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ZERO POINT PRINCIPLE OF BALL-NEAR IDENTITY OPERATORS AND APPLICATIONS TO IMPLICIT OPERATOR PROBLEM

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Abstract. In this paper we give a global zero point principle for operators on a Banach space in terms of ball-near identity operator condition. The techniques of the proof are some variants of the saturated contraction principle. So, we study the well posedness of zero point problem, the Ostrowski property, data dependence and Ulam stability of zero point equations. Some relevant examples are given. Applications to the implicit operator problem are also presented.
Key Words and Phrases: Banach space, ball-near identity operator, saturated contraction principle, zero point principle, well posedness of zero point problem, Ostrowski property, data dependence, Ulam stability of zero point equations, implicit function problem, implicit operator problem.
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1. Introduction

Let X be a linear space over $\mathbb{K} := \mathbb{R} \vee \mathbb{C}$ and $f : X \to X$ be an operator. We denote by

$$Z_f = \{x \in X \mid \ f(x) = 0\}$$

the zero point set of f and by

$$F_f = \{x \in X \mid \ f(x) = x\}$$

the fixed point set of f. The fixed point techniques in the zero point theory, in general, consist as follow (see: [18], [21], [32], [37], [38], [53], [55], [63], [7], [39], [61], ...):
Given $f : X \to X$, the problem is to find an operator $g : X \to X$ such that $F_g = Z_f$. Here are some examples:

(1) $g = 1_X - f$;
(2) $g = 1_X - \gamma f$, where $\gamma \in \mathbb{K}$ with $\gamma \neq 0$;

(3) $g = (1-\lambda) 1_X - \lambda f$, where $\lambda \in \mathbb{R}$, $\lambda \neq 0$;
(4) X is a Banach space, $f : X \to X$ is differentiable with $df(x)$ and $(df(x))^{-1} \in L(X)$, $\forall x \in X$. In this case we take $g(x) = x - (df(x))^{-1} f(x)$, $x \in X$.

In this paper we give new fixed point technique for the global uniqueness zero principle. The main ingredient of this technique is the ball-near identity condition.

2. Preliminaries

2.1. **Notations.** Throughout this paper the notations and terminologies in [49], [54] and [9] are used.

2.2. **Some variants of contraction principle.** In this paper we need the following variant of contraction principle.

Theorem 2.1 (Saturated principle of contraction [49]). *Let (X,d) be a complete metric space and $f : X \to X$ be an l-contraction. Then we have that:*

(i) *There exists $x^* \in X$ such that $F_{f^n} = \{x^*\}$, $\forall n \in \mathbb{N}$.*
(ii) *For all $x \in X$, $f^n(x) \to x^*$ as $n \to +\infty$.*
(iii) *$d(x, x^*) \leq \psi(d(x, f(x)))$, $\forall x \in X$, where $\psi(t) = \frac{t}{1-l}$, $t \geq 0$, i.e., f is a ψ-Picard operator.*
(iv) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $d(y_n, f(y_n)) \to 0$ as $n \to +\infty$ then $y_n \to x^*$ as $n \to +\infty$, i.e., the fixed point problem for f is well posed.*
(v) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to +\infty$ then $y_n \to x^*$ as $n \to +\infty$, i.e., the operator f has the Ostrowski property.*

Theorem 2.2 (Saturated principle of nonself contraction [9]). *Let (X,d) be a metric space, $Y \subset X$ and $f : Y \to X$ an operator. We suppose that:*

(a) *f is an l-contraction.*
(b) *$F_f \neq \varnothing$.*

Then we have:

(i) *$F_f = \{x^*\}$.*
(ii) *$d(x, x^*) \leq (1-l)^{-1} d(x, f(x))$, $\forall x \in Y$.*
(iii) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y such that $d(y_n, f(y_n)) \to 0$ as $n \to +\infty$ then $y_n \to x^*$ as $n \to +\infty$.*
(iv) *If $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in Y such that $d(y_{n+1}, f(y_n)) \to 0$ as $n \to +\infty$ then $y_n \to x^*$ as $n \to +\infty$.*

Remark 2.1. For the Picard operator theory see: [53], [54], [42], [50], [51], ...

Remark 2.2. For the well posedness of the fixed point problem see: [9], [49], [53], ...

Remark 2.3. For the Ostrowski property see: [39], [38], [37], [49], [50], [54], [53], ...

2.3. **Diagonally dominant matrices.** For a better understanding of the examples in section 6, in what follow we present some well known results for the diagonally dominant matrices.

