completion of (X, d).*
2. *The completion is unique up to an isometry, in the sense that if (X_1, d_1), (X_2, d_2) are two strong b-metric spaces which are completions of (X, d), then (X_1, d_1) and (X_2, d_2) are isometric.*

Proof. The proof follows the ideas from the metric case. On the family $\mathcal{C}(X)$ of Cauchy sequences in X one considers the equivalence relation

$$(x_n) \sim (y_n) \iff \lim_n d(x_n, y_n) = 0\,.$$

On the quotient space $\tilde{X} = \mathcal{C}(X)/\sim$ one defines $\tilde{d}$ by $\tilde{d}(\xi, \eta) = \lim_n d(x_n, y_n)$, where $(x_n) \in \xi$ and $(y_n) \in \eta$. One shows that $(\tilde{X}, \tilde{d})$ is a complete strong b-metric space containing X isometrically as a dense subset. □

Remark 2.11. As it is mentioned in [4], the existence of a completion of an arbitrary b-metric space is still an important open problem.

3. Generalized b-metric spaces

The notion of *generalized metric*, meaning a mapping $d : X \times X \to [0, \infty]$ satisfying the axioms of a metric, and generalized metric space (X, d) were introduced by W. A. J. Luxemburg in [35]–[37] in connection with the method of successive approximation and fixed points. These results were completed by A. F. Monna [41] and M. Edelstein [23]. Further results were obtained by J. B. Diaz and B. Margolis [22, 40] and C. F.

K. Jung [30]. G. Dezső [21] considered generalized vector metrics, i.e. metrics with values in $\mathbb{R}_+^m \cup \{(+\infty)^m\}$, and extended to this setting Perov's fixed point theorem (see [43] – [45]) as well as other fixed point results (Luxemburg, Jung, Diaz-Margolis, Kannan). For some recent results on generalized metric spaces see [7] and [16].

Recently, G. Beer and J. Vanderwerf [8]–[10] considered vector spaces equipped with norms that can take infinite values, called by them "extended norms" (see also [18]).

Following these ideas, we consider here the notion of *generalized* b-*metric* on a nonempty set X as a mapping $d : X \times X \to [0, \infty]$ satisfying the conditions (i)–(iii) from (1.1). If d satisfies further the condition (v), then d is called a *generalized strong* b-*metric* and the pair (X, d) a *generalized strong* b-*metric space.*

Let (X, d) be a generalized b-metric space. As in Jung [30], it follows that

$$x \sim y \overset{d}{\Longleftrightarrow} d(x,y) < +\infty, \quad x, y \in X, \tag{3.1}$$

is an equivalence relation on X. Denoting by X_i, $i \in I$, the equivalence classes corresponding to $\sim$ and putting $d_i = d|_{X_i \times X_i}$, $i \in I$, it follows that (X_i, d_i) is a b-metric space (a strong b-metric space if (X, d) is a generalized strong b-metric space), for every $i \in I$. Therefore, X can be uniquely decomposed into equivalence classes X_i, $i \in I$, called the *canonical decomposition* of X.

By analogy with [30] we have.

Theorem 3.1. *Let (X, d) be a generalized b-metric space and X_i, $i \in I$, its canonical decomposition. Then the following hold.*

1. *The space (X, d) is complete if and only if (X_i, d_i) is complete for every $i \in I$.*
2. *If (Y_i, d_i), $i \in I$, are b-metric spaces (with the same s) and $Y_i \cap Y_j = \emptyset$ for all $i \neq j$ in I, then*

$$d(x,y) := \begin{cases} d_i(x,y) & \text{if} \quad x, y \in Y_i, \text{ for some } i \in I, \\ +\infty & \text{if} \quad x \in Y_i \text{ and } y \in Y_j \\ & \qquad \text{for some } i, j \in I \text{ with } i \neq j, \end{cases} \tag{3.2}$$

is a generalized b-metric on $Y = \bigcup_{i \in I} Y_i$, with $\{Y_i : i \in I\}$ the family of equivalence classes corresponding to the equivalence relation (3.1).

The same results are true for generalized strong b-metric spaces.

3.1. **The completion of generalized b-metric spaces.** In this subsection we shall prove the existence of the completion of strong b-metric spaces. The existence of the completion of a generalized metric space was proved in [17].

We start with the following lemma.

Lemma 3.2. *Let (X, d) be a generalized b-metric space, (Z, D) a complete generalized b-metric space, with continuous generalized b-metrics d, D and Y a dense subset of X. Then for every isometric embedding $f : Y \to Z$ there exists a unique isometric embedding $F : X \to Z$ such that $F|_Y = f$. If, in addition, X is complete and $f(Y)$ is dense in Z, then F is bijective (i.e. F is an isometry of X onto Z).*

Proof. For $x \in X$ let (y_n) be a sequence in Y such that $d(y_n, x) \to 0$. Then (y_n) is a Cauchy sequence in (X, d) and the equalities $D(f(y_n), f(y_m)) = d(y_n, y_m)$, $m, n \in \mathbb{N}$, show that $(f(y_n))$ is a Cauchy sequence in (Z, D). Since (Z, D) is complete, there exists $z \in Z$ such that $D(f(y_n), z) \to 0$. If (y'_n) is another sequence in Y converging to x, then $(f(y'_n))$ will converge to an element $z' \in Z$. By the continuity of the generalized b-metrics d and D,

$$D(z, z') = D(\lim_n f(y_n), \lim_n f(y'_n)) = \lim_n D(f(y_n), f(y'_n)) = \lim_n d(y_n, y'_n) = 0\,,$$

showing that $z = z'$. So we can unambiguously define a mapping $F : X \to Z$ by $F(x) = \lim_n f(y_n)$, where (y_n) is a sequence in Y converging to $x \in X$. For $y \in Y$ taking $y_n = y$, $n \in \mathbb{N}$, it follows $F(y) = y$.

For $x, x' \in X$, let $(y_n), (y'_n)$ be sequences in Y converging to x and x', respectively. Then

$$D(F(x), F(x')) = \lim_n D(f(y_n), f(y'_n)) = \lim_n d(y_n, y'_n) = d(x, x')\,,$$

i.e. F is an isometric embedding.

If $f(Y)$ is dense in Z, then, for any $z \in Z$, there exists a sequence (y_n) in Y such that $D(f(y_n), z) \to 0$. It follows that $(f(y_n))$ is a Cauchy sequence in Z and so, as f is an isometry, (y_n) will be a Cauchy sequence in X. As the space X is complete, (y_n) is convergent to some $x \in X$. But then

$$D(F(x), z) = \lim_n D(F(x), f(y_n)) = \lim_n d(x, y_n) = 0\,,$$

showing that $F(x) = z$. □

Remark 3.3. The proof can be adapted to show that, under the hypotheses of Lemma 3.2, every uniformly continuous mapping $f : Y \to Z$ has a unique uniformly continuous extension to X. The notion of uniform continuity for mappings between generalized b-metric spaces is defined as in the metric case.

Let (X, d) be a generalized strong b-metric space with X_i, $i \in I$, the family of equivalence classes corresponding to (3.1). For every $i \in I$, let (Y_i, D_i) be a completion of the strong b-metric space (X_i, d_i). Denote by $T_i : (X_i, d_i) \to (Y_i, D_i)$ the isometric embedding with $T_i(X_i)$ D_i-dense in Y_i corresponding to this completion.

Replacing, if necessary, Y_i with $\overline{Y_i} = Y_i \times \{i\}, D_i$ with $\overline{D_i}((x, i), (y, i)) = D_i(x, y)$, for $x, y \in Y_i$, and putting $\overline{T_i}(x, i) = (T_i(x), i)$, $x \in Y_i$, we may suppose, without restricting the generality, that

$$Y_i \cap Y_j = \emptyset \text{ for all } i, j \in I \text{ with } i \neq j\,.$$

Put $Y := \bigcup_{i \in I} Y_i$, and define

$$D : Y \times Y \to [0, \infty]$$

according to (3.2) and $T : X \to Y$ by

$$T(x) := T_i(x),$$

where i is the unique element of I such that $x \in X_i$.

We have the following result.