Let $\mathbb{K} := \mathbb{R} \vee \mathbb{C}$ and $A \in \mathbb{K}^{m\times m}$ be a matrix. By definition the matrix $A = [a_{kj}]_m^m$ is called *strictly row diagonally dominant* if

$$|a_{kk}| > \sum_{\substack{j=1 \\ j\neq k}}^{m} |a_{kj}|, \quad k = \overline{1,m}.$$

For this type of matrices we have (see [37], [38], [7], [55], ...) the following known results.

Lemma 2.1. *If A is strictly row diagonally dominant matrix then A is nonsingluar.*

It is clear that the condition of strictly row diagonally dominant is more restrictive than the condition of nonsingularity.

Lemma 2.2. *Let A be a strictly row diagonally dominant matrix with positive diagonal elements. If λ is an eigenvalue of A then* $\operatorname{Re}\lambda > 0$.

Lemma 2.3. *Let A be such that:*

(i) *A is strictly row diagonally dominant matrix.*
(ii) $0 < a_{kk} < 1$ *for* $k = \overline{1,m}$.

Then the norm of the linear operator $I - A : \mathbb{K}^m \to \mathbb{K}^m$ satisfies $\|I - A\|_\infty < 1$.

2.4. **Near operators.** Let $(X, \|\cdot\|)$ be a normed space and $f, g : X \to X$. By definition, (see [16], [14], [15]), the operator f is near g if there exists $\gamma > 0$ and $l \in]0,1[$ such that

$$\|g(x_1) - g(x_2) - \gamma(f(x_1) - f(x_2))\| \le l\,\|g(x_1) - g(x_2)\|, \ \forall x_1, x_2 \in X.$$

From this definition it follows that the operator f is near 1_X if and only if there exists $\gamma > 0$ and $l \in]0,1[$ such that the operator $1_X - \gamma f$ is an l-contraction. Moreover, if f is near 1_X then, for each $y \in X$, the operator $f + y$ is near 1_X with the same constants γ and l as f.

From Lemma 2.3 we have:

Lemma 2.4. *Let $A \in \mathbb{K}^{m\times m}$ be such that:*

(i) *A is strictly row diagonally dominant matrix.*
(ii) $0 < a_{kk} < 1$ *for* $k = \overline{1,m}$.

Then the linear operator from $\mathbb{K}^m \to \mathbb{K}^m$ defined by A is near $1_{\mathbb{K}^m}$ with $\gamma = 1$ and $l = \|I - A\|_\infty$.

3. Zero point principle of ball-near identity operators

We start with

Definition 3.1. Let $(X, \|\cdot\|)$ be a Banach space. An operator $T : X \to X$ is, by definition, ball-near identity with respect to a point $x_0 \in X$, if there exists $R_0 > 0$, $l : [R_0, +\infty[\to]0,1[$ and $\Gamma : [R_0, +\infty[\to L(X)$, such that:

(1) $\Gamma(R)$ is a bijection, $\forall R \ge R_0$.
(2) The operator $S_R := 1_X - \Gamma(R)\,T$ is an $l(R)$-contraction on $\bar{B}(x_0; R)$, $\forall R \ge R_0$.
(3) $S_{R_0}\left(\bar{B}(x_0; R_0)\right) \subset \bar{B}(x_0; R_0)$.

Remark 3.1. The condition

(3') $\|\Gamma(R_0) T(x_0)\| \leq (1 - l(R_0)) R_0$
implies condition (3).

Remark 3.2. If for some $R \geq R_0$ we have that

$$\|\Gamma(R) T(x_0)\| \leq (1 - l(R)) R$$

then

$$S_R\left(\bar{B}(x_0; R)\right) \subset \bar{B}(x_0; R).$$

So, if $T(x_0) = 0$ then for all $R \geq R_0$ we have that $S_R\left(\bar{B}(x_0; R)\right) \subset \bar{B}(x_0; R)$.

In what follows, if T, l and Γ are as in Definition 3.1 then we call T, (Γ, l)-ball-near identity with respect to x_0. For examples of (Γ, l)-ball-near identity operators see section 6 of this paper.

Our main abstract result is the following.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ a (Γ, l)-ball-near identity operator with respect to $x_0 \in X$. Then we have that:*

(i) $Z_T = \{x^*\}$.

(ii) *For each $y_0 \in \bar{B}(x_0; R_0)$, the sequence $\{y_n\}_{n\in\mathbb{N}}$, defined by*

$$y_{n+1} = y_n - \Gamma(R_0) T(y_n),$$

converges to x^* as $n \to +\infty$.

(iii) $\|y - x^*\| \leq (1 - l(R))^{-1} \|\Gamma(R) T(y)\|$, $\forall y \in \bar{B}(x_0; R)$, $\forall R \geq R_0$.