Theorem 3.4. *Let (X, d) be a generalized strong b-metric space and (Y, D) the generalized strong b-metric space defined above. Then*

(i) *(Y, D) is a complete generalized strong b-metric space;*
(ii) *$T : (X, d) \to (Y, D)$ is an isometric embedding with $T(X)$ D-dense in Y;*
(iii) *any other complete generalized strong b-metric space (Z, ρ) that contains a ρ-dense isometric copy of (X, d), is isometric to (Y, D).*

Proof. Since each strong b-metric space (Y_i, D_i) is complete, Theorem 3.1 implies that the generalized strong b-metric space (Y, D) is complete.

Let $x, y \in X$. If $x, y \in X_i$, for some $i \in I$, then

$$D(T(x), T(y)) = D_i(T_i(x), T_i(y)) = d_i(x, y) = d(x, y).$$

If $x \in X_i$, $y \in X_j$ with $i \neq j$, then

$$T(x) = T_i(x) \in Y_i \text{ and } T(y) = T_j(x) \in Y_j\,,$$

so that

$$D(T(x), T(y)) = D(T_i(x), T_j(y)) = +\infty = d(x, y).$$

Now for $\xi \in Y$ there exists a unique $i \in I$ such that $\xi \in Y_i$. Since $T_i(X_i)$ is dense in (Y_i, D_i), there exists a sequence (x_n) in X_i such that

$$0 = \lim_{n\to\infty} D_i(T_i(x_n), \xi) = \lim_{n\to\infty} D(T(x_n), \xi),$$

which means that $T(X)$ is D-dense in (Y, D).

Finally, to verify (iii), let $S : (X, d) \to (Z, \rho)$ be an isometric embedding with $S(X)$ dense in Z. Define $R : T(X) \to X$ by $R(T(x)) = x$, $x \in X$. Then R is an isometry of $T(X)$ onto X and $S \circ R$ is an isometric embedding of $T(X)$ into Z. Since $T(X)$ is dense in Y and $S(R(T(X))) = S(X)$ is dense in Z, Lemma 3.2 yields the existence of an isometry U of Y onto Z, which ends the proof. □

4. Fixed points in b-metric spaces

We shall prove some fixed point results in b-metric and in generalized b-metric spaces.

4.1. Fixed points in b-metric spaces.

The first result is an extension to b-metric spaces of Theorem 4.1 from [28], with an appropriate modification in the definition of the comparison function φ.

Let (X, d) be a b-metric space with d satisfying the s-relaxed triangle inequality. We consider functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions

$$\begin{aligned} &\text{(a)} \quad \varphi \ \text{ is nondecreasing,} \\ &\text{(b)} \quad \lim_{n\to\infty} \varphi^n(t) = 0, \text{ and} \\ &\text{(c)} \quad \varphi(t) < \frac{t}{s}\,, \end{aligned} \tag{4.1}$$

for all $t > 0$.

Remark 4.1. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions (a) and (b) from above, then

$$\varphi(t) < t\,,$$

for all $t > 0$.

Indeed, if $\varphi(t) \geq t$ for some $t > 0$, then, by (a), $\varphi^2(t) \geq \varphi(t) \geq t$ and, in general $\varphi^n(t) \geq t > 0$ for all n, in contradiction to (b).

Theorem 4.2. *Let (X, d) be a complete b-metric space, where d satisfies the s-relaxed triangle inequality and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying the conditions (a) – (c) from* (4.1)*. Then every mapping $f : X \to X$ satisfying the inequality*

$$d(f(x), f(y)) \leq \varphi(d(x, y))\,, \tag{4.2}$$

for all $x, y \in X$, has a unique fixed point z and the sequence $\big(f^n(x)\big)_{n\in\mathbb{N}_0}$ converges to z as $n \to \infty$, for every $x \in X$.

As in [28], the proof is based on the following lemma.

Lemma 4.3. *Let (X, d) be a complete b-metric space and $f : X \to X$ a mapping. Suppose that, for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$d(z, f(z)) < \delta(\varepsilon) \implies f\left(B(z, \varepsilon)\right) \subseteq B(z, \varepsilon)\,. \tag{4.3}$$

If, for some $x \in X$, $\lim_{n\to\infty} d(f^n(x), f^{n+1}(x)) = 0$, then the sequence $\big(f^n(x)\big)$ converges to a fixed point of f.

Proof. Let $x \in X$. Put $z_n = f^n(x)$ for $n \in \mathbb{N}_0$, and suppose that

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0.$$

For $\varepsilon > 0$ let $\delta(\varepsilon) > 0$ be such that (4.3) holds.
Pick $m \in \mathbb{N}$ such that $d(z_m, f(z_m)) = d(z_m, z_{m+1}) < \delta(\varepsilon)$. Then

$$z_{m+1} = f(z_m) \in B(z_m, \varepsilon),\ \ z_{m+2} = f(z_{m+1}) \in B(z_m, \varepsilon),$$

and, by induction, $z_{m+k} = f(z_{m+k-1}) \in B(z_m, \varepsilon)$.
It follows that for all $n, n' \geq m$, $d(z_n, z_{n'}) \leq s(d(z_n, z_m) + d(z_m, z_{n'})) < 2s\varepsilon$.
Consequently, the sequence (z_n) is Cauchy, so it converges to some $z \in X$. If $z \neq f(z)$, then $a := d(z, f(z)) > 0$. Consider $\delta\left(\frac{a}{3s}\right)$ given by the hypothesis of the lemma and let $m \in \mathbb{N}$ be such that

$$d(z_m, z) < \frac{a}{3s} \ \text{ and } \ d(z_m, f(z_m)) = d(z_m, z_{m+1}) < \delta\left(\frac{a}{3s}\right)\,.$$

Then

$$f\left(B\big(z_m, \frac{a}{3s}\big)\right) \subseteq B\big(z_m, \frac{a}{3s}\big)\,.$$

Since $z \in B\big(z_m, \frac{a}{3s}\big)$ it follows $f(z) \in B\big(z_m, \frac{a}{3s}\big)$, leading to the contradiction

$$a = d(z, f(z)) \leq s[d(z, z_m) + d(z_m, f(z))] < \frac{2a}{3}.$$

□

Proof of Theorem 4.2. We first show that, under the hypotheses of Theorem 4.2, the condition (4.3) is satisfied. For $\varepsilon > 0$ choose $\delta(\varepsilon) := \frac{\varepsilon}{s} - \varphi(\varepsilon) > 0$. Then, by condition (c) from (4.1), $\delta(\varepsilon) > 0$ and the inequalities $d(z, f(z)) < \delta(\varepsilon)$ and $d(z, y) < \varepsilon$ imply

$$\begin{aligned} d(z, f(y)) &\leq sd(z, f(z)) + sd(f(z), f(y)) \\ &< s\delta(\varepsilon) + s\varphi(d(z, y)) \\ &\leq s\delta(\varepsilon) + s\varphi(\varepsilon) = \varepsilon\,, \end{aligned}$$

that is, $f\left(B(z, \varepsilon)\right) \subseteq B(z, \varepsilon)$.

For an arbitrary point $x \in X$,

$$d(f^n(x), f^{n+1}(x)) \leq \varphi^n(d(x, f(x)) \longrightarrow 0 \quad \text{as} \quad n \to \infty\,.$$

Therefore, the conclusions of the theorem follows from Lemma 4.3. □

Let (X, d) be a b-metric space with d satisfying the s-relaxed triangle inequality for some $s \geq 1$. An important particular case of a function φ satisfying the conditions (a)–(c) from (4.1) is

$$\varphi(t) = \alpha t,\ t \geq 0\,,$$

where $0 < \alpha < 1/s$. Then

$$\varphi(t) = (\alpha s) \cdot \frac{t}{s} < \frac{t}{s}\,,$$

for all $t > 0$, and

$$\varphi^n(t) = \alpha^n t \longrightarrow 0 \ \text{ as } \ n \to \infty\,,$$

because $0 < \alpha < 1/s \leq 1$. Since φ is strictly increasing, it satisfies the conditions (a)–(c) from (4.1).

The inequality (4.2) becomes in this case

$$d(f(x), f(y)) \leq \alpha d(x, y)\,, \tag{4.4}$$

for all $x, y \in X$.

So, Theorem 4.2 has the following corollary – the analog of Banach contraction principle for b-metric spaces. The corollary illustrates how various types of relaxed triangle inequalities influence the form this principle takes.