(iv) *If $R \geq R_0$ and $\{y_n\}_{n\in\mathbb{N}} \subset \bar{B}(x_0; R)$ is such that*

$$T(y_n) \to 0 \quad as \quad n \to +\infty$$

then $y_n \to x^*$ as $n \to +\infty$, i.e., the zero point problem is well posed for T.

(v) *If $R \geq R_0$ and $\{y_n\}_{n\in\mathbb{N}} \subset \bar{B}(x_0; R)$ is such that*

$$y_{n+1} - y_n + \Gamma(R) T(y_n) \to 0 \quad as \quad n \to +\infty$$

then $y_n \to x^*$ as $n \to +\infty$, i.e., the operator S_R has the Ostrowski property.

Proof. First, we remark that $F_{S_R} = Z_T$, for all $R \geq R_0$.

(i) Since $S_{R_0} : \bar{B}(x_0; R_0) \to \bar{B}(x_0; R_0)$ is a contraction we have that

$$F_{S_{R_0}} \cap \bar{B}(x_0; R_0) = \{x^*\}.$$

But, $\bar{B}(x_0; R_0) \subset \bar{B}(x_0; R)$ for $R \geq R_0$ and $S_R : \bar{B}(x_0; R) \to X$ is a contraction for all $R \geq R_0$. On the other hand $F_{S_R} \cap \bar{B}(x_0; R) = Z_T \cap \bar{B}(x_0; R)$, so $Z_T = \{x^*\}$.

(ii) We apply Theorem 2.1 for $S_{R_0} : \bar{B}(x_0; R_0) \to \bar{B}(x_0; R_0)$.

$(iii) - (v)$ We apply Theorem 2.2 for $S_R : \bar{B}(x_0; R) \to X$, $R \geq R_0$. □

Remark 3.3. Let X be a Banach space, $x_0 \in X$ and $T : X \to X$ be a (Γ, l)-ball-near identity operator with respect to x_0. Then, by Theorem 3.1, there exists $x^* \in \bar{B}(x_0; R_0)$ such that $T(x^*) = 0$. The element x^* is the unique zero of T in X, thus x^* is the unique fixed point of S_R for all $R \geq R_0$. For each $R > 0$ there exists $\tilde{R} > 0$ such that $\bar{B}(x^*; R) \subset \bar{B}\left(x_0; \tilde{R}\right)$ then $S_{\tilde{R}}\left(\bar{B}(x^*; R)\right) \subset \bar{B}(x^*; R)$.

Let $\tilde{\Gamma}(R) := \Gamma\left(\tilde{R}\right)$, $\tilde{l}(R) := l\left(\tilde{R}\right)$ and $S_R := S_{\tilde{R}}$. Then the operator T is $\left(\tilde{\Gamma}, \tilde{l}\right)$-ball-near identity with respect to x^*.

Remark 3.4. The condition (3) in Definition 3.1 is essential, as the following example illustrates.

Let $(X, \|\cdot\|) := (\mathbb{R}, |\cdot|)$, $T(x) := -e^{-x}$ and $\Gamma(R) := \frac{1}{2e^R} 1_{\mathbb{R}}$ for all $R > 0$. Then

$$S_R(x) = x + \frac{1}{2e^R} e^{-x}.$$

Since $S_R'(x) = 1 - \frac{1}{2e^R} e^{-x}$, it is clear that $S_R\big|_{[-R,R]} : [-R, R] \to \mathbb{R}$ is $\left(1 - \frac{e^{-R}}{2e^R}\right)$-contraction, but $F_{S_R} = Z_T = \varnothing$.

We remark that no interval $[-R, R]$ is invariant for S_R, $R > 0$. In this example $T'(x) > 0$, $\forall x \in \mathbb{R}$, and $\inf_{\mathbb{R}} T'(x) = 0$.

For more consideration on this example see Example 6.1 in this paper.

4. Data dependence

Let $(X, \|\cdot\|)$ be a Banach space, $Y \subset X$ and $T, \tilde{T} : Y \to X$ two operators. We suppose that $Z_T = \{x^*\}$ and there exists $\eta > 0$ such that

$$\left\|T(x) - \tilde{T}(x)\right\| \leq \eta, \quad \forall x \in Y.$$

The problem is if there exists an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous in 0, with $\psi(0) = 0$, such that

$$\|x^* - \tilde{x}^*\| \leq \psi(\eta), \ \forall \tilde{x}^* \in Z_{\tilde{T}}.$$

For this problem we have the following result in terms of ball-near identity operators.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a Banach space and $T, \tilde{T} : X \to X$ two operators. We suppose that:*

1. *T is as in Theorem 3.1.*
2. *For each $R \geq R_0$, there exists $\eta_R > 0$ such that*

$$\left\|T(x) - \tilde{T}(x)\right\| \leq \eta_R, \quad \forall x \in \bar{B}(x_0; R).$$