Corollary 4.4. ([6]) *Let (X, d) be a complete b-metric space, where d satisfies the s-relaxed triangle inequality and $f : X \to X$ a mapping such that, for some $\alpha > 0$,*

$$d(f(x), f(y)) \leq \alpha d(x, y), \tag{4.5}$$

for all $x, y \in X$.

1. ([6]) *If $0 < \alpha < 1/s$, then f has a unique fixed point z and, for every $x \in X$, the sequence $\left(f^n(x)\right)_{n\in\mathbb{N}}$ converges to z as $n \to \infty$.*

   Furthermore, the following evaluation of the order of convergence holds

$$d(x_n, z) \leq \frac{sd(x_0, x_1)}{1 - \alpha s}\,\alpha^n\,, \tag{4.6}$$

   for all $n \in \mathbb{N}$.

2. ([34], Theorem 12.4) *If d satisfies the s-relaxed polygonal inequality, then the results from 1 hold for $0 < \alpha < 1$ with the following evaluation of the order of convergence*

$$d(x_n, z) \le \frac{sd(x_0, x_1)}{1-\alpha}\,\alpha^n, \tag{4.7}$$

for all $n \in \mathbb{N}$.

Proof. 1. We sketch the simple direct proof, similar to that from the metric case.

Observe first that, (4.5) implies

$$d(f^n(x), f^n(y)) \le \alpha^n d(x, y)\,, \tag{4.8}$$

for all $n \in \mathbb{N}$ and $x, y \in X$.
For $x_0 \in X$ consider the sequence of iterates

$$x_n = f(x_{n-1}) = f^n(x_0), \quad n \in \mathbb{N}\,.$$

Let us prove that (x_n) is a Cauchy sequence. Successive applications of the s-relaxed triangle inequality yield

$$d(x_0, x_n) \le sd(x_0, x_1) + s^2 d(x_1, x_2) + \cdots + s^n d(x_{n-1}, x_n)\,, \tag{4.9}$$

for all $n \in \mathbb{N}$.
By (4.9) and (4.8),

$$\begin{aligned} d(x_n, x_{n+k}) &\le sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^k d(x_{n+k-1}, x_{n+k}) \\ &\le \big(\alpha^n s + \alpha^{n+1} s^2 + \cdots + \alpha^{n+k-1} s^k\big) d(x_0, x_1) \\ &= \alpha^n s \cdot \frac{1-(\alpha s)^k}{1-\alpha s} d(x_0, x_1) < \alpha^n \cdot \frac{sd(x_0, x_1)}{1-\alpha s}\,, \end{aligned} \tag{4.10}$$

for all $n, k \in \mathbb{N}$. Since $\lim_{n\to\infty} \alpha^{n+1} = 0$, this shows that (x_n) is a Cauchy sequence. By the completeness of (X, d) there exists $z \in X$ such that $\lim_{n\to\infty} d(x_n, z) = 0$. Since, by (4.5), the mapping f is continuous, we can pass to limit in the equality $x_{n+1} = f(x_n)$, $n \in \mathbb{N}_0$, to obtain $z = f(z)$
Suppose now that there exists two points $z, z' \in X$ such that $f(z) = z$ and $f(z') = z'$. Then the relations

$$d(z, z') = d(f(z), f(z')) \le \alpha d(z, z')$$

show that $d(z, z') = 0$, i.e. $z = z'$.
Now, from (4.10),

$$d(x_n, x_{n+k}) < \alpha^n s \cdot \frac{1-(\alpha s)^k}{1-\alpha s} d(x_0, x_1)\,,$$

which yields (4.6) for $k \to \infty$.

2. Let $x_0 \in X$ and $x_n = f(x_{n-1})$, $n \in \mathbb{N}$. Taking into account the relaxed polygonal inequality and (4.8), we obtain

$$\begin{aligned} d(x_n, x_{n+k}) &\le s \sum_{i=0}^{k-1} d(x_{n+i}, x_{n+i+1}) \le s(\alpha^n + \alpha^{n+1} + \cdots + \alpha^{n+k}) d(x_0, x_1) \\ &= s\alpha^n \frac{1-\alpha^{k+1}}{1-\alpha} \cdot d(x_0, x_1) < \frac{sd(x_0, x_1)}{1-\alpha} \cdot \alpha^n. \end{aligned}$$

Based on these relations the proof goes as in case 1. □

Remark 4.5. The proof given here to statement 2 from Corollary 4.4 is simpler than that from [34].