Then we have that

$$\|x^* - \tilde{x}^*\| \leq (1 - l(R))^{-1} \|\Gamma(R)\| \eta_R, \quad \forall \tilde{x}^* \in Z_{\tilde{T}} \cap \bar{B}(x_0; R).$$

Proof. In the conclusion (iii) of Theorem 3.1 we take $y := \tilde{x}^*$. We have

$$\begin{aligned} \|x^* - \tilde{x}^*\| &\leq (1 - l(R))^{-1} \|\Gamma(R) T(\tilde{x}^*)\| \\ &= (1 - l(R))^{-1} \left\|\Gamma(R)\left(T(\tilde{x}^*) - \tilde{T}(\tilde{x}^*)\right)\right\| \\ &\leq (1 - l(R))^{-1} \|\Gamma(R)\| \eta_R. \end{aligned}$$

□

Remark 4.1. For the data dependence of the fixed point in terms of retraction-displacement condition see: [53], [54], [9], [42], [58], ...

5. Ulam stability

Let $(X, \|\cdot\|)$ be a Banach space, $Y \subset X$ and $T : Y \to X$. By definition, the zero point equation

$$T(x) = 0 \tag{5.1}$$

is Ulam stable if for each $\varepsilon > 0$ and each y^* a solution of

$$\|T(y)\| \leq \varepsilon \tag{5.2}$$

there exists a solution x^* of the equation (5.1) and $c > 0$ such that

$$\|x^* - y^*\| \leq c\varepsilon.$$

We have the following result in terms of ball-near identity operators.

Theorem 5.1. *Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ an operator as in the Theorem 3.1. Then we have that:*

$$\|x^* - y^*\| \leq (1 - l_R)^{-1} \|\Gamma(R)\| \varepsilon,$$

for each $y^* \in \bar{B}(x_0; R)$ solution of (5.2), $R \geq R_0$.

Proof. From the conclusion (iii) of Theorem 3.1, with $y := y^* \in \bar{B}(x_0; R)$, we have

$$\|x^* - y^*\| \leq (1 - l_R)^{-1} \|\Gamma(R) T(y^*)\| \leq (1 - l_R)^{-1} \|\Gamma(R)\| \varepsilon. \qquad \square$$

Remark 5.1. For the Ulam stability of operatorial equations see, for example, [44], [48], [54] and the references therein.

6. Some examples

Example 6.1. Let $(X, ||\cdot||)$ be $(\mathbb{R}, |\cdot|)$ and $T \in C^1(\mathbb{R}, \mathbb{R})$. We assume that there exists some $m > 0$ such that

$$T'(x) \geq m, \ \forall x \in \mathbb{R}.$$

Let $x_0 \in \mathbb{R}$ be fixed. For each $R > 0$ we take some

$$M_R \geq T'(x), \ \forall x \in [x_0 - R, x_0 + R].$$

Note that $M_R \geq m$, thus

$$0 < 1 - \frac{m}{2M_R} < 1.$$

Now define

$$\Gamma(R) := \frac{1}{2M_R} 1_{\mathbb{R}} \text{ and } S_R := 1_{\mathbb{R}} - \frac{1}{2M_R} T.$$

Using the Mean Value Theorem one can easily prove that S_R is an $(1 - m/(2M_R))$-contraction on $[x_0 - R, x_0 + R]$. Thus $l(R) = 1 - m/(2M_R)$.

Now we are concerned finding values of R such that condition (3) is fulfilled. Using Remark 3.1 it is sufficient if condition (3') is fulfilled. Since

$$\Gamma(R)T(x_0) = \frac{T(x_0)}{2M_R} \ \ \text{and} \ \ (1 - l(R))R = \frac{mR}{2M_R}$$

it can proved that, in this situation, condition (3') is fulfilled for each $R \geq R_0$, where

$$R_0 = \frac{|T(x_0)|}{m} \ \ if \ \ T(x_0) \neq 0,$$

respectively,

$$R_0 > 0 \ \ fixed \ \ if \ \ T(x_0) = 0.$$

Thus

$$S_R\left([x_0 - R, x_0 + R]\right) \subset [x_0 - R, x_0 + R], \quad \forall R \geq R_0.$$

We also deduce that the function T is ball near-identity with respect to any $x_0 \in \mathbb{R}$. Remark that the condition that the interval $[x_0 - R, x_0 + R]$ is invariant by the function S_R for each $R \geq R_0$ is not necessary in the definition of ball near-identity map.

Taking into account all the above comments, it is not difficult to prove that the Saturated Contraction Principle 2.1 and Theorem 3.1 have the following consequence in this case.

Theorem 6.1. *Let $T \in C^1(\mathbb{R}, \mathbb{R})$ be such that there exists some $m > 0$ with $T'(x) \geq m$ for all $x \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ and $R_0 = |T(x_0)|/m$. For each $R > 0$ we take $M_R = \max\limits_{x \in [x_0 - R, x_0 + R]} T'(x)$. Then the following affirmations are valid.*

(i) $Z_T = \{x^*\}$.

(ii) *For each $y_0 \in \mathbb{R}$ and $R \geq R_0$ such that $|x_0 - y_0| \leq R$, the sequence $\{y_n\}_{n \in \mathbb{N}}$, defined by*

$$y_{n+1} = y_n - \frac{1}{2M_R} T\left(y_n\right), \quad n \geq 0$$

converges to x^* as $n \to +\infty$.

(iii) $|y - x^*| \leq \frac{1}{m}|T(y)|$, *for all $y \in \mathbb{R}$.*

(iv) *If the sequence of reals $\{y_n\}_{n \in \mathbb{N}}$ is bounded and such that*

$$T\left(y_n\right) \to 0 \ \ as \ \ n \to +\infty$$

then $y_n \to x^*$ as $n \to +\infty$.

(v) *For each $R \geq R_0$, if the sequence $\{y_n\}_{n \in \mathbb{N}} \subset [x_0 - R, x_0 + R]$ is such that*

$$y_{n+1} - y_n + \frac{1}{2M_R} T\left(y_n\right) \to 0 \ \ as \ \ n \to +\infty$$

then $y_n \to x^*$ as $n \to +\infty$.

Example 6.2 (*Functions with strictly diagonally dominant Jacobian matrices*)**.** Let $d \geq 1$, $(X, \|\cdot\|)$ be $(\mathbb{R}^d, \|\cdot\|_\infty)$ and $T \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Denote by T_1,..., T_d the components of T. We assume that there exists some $m > 0$ such that

$$\frac{\partial T_k}{\partial x_k}(x) - \sum_{j \neq k} \left| \frac{\partial T_k}{\partial x_j}(x) \right| \geq m, \quad \forall\, k = \overline{1, d}, \, \forall\, x \in \mathbb{R}^d. \tag{6.1}$$

Let $x_0 \in \mathbb{R}^d$ be fixed. For each $R > 0$ we take some $M_R > 0$ such that

$$\frac{\partial T_k}{\partial x_k}(x) \leq M_R, \quad \forall\, k = \overline{1, d}, \, \forall\, x \in \overline{B}(x_0, R). \tag{6.2}$$

Note that $M_R \geq m$, thus $0 < 1 - \frac{m}{2M_R} < 1$. Now define

$$\Gamma(R) := \frac{1}{2M_R} 1_{\mathbb{R}^d} \quad \text{and} \quad S_R := 1_{\mathbb{R}^d} - \frac{1}{2M_R} T.$$

Remind that, for a vector $u \in \mathbb{R}^d$ with the components $u_1, ..., u_d$ we have

$$\|u\|_\infty = \max\{|u_1|, ..., |u_d|\},$$

while for a matrix $A \in \mathbb{R}^{d\times d}$ we have

$$\|A\|_\infty = \sup\left\{\frac{\|Au\|_\infty}{\|u\|_\infty} : \quad u \in \mathbb{R}^d,\ u \neq 0\right\}.$$

By $DT(x)$ we denote the Jacobian matrix of T computed in $x \in \mathbb{R}^d$.

Lemma 6.1. *For each $R > 0$ the function S_R is an $(1 - \frac{m}{2M_R})$-contraction on $\overline{B}(x_0, R)$.*

Proof. For all $\xi, u \in \mathbb{R}^d$ we have that

$$\begin{aligned}
\|DS_R(\xi)u\|_\infty &= \left\|u - \frac{1}{2M_R}DT(\xi)u\right\|_\infty = \max_{k=\overline{1,d}}\left|u_k - \frac{1}{2M_R}\sum_{j=1}^{d}\frac{\partial T_k}{\partial x_j}(\xi)u_j\right| \\
&\leq \max_{k=\overline{1,d}}\left\{\left|1 - \frac{1}{2M_R}\frac{\partial T_k}{\partial x_k}(\xi)\right| + \frac{1}{2M_R}\sum_{j=1,j\neq k}^{d}\left|\frac{\partial T_k}{\partial x_j}(\xi)\right|\right\}\|u\|_\infty \\
&= \max_{k=\overline{1,d}}\left\{1 - \frac{1}{2M_R}\left[\frac{\partial T_k}{\partial x_k}(\xi) - \sum_{j=1,j\neq k}^{d}\left|\frac{\partial T_k}{\partial x_j}(\xi)\right|\right]\right\}\|u\|_\infty \\
&\leq \left(1 - \frac{m}{2M_R}\right)\|u\|_\infty.
\end{aligned}$$