Based on Theorem 2.2, one can show that Banach's fixed point theorem actually holds for arbitrary contractions on complete b-metric spaces.

Theorem 4.6. ([2]) *Let* (X, d) *be a complete b-metric space and* $0 < \alpha < 1$. *If* $f : X \to X$ *satisfies the inequality*

$$d(f(x), f(y)) \le \alpha d(x, y)\,, \tag{4.11}$$

for all $x, y \in X$, *then* f *has a unique fixed point* z *and the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *converges to* z *for every* $x \in X$.

Proof. Suppose that d satisfies the s-relaxed triangle inequality, for some $s \ge 1$. If $0 < p \le 1$ is given by the equation $(2s)^p = 1$, then, by Theorem 2.2, the functional ρ_p given by (2.2) is a metric on X satisfying the inequalities

$$\rho_p \le d^p \le 2\rho_p\,. \tag{4.12}$$

For $x, y \in X$ let $x = x_0, x_1, \ldots, x_n = y$ be an arbitrary chain in X connecting x and y. Then $y_i = f(x_i)$, $i = 0, 1, \ldots, n$, is a chain in X connecting $f(x)$ and $f(y)$. Consequently, by (2.2) and (4.11),

$$\rho_p(f(x), f(y)) \le \sum_{i=0}^{n-1} d(y_i, y_{i+1})^p \le \alpha^p \sum_{i=0}^{n-1} d(x_i, x_{i+1})^p. \tag{4.13}$$

Since the inequality between the extreme terms in (4.13) holds for all chains $x = x_0, x_1, \ldots, x_n = y$, $n \in \mathbb{N}$, connecting x and y, it follows

$$\rho_p(f(x), f(y)) \le \alpha^p \rho_p(x, y)\,,$$

for all $x, y \in X$, where $0 < \alpha^p < 1$. Consequently, f is a contraction with respect toρ_p. The inequalities (4.12) and the completeness of (X, d) imply the completeness of (X, ρ_p) and so, by Banach's contraction principle, f has a unique fixed point $z \in X$ and the sequence of iterates $(f^n(x))_{n\in\mathbb{N}}$ is ρ_p-convergent to z, for every $x \in X$. Appealing again to the inequalities (4.12), it follows that $(f^n(x))_{n\in\mathbb{N}}$ is also d-convergent to z for every $x \in X$. □

Remark 4.7. In [14] and [34], Theorem 4.2 appears under the hypothesis that the function φ satisfies only the conditions (a) and (b) from (4.1). In both cases, the proof goes in the following way.

Let x be a fixed element of X and $\varepsilon > 0$. By (4.1).(b) there exists $m = m_\varepsilon \in \mathbb{N}$ such that

$$\varphi^m(\varepsilon) < \frac{\varepsilon}{2s}\,. \tag{4.14}$$

One considers the sequence $x_k = f^{km}(x)$, $k \in \mathbb{N}$, and one shows that there exists $k_0 \in \mathbb{N}$ such that

$$d(x_k, x_{k'}) < 2s\varepsilon\,, \tag{4.15}$$

for all $k, k' \geq k_0$. One affirms that the inequality (4.15) shows that (x_k) is a Cauchy sequence, which is not surely true, because the inequality is true only for this specific ε.

Taking another ε, say $0 < \varepsilon' < \varepsilon$, we find another number $m' = m_{\varepsilon'}$ (possibly different from m), such that

$$\varphi^{m'}(\varepsilon') < \frac{\varepsilon'}{2s} . \tag{4.16}$$

The above procedure yields a sequence $x'_k = f^{km'}(x)$, $k \in \mathbb{N}$, satisfying, for some $k_1 \in \mathbb{N}$,

$$d(x_k, x_{k'}) < 2s\varepsilon' , \tag{4.17}$$

for all $k, k' \geq k_1$.

But the sequences (x_k) and (x'_k) can be totally different, so we cannot infer that the sequence (x_k) is Cauchy.[1]

It seems that, besides (a) and (b) from (4.1), some supplementary conditions on the comparison function φ are needed in order to obtain some fixed point results in b-metric spaces for mappings satisfying (4.2).