In the above estimations we used (6.2) and (6.1). In short, we proved that

$$\|DS_R(\xi)u\|_\infty \leq \left(1 - \frac{m}{2M_R}\right)\|u\|_\infty, \quad \forall\, \xi, u \in \mathbb{R}^d. \tag{6.3}$$

Now let $x, y \in \overline{B}(x_0, R)$ and denote $\xi_s = (1 - s)x + sy$ for each $s \in [0, 1]$. From the Mean value theorem in integral form and (6.3) we have

$$\begin{aligned}
\|S_R(y) - S_R(x)\|_\infty &= \left\|\int_0^1 DS_R(\xi_s)(y - x)\,ds\right\|_\infty \leq \int_0^1 \|DS_R(\xi_s)(y - x)\|_\infty \\
&\leq \left(1 - \frac{m}{2M_R}\right)\|y - x\|_\infty.
\end{aligned}$$

The proof is done. □

Let $R_0 > \|T(x_0)\|_\infty/m$ be fixed. The comments follow now exactly like in the previous example such that we conclude that the function T is ball near-identity with respect to each $x_0 \in \mathbb{R}^d$ and, in addition, the ball $\overline{B}(x_0, R)$ is invariant by S_R for all $R \geq R_0$. Thus, the Saturated Contraction Principle 2.1 and Theorem 3.1 have the following consequence in this case.

Theorem 6.2. *Let $T \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ be such that there exists some $m > 0$ satisfying* (6.1)*. Let $x_0 \in \mathbb{R}^d$ and $R_0 = \|T(x_0)\|_\infty/m$. For each $R > 0$ we take $M_R > 0$ satisfying* (6.2)*. Then the following affirmations are valid.*

(i) $Z_T = \{x^*\}$.

(ii) *For each* $y_0 \in \mathbb{R}^d$ *and* $R \geq R_0$ *such that* $y_0 \in \overline{B}(x_0, R)$, *the sequence* $\{y_n\}_{n\in\mathbb{N}}$, *defined by*

$$y_{n+1} = y_n - \frac{1}{2M_R} T(y_n), \quad n \geq 0$$

converges to x^* *as* $n \to +\infty$.

(iii) $\|y - x^*\|_\infty \leq \frac{1}{m} \|T(y)\|_\infty$, *for all* $y \in \mathbb{R}^d$.

(iv) *If the sequence* $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ *is bounded and such that*

$$T(y_n) \to 0 \quad as \quad n \to +\infty$$

then $y_n \to x^*$ *as* $n \to +\infty$.

(v) *For each* $R \geq R_0$, *if the sequence* $\{y_n\}_{n\in\mathbb{N}} \subset \overline{B}(x_0, R)$ *is such that*

$$y_{n+1} - y_n + \frac{1}{2M_R} T(y_n) \to 0 \quad as \quad n \to +\infty$$

then $y_n \to x^*$ *as* $n \to +\infty$.

Remark 6.1. Theorem 6.1 is the particular case for $d = 1$ of Theorem 6.2.

Remark 6.2. These examples are inspired by Theorem 1 in the paper [67] by Zhang and Ge, and our effort to understand its proof. Note that Theorem 6.2 (i) is a zero-point result, while Theorem 1 in [67] is a global implicit function theorem result. The strategy to prove Theorem 1 for some map $T : \mathbb{R}^p \times \mathbb{R}^d \to \mathbb{R}^d$ is to prove that $T(x, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ has a zero for all $x \in \mathbb{R}^p$. We noted that in some places the proof of Theorem 1 in [67] lacks of rigor and that the proof of the solution uniqueness is missing. Moreover, the corresponding hypothesis on $T(x, \cdot)$ used in [67] is weaker than (6.1) but their argument works only if (6.1) is assumed and fails under their hypothesis. We noted also that the hypotheses of Corollaries 1 and, respectively, 2 in [67] are contradictory, thus there is no function to which these corollaries apply. Based on Example 6.2, in Example 7.1 we complete the proof of the global implicit function Theorem 1 in [67] under the hypothesis (6.1) for $T(x, \cdot)$.

Example 6.3. Let X be a real Hilbert space and $T : X \to X$ an operator. We suppose that:

(i) There exists $m > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq m \|x - y\|^2, \ \forall x, y \in X.$$

(ii) For each $R > 0$, there exists $M(R) > 0$ such that:

$$\|T(x) - T(y)\| \leq M(R) \|x - y\|, \ \forall x, y \in \bar{B}(\theta; R).$$

Let us choose m and $M(R)$ such that $m < 1$ and $M(R) > 1$.

First, we are looking for $\gamma(R) > 0$ such that for the operator $S_R : X \to X$, $S_R := 1_X - \gamma(R) T$ we have that $S_R \big|_{\bar{B}(\theta;R)} : \bar{B}(\theta; R) \to X$ is a contraction. From $(i) - (ii)$ we have

$$\begin{aligned}\|S_R(x) - S_R(y)\|^2 &= \|x - y\|^2 - 2\gamma(R) \langle T(x) - T(y), x - y \rangle + \gamma^2(R) \|T(x) - T(y)\|^2 \\ &\leq \left(1 - 2\gamma(R) m + \gamma^2(R) M^2(R)\right) \|x - y\|^2.\end{aligned}$$

If we take $\gamma(R) := \frac{m}{M^2(R)}$ then

$$\|S_R(x) - S_R(y)\| \le l(R)\|x - y\|,$$

with $l(R) = \left(1 - \frac{m^2}{M^2(R)}\right)^{\frac{1}{2}}$.

Now, we are looking for $R_0 > 0$ such that $S_{R_0}\left(\bar{B}(\theta; R_0)\right) \subset \bar{B}(\theta; R_0)$. For example, we have a such R_0 if

(iii) $\|\gamma(R) T(\theta)\| \le (1 - l(R_0)) R_0$.

So, in the above conditions $(i) - (iii)$ the operator T is $(\gamma(R), l(R))$-near the 1_X and we have for T a corresponding theorem as Theorem 3.1.

Remark 6.3. For the condition (i) see [13], [56], [18], ...

7. Applications to the implicit operator problem

There are various techniques in the theory of implicit function and of implicit operators (see, for example, [18], [21], [24], [28], [31], [32], [34], [47], [62], [26], [25], [3], [36], [37], [38], [2], [19], [22], [30], [40], [20], [66], [8], [1], [17], [46], ...). In what follows we give some application of Theorem 3.1.

Let X be a nonempty set, $(Y, \|\cdot\|)$ a Banach space and $T : X \times Y \to Y$ such that $T(x, \cdot) : Y \to Y$ is a (Γ, l)-ball-near identity operator with respect to $y_0 \in Y$. In general, Γ, l, y_0 and R_0 are depending on x. Let us use in this case the notations: $\Gamma(R; x)$, $l(R; x)$, $y_0(x)$ and $R_0(x)$. From Theorem 3.1, we have:

Theorem 7.1. *Let X be a nonempty set, $(Y, \|\cdot\|)$ a Banach space and $T : X \times Y \to Y$ an operator. We suppose that:*

(1) *T is as above;*
(2) $\|\Gamma(R_0(x); x) T(x, y_0(x))\| \le (1 - l(R_0(x); x)) R_0(x)$, $\forall x \in X$.

Then there exists a unique $\Phi : X \to Y$ such that

$$T(x, \Phi(x)) = 0, \quad \forall x \in X.$$

To study some properties of the operator Φ, the following result is useful.

Theorem 7.2. *Let X be a nonempty set, $(Y, \|\cdot\|)$ a Banach space and $T : X \times Y \to Y$ an operator. We suppose that:*

(1) *$T(x, \cdot) : Y \to Y$ is a (Γ, l)-ball-near identity operator with respect to $y_0 \in Y$, for all $x \in X$.*
(2) *(Γ, l) and y_0 do not depend on $x \in X$, and*

$$\|\Gamma(R_0) T(x, y_0)\| \le (1 - l(R_0)) R_0, \quad \forall x \in X.$$

Then there exists a unique operator $\Phi : X \to Y$, such that

$$T(x, \Phi(x)) = 0, \ \forall x \in X.$$

Moreover we have that:

(i) *For each* $x \in X$ *and* $z_0 \in \bar{B}(y_0; R_0)$, *for the sequence* $\{z_n\}_{n\in\mathbb{N}}$ *defined by*

$$z_{n+1} = z_n - \Gamma(R_0) T(x, z_n), \ n \in \mathbb{N},$$

we have that

$$z_n \to \Phi(x) \ \text{as } n \to +\infty.$$

(ii) $\|y - \Phi(x)\| \leq \frac{1}{1-l(R)} \|\Gamma(R) T(x, y)\|$, $\forall y \in \bar{B}(y_0; R)$, $\forall R \geq R_0$.