For instance, Berinde [11] considers comparison functions satisfying a condition stronger than (c) from (4.1), namely $\sum_{k=1}^{\infty} \varphi^k(t) < \infty$, allowing estimations of the order of convergence similar to (4.6). He also shows that the sequence $x_n = f^n(x_0)$, $n \in \mathbb{N}_0$, is convergent to a fixed point of f if and only if it is bounded. For various kinds of comparison functions, the relations between them and applications to fixed points, see [46, §3.0.3].

4.2. **Fixed points in generalized b-metric spaces.** Theorem 4.2 admits the following extension to generalized b-metric spaces.

Theorem 4.8. *Let (X, d) be a complete generalized b-metric space and suppose that the mapping $f : X \to X$ is such that*

$$d\left(f(x), f(y)\right) \leq \varphi\left(d(x, y)\right) , \tag{4.18}$$

for all $x, y \in X$ with $d(x, y) < \infty$, where the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions (a)–(c) from (4.1).

Consider, for some $x \in X$, the sequence of successive approximations $\left(f^n(x)\right)_{n\in\mathbb{N}_0}$. Then either

(A) *$d(f^k(x), f^{k+1}(x)) = +\infty$ for all $k \in \mathbb{N}_0$,*

or

(B) *the sequence $\left(f^n(x)\right)_{n\in\mathbb{N}}$ is convergent to a fixed point of f.*

Proof. Let $X = \bigcup_{i\in I} X_i$ be the canonical decomposition of X corresponding to the equivalence relation (3.1). Assume that (A) does not hold. Then

$$d(f^m(x), f^{m+1}(x)) < +\infty ,$$

[1] This flow was fixed by S. Kajántó and A. Lukács, A note on the paper "Contraction mappings in b-metric spaces" by Czerwik, Acta Univ. Sapientiae Math. 10 (2018), no. 1, 85-89.

for some $m \in \mathbb{N}_0$. If $i \in I$ is such that $f^m(x), f^{m+1}(x) \in X_i$, then

$$d\left(f^{m+1}(x), f^{m+2}(x)\right) \le \varphi\left(d\left(f^m(x), f^{m+1}(x)\right)\right) < \infty\,,$$

implies $f^{m+2}(x) \in X_i$, and so, by mathematical induction, $f^{m+k}(x) \in X_i$ for all $k \in \mathbb{N}_0$. Since

$$z \in X_i \iff d(z, f^m(x)) < \infty\,,$$

the inequality

$$d(f(z), f^{m+1}(x)) \le \varphi(d(z, f^m(x)) < \infty\,,$$

shows that the restriction $f_i = f|_{X_i}$ of f to X_i is a mapping from X_i to X_i satisfying

$$d(f_i(y), f_i(z)) \le \varphi(d(y,z))\,,$$

for all $y, z \in X_i$. By Theorem 3.1, X_i is complete, so that, by Theorem 4.2, the sequence $\left(f^{m+k}(x)\right)_{k\in\mathbb{N}_0}$ is convergent to a fixed point of f_i, which is a fixed point for f. □

Remark 4.9. For $s = 1$ and $\varphi(t) = \alpha t$, $t \ge 0$, where $0 \le \alpha < 1$, we get the Diaz and Margolis [22] fixed point theorem of the alternative. At the same time these extend Theorem 2 from [19] and give simpler proofs to Theorems 2.1 and 3.1 from [5].

Corollary 4.4 and Theorem 4.6 also admit extensions to this setting as results of the alternative. We formulate only one of these results.

Corollary 4.10. *Let (X, d) be a complete generalized b-metric space, where d satisfies the s-relaxed triangle inequality and let*

$$0 < \alpha < \frac{1}{s}. \tag{4.19}$$

Then, for every mapping $f : X \to X$ satisfying the inequality

$$d(f(x), f(y)) \le \alpha d(x,y)\,, \tag{4.20}$$

for all $x, y \in X$ with $d(x,y) < \infty$, either

(A′) $d(f^k(x), f^{k+1}(x)) = +\infty$ *for all* $k \in \mathbb{N}_0$,

or

(B′) *the sequence* $(f^n(x))_{n\in\mathbb{N}}$ *is convergent to a fixed point of* f.

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