(iii) *If* $R \geq R_0$, *for each* $\{z_n\}_{n\in\mathbb{N}} \subset \bar{B}(y_0; R)$ *such that*

$$T(x, z_n) \to 0 \ \text{as } n \to +\infty,$$

for some $x \in X$, *we have that*

$$z_n \to \Phi(x) \ \text{as } n \to +\infty.$$

(iv) *If* $R \geq R_0$ *and* $\{z_n\}_{n\in\mathbb{N}} \subset \bar{B}(y_0; R)$ *is such that*

$$z_{n+1} - z_n + \Gamma(R) T(x, z_n) \to 0 \ \text{as } n \to +\infty,$$

for some $x \in X$, *then*

$$z_n \to \Phi(x) \ \text{as } n \to +\infty.$$

Proof. We apply Theorem 3.1 in the case of operators $T(x, \cdot) : Y \to Y$, $x \in X$. □

Remark 7.1. If, in Theorem 7.2, X is a topological space and, in addition, $T : X \times Y \to Y$ is continuous then the operator $\Phi : X \to Y$, defined by

$$T(x, \Phi(x)) = 0, \ \forall x \in X,$$

is continuous.

Indeed, let $x_n \to x^*$. Then $\Phi(x_n)$ and $\Phi(x^*) \in \bar{B}(y_0; R_0)$. From conclusion (ii) of Theorem 7.2 we have that

$$\begin{aligned}
\|\Phi(x^*) - \Phi(x_n)\| &\leq \frac{1}{1 - l(R_0)} \|\Gamma(R_0) T(x_n, \Phi(x^*))\| \\
&\leq \frac{1}{1 - l(R_0)} \|\Gamma(R_0)\| \|T(x_n, \Phi(x^*))\| \\
&\to \frac{1}{1 - l(R_0)} \|\Gamma(R_0)\| \|T(x^*, \Phi(x^*))\| = 0.
\end{aligned}$$

Remark 7.2. If, in Theorem 7.2, (X, d) is a metric space and, in addition, there exists $L > 0$ such that

$$\|T(x_1, y) - T(x_2, y)\| \leq L d(x_1, x_2), \ \forall x_1, x_2 \in X, \ \forall y \in Y,$$

then

$$\|\Phi(x_1) - \Phi(x_2)\| \leq \frac{L \|\Gamma(R_0)\|}{1 - l(R_0)} d(x_1, x_2).$$

Indeed, from conclusion (ii) of Theorem 7.2 we have that

$$\begin{aligned}\|\Phi(x_1)-\Phi(x_2)\| &\leq \frac{1}{1-l(R_0)}\|\Gamma(R_0)T(x_1,\Phi(x_2))\| \\ &\leq \frac{1}{1-l(R_0)}\|\Gamma(R_0)\|\|T(x_1,\Phi(x_2))-T(x_2,\Phi(x_2))\| \\ &\leq \frac{L\|\Gamma(R_0)\|}{1-l(R_0)}d(x_1,x_2).\end{aligned}$$

To illustrate the relevance of the Theorems 7.1 and 7.2, let us consider the following example.

Example 7.1. Let X be a nonempty set, $(Y,\|\cdot\|):=\left(\mathbb{R}^d,\|\cdot\|_\infty\right)$ and $T:X\times\mathbb{R}^d\to\mathbb{R}^d$ be an operator. We suppose that $T(x,\cdot)\in C^1\left(\mathbb{R}^d,\mathbb{R}^d\right)$ and there exists $m>0$ such that

$$\frac{\partial T_k}{\partial y_k}(x,y)-\sum_{j\neq k}\left|\frac{\partial T_k}{\partial y_j}(x,y)\right|\geq m,\quad \forall\, k=\overline{1,d},\ \forall\, x\in X,\ \forall y\in\mathbb{R}^d.$$

Let $y_0\in\mathbb{R}^d$ be fixed. Let $x\in X$ arbitrary. For each $R>0$ we remark that (see Example 6.2) there exists $M(R;x)>0$ such that

$$\frac{\partial T_k}{\partial y_k}(x,y)\leq M(R;x)\,,\quad \forall\, k=\overline{1,d},\ \forall\, y\in\overline{B}(y_0,R).$$

Now we use Example 6.2, hence we take

$$\Gamma(R;x)=\frac{1}{2M(R;x)}\quad \textit{and}\quad S(R;x):=1\,|_{\mathbb{R}^d}-\frac{1}{2M(R;x)}T(x,\cdot)\,.$$

If we take some

$$R_0(x)>\frac{\|T(x,y_0)\|_\infty}{m}$$

then we are in the conditions of Theorem 7.1. If we take $X:=(\mathbb{R}^p,\|\cdot\|_\infty)$ and, in addition, we suppose that $T\in C^1\left(\mathbb{R}^p\times\mathbb{R}^d,\mathbb{R}^d\right)$ then, from Theorem 7.1 and the classical implicit function theorem we have a complete proof of the Theorem 1 in [67].

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